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Probability density equivalent linearization and non-linearization techniques

L. SOCHA

Institute of Transport, Silesian Technical University, ul. Krasińskiego 8, 40–019 Katowice, POLAND

THE CONCEPT of equivalent linearization and non-linearization for dynamic systems under Gaussian excitations with criteria in probability density space is considered in this paper. The term non-linearization used in the literature means the procedure of finding an equivalent nonlinear system for stochastically excited Hamiltonian system. New criteria of linearization and non-linearization and two approximate approaches are proposed. In the first one, the direct minimization of a criterion is applied and the approximation of the probability density function by the truncated Gram-Charlier expansion is used. In the second approach, equivalent linearization or non-linearization is made for the Fokker-Planck equations corresponding to the original nonlinear and linearized or non-linearized dynamic systems, respectively. Two examples are given to illustrate the results obtained.

1. Introduction

EQUIVALENT LINEARIZATION was first proposed by CAUGHEY [3] who considered the replacement of a nonlinear oscillator by a linear one for which coefficients of linearization can be found from the mean square criterion. These coefficients depend on the first and second order moments of the response. Equivalent linearization has been developed in the field of control, mechanical and structural engineering, and has been generalized by many authors. Numerous studies have been performed in the context of this method, and they are summarized in the monograph by ROBERTS and SPANOS [11] and the review article by SOCHA and SOONG [12]. In almost all studies of different versions of stochastic linearization, the difference between variances of nonlinear and linearized systems has been taken as a measure of the accuracy of the considered version. The most important reason for considering only this measure is the fact that stochastic linearization is treated as one of the numerous approximate methods used in studies of vibration systems, where the well known and simple covariance analysis can be used. From theoretical point of view, the best linearization technique in the sense of previously discussed measure should be the "true linearization" proposed by KOZIN [7], where the variances of the outputs of nonlinear and linearized systems are the same. In fact, this true linearization is not true with respect to other criteria, for instance, higher order moments, correlation functions or spectral densities of responses of nonlinear and linearized systems. The difference between the correlation functions and the spectral densities of responses of nonlinear and linearized systems as a measure of accuracy was considered, for instance, by Kazakov and Dostupov [6] and by Apetaur and Opicka [1], Iyengar [5], respectively. The idea of equivalent linearization has been developed to the case when the original nonlinear system is replaced by another equivalent nonlinear system for which the exact probability density function of the stationary solution is known. Numerous studies have been performed in the context of this method, see, for instance, [4, 8, 14, 16], where mainly the mean-square criteria were used.

Since the complete information about random variable is contained in probability density function, it would be reasonable to consider a criterion depending on the difference between probability densities of responses of nonlinear and linearized or non-linearized systems. Therefore in this paper a new philosophy for stochastic equivalent linearization and non-linearization is proposed. We introduce new criteria of linearization and non-linearization and we discuss two approximate approaches. In the first one, the direct minimization of a criterion is applied and the approximation of the probability density function by the Gram-Charlier expansion is used. In the second approach, the linearization or non-linearization is made for the Fokker-Planck equations corresponding to the original nonlinear and linearized or non-linearized dynamic systems, respectively. The proposed approach is a generalization of the considerations given in [13]. The detailed analysis for two-dimensional systems is given to illustrate the results obtained. To compare characteristics of the responses obtained by the proposed methods and other equivalent linearization or non-linearization techniques, the examples with exactly known stationary probability density functions have been chosen.

2. Equivalent linearization

We consider a nonlinear stochastic model of dynamic system described by the Ito vector differential equation

$$d\mathbf{x}(t) = \mathbf{\Phi}(x,t)dt + \sum_{k=1}^{M} \mathbf{G}_k(t)d\xi_k(t), \tag{2.1}$$

where $\mathbf{x} = [x_1, \ldots, x_n]^T$ is vector state, $\mathbf{\Phi} = [\Phi_1, \ldots, \Phi_n]^T$ is a vector nonlinear function, $\mathbf{G}_k = [G_{k1}, \ldots, G_{kn}]^T$ are deterministic vectors, ξ_k are independent standard Wiener processes.

We assume that the unique solution of Eq. (2.1) exists and an equivalent linear system has the form

$$dx(t) = [\mathbf{A}(t)x(t) + \mathbf{C}(t)]dt + \sum_{k=1}^{M} \mathbf{G}_k(t)d\xi_k(t), \tag{2.2}$$

where $\mathbf{A} = [\mathbf{a}_{ij}]$ is a matrix and $\mathbf{C} = [C_1, \dots, C_n]^T$ is a vector of linearization coefficients.

The objective of the probability density equivalent linearization is to find the elements a_{ij} and C_i which minimize the criterion

$$I_1 = \int_{-\infty}^{+\infty} w(\mathbf{x})\Psi(g_N(\mathbf{x}) - g_L(\mathbf{x}))d\mathbf{x}, \tag{2.3}$$

where Ψ is a convex function, $w(\mathbf{x})$ is a weight function, $g_N(\mathbf{x})$ and $g_L(\mathbf{x})$ are probability density functions of stationary solutions of nonlinear system (2.1) and linearized system (2.2), respectively. It means that the discussed equivalent linearization method is made for criteria in the space of probability densities. In the case of linearized system, the probability density of the solution of system (2.2) is known and can be expressed as follows:

$$g_L(x) = [(2\pi)^n|\mathbf{K}_L|]^{-1/2} \exp\left\{-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T\mathbf{K}_L^{-1}(\mathbf{x}-\mathbf{m})\right\}, \tag{2.4}$$

where $\mathbf{m} = \mathbf{m}(t) = E[\mathbf{x}(t)]$ and $\mathbf{K}_L = \mathbf{K}_L(t) = E[\mathbf{x}(t)\mathbf{x}(t)^T] - \mathbf{m}(t)\mathbf{m}(t)^T$ are the mean value and the covariance matrix of the solution $\mathbf{x} = \mathbf{x}(t)$ of system (2.2), respectively, $|\mathbf{K}_L|$ denotes the determinant of the matrix $\mathbf{K}_L$. The vector $\mathbf{m}$ and matrix $\mathbf{K}_L$ satisfy the following equations:

$$\frac{d\mathbf{m}}{dt} = A(t)m + C(t), \tag{2.5}$$

$$\frac{d\mathbf{K}_L}{dt} = \mathbf{K}_L\mathbf{A}^T(t) + \mathbf{A}(t)\mathbf{K}_L + \sum_{k=1}^{M} \mathbf{G}_k(t)\mathbf{G}_k^T(t). \tag{2.6}$$

To apply the proposed criterion (2.3) we have to find the probability density $g_N(\mathbf{x})$. Unfortunately, except for some special cases, it is impossible to find the function $g_N(\mathbf{x})$ in analytical form. However, it can be done by approximation methods or by simulations.

To obtain approximate the probability density function of the stationary solution of a nonlinear dynamic system one can use for instance, the Gram-Charlier expansion. For n-dimensional system the one-dimensional density has the following truncated form [10]

$$g_N(\mathbf{x}) = g_{GC}(\mathbf{x}) = g_G(\mathbf{x}) \left[1 + \sum_{k=3}^{N} \sum_{\sigma(\nu)=k} \frac{c_\nu H_\nu(\mathbf{x}-\mathbf{m})}{\nu_1!...\nu_n!} \right], \tag{2.7}$$

where $g_G(\mathbf{x})$ is the probability density of a vector Gaussian random variable $\mathbf{x} \in R^n$,

$$g_G(\mathbf{x}) = [(2\pi)^n |\mathbf{K}_G|]^{-1/2} \exp\left\{ -\frac{1}{2}(\mathbf{x}-\mathbf{m})^T \mathbf{K}_G^{-1} (\mathbf{x}-\mathbf{m}) \right\}, \tag{2.8}$$

$\mathbf{m}$ and $\mathbf{K}_G$ are the mean value and covariance matrix of vector variable $\mathbf{x}$, ν is the multi-index $\nu = [\nu_1, ..., \nu_n]^T$, $\sigma(\nu) = \sum_{i=1}^{n} \nu_i$, N is a number of elements in truncation series, $c_\nu = E[G_\nu(\mathbf{x}-\mathbf{m})]\nu_i$ are quasimoments. H_ν and G_ν are Hermite's polynomials defined by

$$H_m(x) = (-1)^{\sigma(m)} \exp\left\{ \frac{1}{2}\mathbf{x}^T \mathbf{K}^{-1} \mathbf{x} \right\} \frac{\partial^{\sigma(m)}}{\partial x_1^{m_1} \ldots \partial x_N^{m_n}} \exp\left\{ -\frac{1}{2}\mathbf{x}^T \mathbf{K}^{-1} \mathbf{x} \right\}, \tag{2.9}$$

$$G_m(x) = (-1)^{\sigma(m)} \exp\left\{ \frac{1}{2}\mathbf{x}^T \mathbf{K}^{-1} \mathbf{x} \right\} \left[\frac{\partial^{\sigma(m)}}{\partial y_1^{m_1} \ldots \partial y_N^{m_n}} \exp\left\{ -\frac{1}{2}\mathbf{y}^T \mathbf{K}^{-1} \mathbf{y} \right\} \right]_{\mathbf{y}=\mathbf{K}^{-1}\mathbf{x}} \tag{2.10}$$

where $\mathbf{K}$ is a real positive definite matrix.

To obtain quasimoments c_ν, first we must derive the moment equations for the system (2.1) which can be closed, for instance, by the cumulant closure technique, and next we use the algebraic relationships between quasimoments and moments.

3. Direct optimization method

Since the probability density of the linearized system $g_L(x)$ is a function of coefficients of linearization a_{ij} and C_i, therefore in the case when the function $\Psi(x)$ is differentiable, the necessary conditions of minimization one can find, for instance, from conditions

$$\frac{\partial I_1}{\partial a_{ij}} = 2 \int_{-\infty}^{+\infty} w(x) \frac{\partial \Psi(g_N, g_L)}{\partial g_L} \frac{\partial g_L(x)}{\partial a_{ij}} dx = 0, \tag{3.1}$$

$$\frac{\partial I_1}{\partial C_i} = 2 \int_{-\infty}^{+\infty} w(x) \frac{\partial \Psi(g_N, g_L)}{\partial g_L} \frac{\partial g_L(x)}{\partial C_i} dx = 0. \tag{3.2}$$

In this paper we consider both differentiable and non-differentiable functions, for instance if $w(x) = 1$ and $\Psi(x) = x^2$ then we have

$$I_2 = \int_{-\infty}^{+\infty} (g_N(\mathbf{x}) - g_L(\mathbf{x}))^2 d\mathbf{x}. \tag{3.3}$$

Another criterion with non-differentiable function we propose is

$$I_3 = \int_{-\infty}^{+\infty} |x|^{2l} |g_N(\mathbf{x}) - g_L(\mathbf{x})| d\mathbf{x}, \qquad l = 1, 2 \ldots \tag{3.4}$$

Since the criteria I_2 and I_3 are known in the mathematical literature of probabilistic metrics, as square metric and pseudo-moment metric, respectively [15], we propose to call the corresponding linearization techniques by square metric equivalent linearization and $2l$-order pseudo-moment equivalent linearization, respectively. The necessary conditions (3.1) and (3.2) for criterion (3.3) will be shown in details in Sec. 5 and 6.

4. The Fokker – Planck equation approach

When the probability density function of nonlinear system is unknown and for some reason the direct optimization technique can not be applied, we propose instead of the state equations (2.1) and (2.2) to consider the corresponding reduced Fokker-Planck equations

$$\frac{\partial g_N}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} [\Phi_i(\mathbf{x}, t) g_N] + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} [b_{ij} g_N] = 0 \tag{4.1}$$

and

$$\frac{\partial g_L}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[\left(A_i^T \mathbf{x} + C_i \right) g_L \right] + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} [b_{ij} g_L] = 0, \tag{4.2}$$

where A_i^T is i-th row of matrix $\mathbf{A}$, $\mathbf{B} = [b_{ij}]$ is the diffusion matrix

$$b_{ij} = \sum_{k=1}^{M} G_{ki} G_{kj}. \tag{4.3}$$

If we denote $p_1 = g_N$, $p_{2i} = \dfrac{\partial g_N}{\partial x_i}$ and $q_1 = g_L$, $q_{2i} = \dfrac{\partial g_L}{\partial x_i}$ then the Eqs. (4.1) and (4.2) can be transformed to the following two-dimensional vector systems:

$$\frac{\partial p_1}{\partial x_i} = p_{2i}$$

$$\sum_{i=1}^{n}\left[\frac{\partial \Phi}{\partial x_i}p_1 + \Phi_i p_{2i}\right] - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial^2 b_{ij}}{\partial x_i \partial x_j}p_1 + \frac{\partial b_{ij}}{\partial x_j}p_{2i} + \frac{\partial b_{ij}}{\partial x_i}p_{2j} + b_{ij}\frac{\partial p_{2j}}{\partial x_i}\right] = 0, \tag{4.4}$$

$$\frac{\partial q_1}{\partial x_i} = q_{2i},$$

$$\sum_{i=1}^{n}\left[a_{ii}q_1 + A_i^T x q_{2i}\right] - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial^2 b_{ij}}{\partial x_i \partial x_j}q_1 + \frac{\partial b_{ij}}{\partial x_j}q_{2i} + \frac{\partial b_{ij}}{\partial x_i}q_{2j} + b_{ij}\frac{\partial q_{2j}}{\partial x_i}\right] = 0. \tag{4.5}$$

Comparing the system Eqs. (4.4) with (4.5) we find that g_N and $\dfrac{\partial g_N}{\partial x_i}$ will be approximated by g_L and $\dfrac{\partial g_L}{\partial x_i}$ respectively, when the error ε defined by

$$\varepsilon = \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left[(\Phi_i - \mathbf{A}_i^T\mathbf{x} - C_i)g_L\right] \tag{4.6}$$

will be minimal "in some sense" for all x. We note that $g_L(x)$ is the probability density of linearized system and depends on the parameters of linearized system i.e. on a_{ij} and C_i. Since $\varepsilon = \varepsilon(x)$ is a function of x, the criterion I_4 and the necessary conditions of minimum can be proposed, for instance, as follows:

$$I_4 = \int_{-\infty}^{+\infty} \varepsilon^2(\mathbf{x})d\mathbf{x}, \tag{4.7}$$

$$\frac{\partial I_4}{\partial a_{ij}} = 0, \qquad \frac{\partial I_4}{\partial C_i} = 0. \tag{4.8}$$

5. Application to a nonlinear oscillator

Consider a nonlinear oscillator described by

$$(5.1)\qquad \begin{aligned} dx_1 &= x_2 dt, \\ dx_2 &= [-2hx_2 - f(x_1)]dt + gd\xi, \end{aligned}$$

where h and q are constant parameters, $f(x_1)$ is a nonlinear function such that $f(0) = 0$, ξ is a standard Wiener process, and an equivalent linearized oscillator

$$(5.2)\qquad \begin{aligned} dx_1 &= x_2 dt, \\ dx_2 &= [-2hx_2 - k_1 x_1]dt + qd\xi, \end{aligned}$$

where k_1 is a linearization coefficient.

First we show the application of criterion I_2. In that case the probability densities of stationary solutions of nonlinear and linearized oscillators are known and have the form

$$(5.3)\qquad g_N(x) = \frac{1}{C_N} \exp\left\{-\frac{4h}{q^2}\left(\int_0^{x_1} f(s)ds + \frac{x_2^2}{2}\right)\right\},$$

$$(5.4)\qquad g_L(x_1, x_2, k_1) = \frac{1}{C_L}\frac{4h\sqrt{k_1}}{q^2} \exp\left\{-\frac{2h}{q^2}\left(k_1 x_1^2 + x_2^2\right)\right\},$$

where C_N and C_L are normalized constants.

Since in this case $E[x_1] = E[x_2] = 0$, the application of the Gram-Charlier expansion leads to the following formula of approximate probability density function:

$$(5.5)\qquad g_N(x) = g_{GC}(x) = g_G(x)\left[1 + \sum_{k=3}^{N} \sum_{\sigma(\nu)=k} \frac{c_{\nu_1\nu_2} H_{\nu_1\nu_2}(x)}{\nu_1! \nu_2!}\right]$$

where

$$(5.6)\qquad g_G(x) = \frac{1}{2\pi\sqrt{k_{11}k_{22} - k_{12}^2}} \exp\left\{-\frac{k_{11}(x_2)^2 - 2k_{12}x_1x_2 + k_{22}(x_1)^2}{2(k_{11}k_{22} - k_{12}^2)}\right\},$$

$c_{\nu_1\nu_2} = E[G_{\nu_1\nu_2}(\mathbf{x})]$ are quasimoments, ν_1, $\nu_2 = 0, 1, ..., N$, $\nu_1 + \nu_2 = 3, 4, \ldots, N$, $H_{\nu_1\nu_2}(x)$ and $G_{\nu_1\nu_2}(x)$ are Hermite's polynomials defined by

$$H_{pr}(x_1,x_2) = (-1)^{p+r}\exp\left\{\frac{1}{2}\left(q_{11}x_1^2 + 2q_{12}x_1x_2 + q_{22}x_2^2\right)\right\} \times \frac{\partial^{p+r}}{\partial x_1^p \partial x_2^r}\exp\left\{-\frac{1}{2}\left(q_{11}x_1^2 + 2q_{12}x_1x_2 + q_{22}x_2^2\right)\right\}, \tag{5.7}$$

$$G_{pr}(x_1,x_2) = (-1)^{p+r}\exp\left\{\frac{1}{2}\left(q_{11}x_1^2 + 2q_{12}x_1x_2 + q_{22}x_2^2\right)\right\} \times \left[\frac{\partial^{p+r}}{\partial y_1^p \partial y_2^r}\exp\left\{-\frac{1}{2}\left(k_{11}y_1^2 + 2k_{12}y_1y_2 + k_{22}y_2^2\right)\right\}\right]_{y=Qx}, \tag{5.8}$$

where

$$\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}, \qquad \mathbf{K}^{-1} = Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}, \tag{5.9}$$

$$k_{ij} = E[x_i x_j], \qquad i,j = 1,2.$$

In the case of stationary probability density function the corresponding moments are as follows:

$$k_{12} = 0, \qquad k_{22} = \frac{1}{q_{22}}, \qquad q_{12} = 0, \qquad k_{11} = \frac{1}{q_{11}}. \tag{5.10}$$

The second moment k_{11}, the quasimoments $c_{\nu_1\nu_2}$ and two-dimensional Hermite's polynomials $H_{\nu_1\nu_2}(x_1,x_2)$ and $G_{\nu_1\nu_2}(x_1,x_2)$ are presented in the Appendix. The moment k_{11} has to be found from moment equations. The necessary condition of minimum of I_2 defined by (3.3) takes the form

$$\frac{\partial I_2}{\partial k_1} = 2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(g_N(x) - g_L(x))\left(\frac{1}{2k_1} - 2h\frac{x_1^2}{q^2}\right)g_L(x)dx_1dx_2 = 0. \tag{5.11}$$

In the case of pseudo-moment equivalent linearization, the linearization coefficient we find by minimization of the following criterion:

$$I_3 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1^{2l}|g_N(x_1,x_2) - g_L(x_1,x_2,k_1)|dx_1dx_2, \qquad l = 1,3. \tag{5.12}$$

The linearization coefficient k_1 can be calculated numerically in two considered cases from equality (3.3) and directly from (3.4).

Next we consider the application of the Fokker-Planck equation approach to the system (5.1) – (5.2). In that case the transformed Fokker-Planck Eqs. (4.4)

– (4.5) are as follows: for nonlinear system

$$
\begin{aligned}
&\frac{\partial p_1}{\partial x_2} = p_2, \\
(5.13)\qquad &\frac{\partial p_2}{\partial x_2} = -\frac{2}{q^2}\frac{\partial}{\partial x_2}\left[(2hx_2 + f(x_1))p_1\right] + \frac{2x_2}{q^2}\frac{\partial p_1}{\partial x_1};
\end{aligned}
$$

for linearized system

$$
\begin{aligned}
&\frac{\partial q_1}{\partial x_2} = q_2, \\
(5.14)\qquad &\frac{\partial q_2}{\partial x_2} = -\frac{2}{q^2}\frac{\partial}{\partial x_2}\left[(2hx_2 + k_1x_1)q_1\right] + \frac{2x_2}{q^2}\frac{\partial q_1}{\partial x_1}.
\end{aligned}
$$

Then the following criterion is proposed

$$
(5.15)\qquad I_4 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{\frac{\partial}{\partial x_2}\left[(f(x_1) - k_1x_1)g_L(x_1, x_2, k_1)\right]\right\}^2 dx_1dx_2,
$$

where g_L is defined by (5.4).

The necessary condition of minimum of criterion (5.15) is the following:

$$
\begin{aligned}
(5.16)\qquad \frac{\partial I_4}{\partial k_1} = 2\int_{-\infty}^{+\infty}&\left\{\frac{\partial}{\partial x_2}\left[(f(x_1) - k_1x_1)g_L(x_1, x_2, k_1)\right]\right. \\
&\left[f(x_1)\frac{\partial}{\partial k_1}\left(\frac{\partial}{\partial x_2}g_L(x_1, x_2, k_1)\right) - x_1\frac{\partial}{\partial x_2}g_L(x_1, x_2, k_1)\right. \\
&\left.\left. -kx_1\frac{\partial}{\partial k_1}\left(\frac{\partial}{\partial x_2}g_L(x_1, x_2, k_1)\right)\right]\right\} dx_1dx_2 = 0
\end{aligned}
$$

where

$$
\begin{aligned}
&\frac{\partial g_L}{\partial x_2} = -\frac{4hx_2}{q^2}g_L(x_1, x_2, k_1), \\
(5.17)\qquad &\frac{\partial}{\partial k_1}\left(\frac{\partial g_L}{\partial x_2}\right) = \left[-\frac{1}{2k_1} + \frac{2hx_1^2}{q^2}\right]\frac{4hx_2}{q^2}g_L(x_1, x_2, k_1).
\end{aligned}
$$

EXAMPLE 1.

Consider a nonlinear Duffing oscillator excited by a stationary white noise described by Eq. (5.1) for

$$f(x_1) = \omega_0^2 x_1 + \varepsilon x_1^3, \tag{5.18}$$

where ω_0^2 and ε are constant parameters.

The probability density of stationary solution of the system (5.1) with conditions (5.18) has the form

$$g_N(x) = \frac{1}{C_N} \exp\left\{-\frac{2h}{q^2}\left(\omega_0^2 x_1^2 + \varepsilon \frac{x_1^4}{2} + x_2^2\right)\right\}, \tag{5.19}$$

where C_N is a normalized constant.

To apply the approximate probability density function we use formula (5.5) for $N = 6$ and $m = 0$. The second moment k_{11}, the quasimoments $c_{\nu_1\nu_2}$ and two-dimensional Hermite's polynomials $H_{\nu_1\nu_2}(x_1, x_2)$ and $G_{\nu_1\nu_2}(x_1, x_2)$ are presented in the Appendix.

The application of criterion I_2 leads to equality (5.11) with $g_N(x)$ and $g_L(x)$ given by (5.19) and (5.4), respectively, i.e.

$$\begin{aligned}\frac{\partial I_2}{\partial k_1} = 2 \int\limits_{-\infty}^{+\infty} \Bigg[\frac{1}{C_N} \exp\left\{-\frac{2h}{q^2}\left(\omega_0^2 x_1^2 + \varepsilon \frac{x_1^4}{2} + x_2^2\right)\right\} \\ - \frac{1}{C_L}\frac{4h\sqrt{k_1}}{q^2} \exp\left\{-\frac{2h}{q^2}\left(k_1 x_1^2 + x_2^2\right)\right\}\Bigg] \\ \times \left(\frac{1}{2k_1} - \frac{2hx_1^2}{q^2}\right) \frac{1}{C_L}\frac{4h\sqrt{k_1}}{q^2} \exp\left\{-\frac{2h}{q^2}\left(k_1 x_1^2 + x_2^2\right)\right\} dx_1 dx_2 = 0.\end{aligned} \tag{5.20}$$

In the case of pseudo-moment equivalent linearization, we find the linearization coefficient by minimization of the following criterion:

$$I_3 = \int\limits_{-\infty}^{+\infty} (x_1)^{2l} |g_N(x_1, x_2) - g_L(x_1, x_2, k_1)| dx_1 dx_2, \qquad l = 1, 3. \tag{5.21}$$

The linearization coefficient k_1 can be calculated numerically in two considered cases from equality (5.20) and directly from (5.21). The application of criterion I_4 leads to equality (5.16) with conditions (5.17) and with $f(x)$ given by (5.18), i.e.

(5.22)
$$\int_{-\infty}^{+\infty} \left[(\omega_0^2 - k_1)x_1 + \varepsilon x_1^3\right]^2 \left(\frac{1}{2k_1} - \frac{2hx_1^2}{q^2}\right) x_2^2 g_L^2(x_1, x_2, k_1) dx_1 dx_2$$
$$- \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left[(\omega_0^2 - k_1)x_1 + \varepsilon x_1^3\right] x_2^2 x_1 g_L^2(x_1, x_2, k_1) dx_1 dx_2 = 0.$$

Using the properties of Gaussian process we calculate from Eq. (5.22) the integrals and we show (see Appendix) that the linearization coefficient k_1 satisfies the following algebraic equation:

(5.23)
$$192h^2k_1^4 - 128\omega_0^2 h^2 k_1^3 - 64\omega_0^2 h^2 k_1^2 + 48\varepsilon h q^2 k_1$$
$$-144\varepsilon h q^2 \omega_0^2 k_1 - 75\varepsilon^2 q^4 = 0.$$

To illustrate all the methods discussed we compare stationary mean-square displacements of linearized systems (for different linearization coefficients) obtained by applying the exact and the approximate (by the Gram-Charlier expansion) probability density functions of the Duffing oscillator, and by applying the Fokker-Planck equation approach. The numerical results for parameters $\omega_0^2 = 1.85$, $\varepsilon = 1.85 \times i$, $i = 1, \ldots, 10$, $q^2 = 0.2$, $h = 0.05$ are presented in Fig. 1.

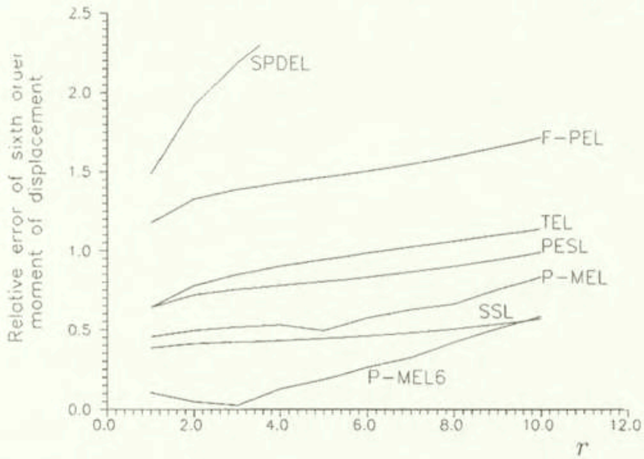

FIG. 1. a) Comparison of the relative errors of the displacement variance $E[x_1^2]$ versus the ratio of parameters $r = \varepsilon/\omega_0^2$ for $\omega_0^2 = 1.85$, $\varepsilon = 1.85 \times i$, $i = 1, \ldots, 10$ and $q^2 = 0.2$, $h = 0.05$ with notations: standard statistical linearization (SSL), potential energy statistical linearization (PESL), Fokker-Planck equation linearization (F-PEL), second order pseudo-moment equivalent linearization (P-MEL2), sixth order pseudo-moment equivalent linearization (P-MEL6), square metric probability density equivalent linearization (SPDEL).

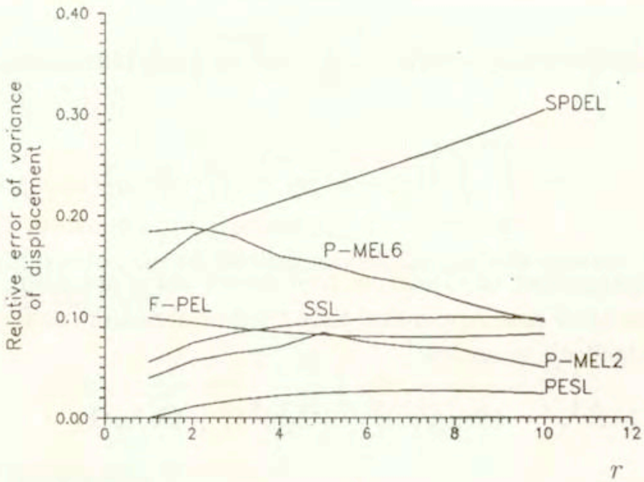

FIG. 1. b) Comparison of the relative errors of the sixth order moments of the displacement $E[x_1^6]$ versus the ratio of parameters $r = \varepsilon/\omega_0^2$ for $\omega_0^2 = 1.85$, $\varepsilon = 1.85 \times i$, $i = 1, \ldots, 10$ and $q^2 = 0.2$, $h = 0.05$ with notations: standard statistical linearization (SSL), potential energy statistical linearization (PESL), true equivalent linearization (TEL), Fokker-Planck equation linearization (F-PEL), second order pseudo-moment equivalent linearization (P-MEL2), sixth order pseudo-moment equivalent linearization (P-MEL6), square metric probability density equivalent linearization (SPDEL).

Figures 1a and 1b show that for the second order moments of the displacement, the relative error obtained by (TEL) is equal to zero and the errors obtained by (PESL) and (P-MEL2) are almost zero while the errors obtained by the other methods are significantly greater. The opposite situation is observed for sixth order moments of the displacement.

6. Equivalent non-linearization

As it was mentioned in the Introduction, the idea of finding an equivalent model for a nonlinear system has not been limited to a linear model. It has been extended to the case when the original nonlinear system is replaced by another equivalent nonlinear system for which the probability density of the exact stationary solution is known (see, for instance, [4, 8, 9, 14, 16]), and mainly the mean-square criterion was used. In all these approaches the most important role is played by the moments of response. In this paper a new philosophy for equivalent non-linearization is proposed. Instead of moment equations of responses of original nonlinear and non-linearized systems, the corresponding probability density functions are considered. The detailed discussion is given for a nonlinear oscillator described by the Ito vector differential equation

$$
\begin{aligned}
dx_1 &= x_2 dt, \\
dx_2 &= [-f(x_1, x_2) - g(x_1]dt + q d\xi(t),
\end{aligned}
\tag{6.1}
$$

where $f(.,.)$ and $g(.)$ are known nonlinear functions, $\xi(t)$ is a standard Wiener process, q is a parameter intensity of noise. An equivalent nonlinear system is proposed in the form

$$
\begin{aligned}
dx_1 &= x_2 dt, \\
dx_2 &= [-f_E(H)x_2 - g(x_1)]dt + q dw(t),
\end{aligned}
\tag{6.2}
$$

where H is the Hamiltonian

$$
H = \frac{1}{2}x_2^2 + \int_0^{x_1} g(s)ds,
\tag{6.3}
$$

f_E is a nonlinear function.

For further consideration we assume that the integral $\int_0^H f_E(s)ds$ has the polynomial form, i.e.

$$
\int_0^H f_E(s)ds = \sum_{i=0}^{N_2} \alpha_i H^i
\tag{6.4}
$$

where α_i are non-linearization coefficients. They have to be determined from a proposed criterion, for instance

$$
I = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left[f(x_1, x_2) - \left(\sum_{i=1}^{N_2} \alpha_i H^i \right) x_2 \right] \rho(x_1, x_2) dx_1 dx_2
\tag{6.5}
$$

where $\rho(x_1, x_2)$ is a weight function. Taking this function in a particular form we obtain the earlier literature results. For instance, in one of the best approximations obtained by POLIDORI and BECK [9], the weight function is given by

$$
\rho(x_1, x_2) = g_{\text{Lin}}(x_1, x_2) = \frac{1}{2\pi\sigma_{x_1}\sigma_{x_2}} \exp\left\{ -\frac{x_1^2}{2\sigma_{x_1}^2} - \frac{x_2^2}{2\sigma_{x_2}^2} \right\},
\tag{6.6}
$$

where $\sigma_{x_1}^2$ and $\sigma_{x_2}^2$ are constant parameters. As for the case of equivalent linearization, new criteria of non-linearization and two approximate approaches are proposed. In the first one the direct minimization of a criterion is applied and the approximation of the probability density function of the stationary solution of Eq. (6.1) by the Gram-Charlier expansion is proposed.

The probability density of a stationary solution of Eq. (6.2) is known exactly and has the form

$$g_{EN}(x) = \frac{1}{C_N}\exp\left\{-\frac{2}{q^2}\int_0^H f_E(s)ds\right\}. \tag{6.7}$$

where C_N is a normalized constant.

As in the case of equivalent linearization, we consider two particular cases of modified criterion (2.3) for both differentiable and non-differentiable functions $\Psi(x)$ and $w(x)$, namely

$$I_2 = \int_{-\infty}^{+\infty}(g_N(x) - g_{EN}(x))^2 dx, \tag{6.8}$$

$$I_3 = = \int_{-\infty}^{+\infty}|x|^{2l}|g_N(x) - g_{EN}(x)|dx, \quad l = 1, 2\ldots \tag{6.9}$$

Also, when for some reason the direct optimization technique can not be applied we propose, instead of state Eqs. (6.1) and (6.2), to consider the corresponding reduced Fokker-Planck equations in vector form

$$\begin{aligned} \frac{\partial p_1}{\partial x_2} &= p_2, \\ \frac{\partial p_2}{\partial x_2} &= -\frac{2}{q^2}\frac{\partial}{\partial x_2}\left[(-f(x_1,x_2) - g(x_1))p_1\right] + \frac{2x_2}{q^2}\frac{\partial p_1}{\partial x_1}, \end{aligned} \tag{6.10}$$

$$\begin{aligned} \frac{\partial q_1}{\partial x_2} &= q_2, \\ \frac{\partial q_2}{\partial x_2} &= -\frac{2}{q^2}\frac{\partial}{\partial x_2}\left[(-f_E(H)x_2 - g(x_1)q_1\right] + \frac{2x_2}{q^2}\frac{\partial q_1}{\partial x_1}, \end{aligned} \tag{6.11}$$

where $p_1 = p_1(x_1, x_2) = g_N(x_1, x_2)$, $q_1 = q_1(x_1, x_2) = g_{EN}(x_1, x_2)$.

Comparing the system Eqs. (6.10) with (6.11) we find that g_N and $\partial g_N/\partial x_i$ will be approximated by g_{EN} and $\partial g_{EN}/\partial x_i$ respectively, when the error ε defined by

$$\varepsilon = \frac{\partial}{\partial x_2}\left[(-f(x_1,x_2) + f_E(H)x_2)g_{EN}\right] \tag{6.12}$$

will be minimal " in some sense" for all x. We note that $g_{EN}(x)$ is the probability density of non-linearized system and depends on parameters of non-linearized system i.e. on α_i. Since $\varepsilon = \varepsilon(x)$ is a function of x, the criterium I_{FP} and the necessary conditions of minimum can be proposed, for instance, as follows:

$$I_{FP} = \int_{-\infty}^{+\infty} \varepsilon^2(x)dx, \tag{6.13}$$

$$\frac{\partial I_{FP}}{\partial \alpha_i} = 0, \qquad i = 1, 2, \ldots, N_2. \tag{6.14}$$

EXAMPLE 2.

We consider a nonlinearly damped Duffing oscillator excited by the white noise previously studied in the literature

$$\begin{aligned} dx_1 &= x_2 dt, \\ dx_2 &= \left[-\beta x_2 - \alpha x_2^2 - \gamma x_1 - \varepsilon x_1^3\right] dt + q d\xi(t). \end{aligned} \tag{6.15}$$

The equivalent nonlinear system is proposed in the form

$$\begin{aligned} dx_1 &= x_2 dt, \\ dx_2 &= \left[-f_E(H) x_2 - \gamma x_1 - \varepsilon x_1^3\right] dt + q d\xi(t). \end{aligned} \tag{6.16}$$

where H is the Hamiltonian

$$H = \frac{1}{2}x_2^2 + \gamma\frac{x_1^2}{2} + \varepsilon\frac{x_1^4}{4}. \tag{6.17}$$

The exact probability density of stationary solution of Eq. (6.16) is given by

$$g_{EN}(x) = \frac{1}{C_N}\exp\left\{-\frac{2}{q^2}\int_0^H f_E(s)ds\right\}. \tag{6.18}$$

However, the function f_E is unknown. We seek an approximation of the form

$$f_E(H) = b_0 + b_1 H. \tag{6.19}$$

Then the approximate probability density function is given by

$$g_{EN}(x) = \frac{1}{C_N}\exp\left\{-\frac{2}{q^2}\left(b_0 H + \frac{b_1}{2}H^2\right)\right\}, \tag{6.20}$$

In the case of the Fokker-Planck approach, the following criterion is proposed:

$$(6.21)\qquad I_{FP} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left[\frac{\partial}{\partial x_2}\left[(-\beta x_2 - \alpha x_2^3 + b_0 x_2 + b_1 H x_2) g_{EN}(x_1, x_2, h_0, h_1)\right]\right]^2 dx_1 dx_2.$$

To illustrate all the methods discussed we compare stationary mean-square displacements of non-linearized systems versus parameters ε and α, using the approximation of probability density function in the form of the Gram-Charlier expansion (for $N = 6$) and also by applying the Fokker-Planck equation approach. Since in this case exact response characteristics does not exist, the comparison is given for simulations. The numerical results for parameters $\alpha = 0.5$, $\beta = 0.1$, $\gamma = 1$, $q^2 = 2\pi$ are presented in Fig. 2.

Figures 2a and 2b show that for the second order moments of the displacement there are no significant differences between the methods proposed in this paper and the best techniques known from the literature.

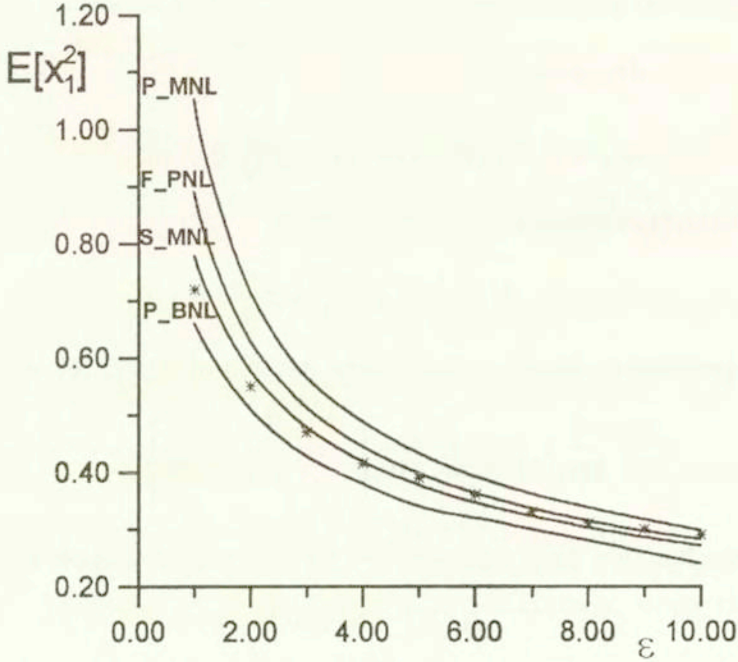

FIG. 2. a) Mean-square displacement variance $E[x_1^2]$ versus parameter ε for $\alpha = 0.5$, $q^2 = 2\pi$, $\beta = 0.1$, $\gamma = 1$ with notations: Polidori-Beck non-linearization (P-BNL), Fokker-Planck equation non-linearization (F-PNL), pseudo-moment metric equivalent non-linearization (P-MNL), square metric equivalent non-linearization (S-MNL), simulations (stars).

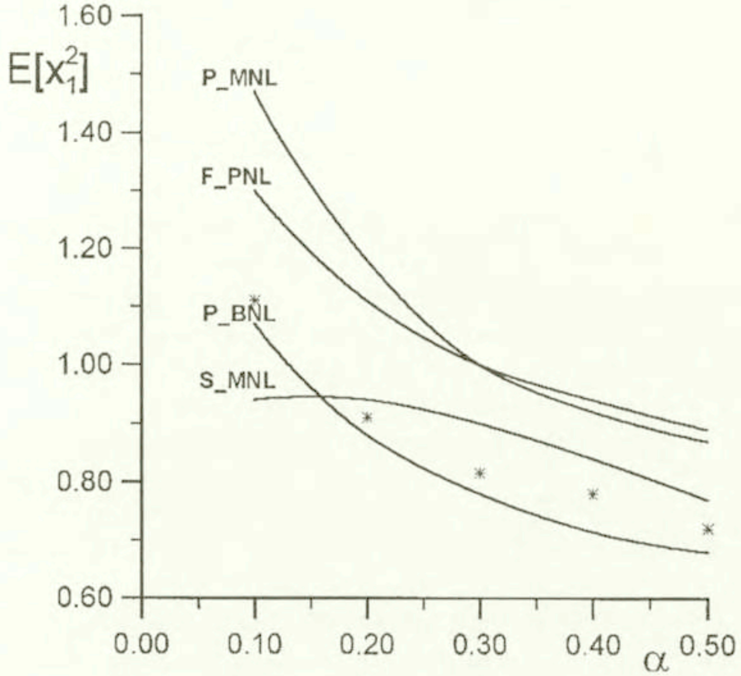

FIG. 2b) Mean-square displacement variance $E[x_1^2]$ versus parameter α for $\varepsilon = 1$, $q^2 = 2\pi$, $\beta = 0.1$, $\gamma = 1$ with notations: Polidori-Beck non-linearization (P-BNL), Fokker-Planck equation non-linearization (F-PNL), pseudo-moment metric equivalent non-linearization (P-MNL), square metric equivalent non-linearization (S-MNL), simulations (stars).

7. Conclusions and generalizations

The probability density equivalent linearization and non-linearization techniques applied to dynamic systems subjected to external Gaussian excitations have been considered. Two different approaches: direct optimization method and Fokker-Planck equation method have been examined on two examples of nonlinear oscillators. It has been shown in Example 1, that although some stochastic linearization techniques such as (PESL), (TEL) and (P-MEL2) approximate very well the mean-square error of the response, they fail in the case of higher order moments. In contrast to them, (F-PEL) and (P-MEL6) methods are more accurate for higher order moments than for the second order. It means that some of the proposed probability density equivalent linearization techniques are more accurate for second order moments of the response while the other – for higher order moments. Here it should be stressed that the standard statistical linearization technique which was criticized by many authors because it did not approximate well the second order moment of the response, is in fact "not so bad". In both comparisons (second and sixth order moments of the response) the results are "in the middle". It means that the (SSL) method can be treated as a kind of compromise between methods which approximate very well the second order and higher order moments. In contrast to the SSL method, the relative error obtained by the SPDL method was significantly greater for second and higher order moments of the response.

In the case of equivalent non-linearization, from numerical results it follows that for the second order moments of the displacement there are no significant differences between the methods proposed in this paper and the best techniques known from the literature. However, it should be stressed that the comparison was given for the second order moments while the criteria proposed in this paper were given in the space of probability density functions.

We note that, similarly to the generalization obtained for standard equivalent linearization and non-linearization techniques, several new approaches of probability density equivalent linearization and non-linearization methods can be considered. It includes the cases of criteria depending on the probability density of energy of the response and linearization of stochastic dynamic systems under parametric excitations. Also other probabilistic measures (metrics) discussed in mathematical literature [15] can be analyzed. Another generalization can be done with application of the idea of equivalent systems derived by Cai and Lin [2].

Acknowledgments

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Appendix

1. The second moment k_{11}, the quasimoments $c_{\nu_1\nu_2}$ and two-dimensional Hermite's polynomials $H_{\nu_1\nu_2}(x_1, x_2)$:

$$
\begin{aligned}
\text{(A.1)}\qquad & k_{11} = E[x_1^2], \quad c_{\nu_1\nu_2} = 0 \quad \text{for} \quad \nu_1 + \nu_2 = 3, 5, \\
& c_{40} = E[x_1^4] - 3k_{11}^2, \quad c_{31} = E[x_1^3 x_2] = 0, \quad c_{22} = E[x_1^2 x_2^2] - k_{22}k_{11}, \\
& c_{13} = E[x_1 x_2^3], \quad c_{04} = E[x_2^4] - 3k_{22}^2, \\
& c_{60} = E[x_1^6] - 15k_{11}E[x_1^4] + 30k_{11}^3, \\
& c_{51} = E[x_1^5 x_2] = 0,
\end{aligned}
$$

$$
\begin{aligned}
c_{42} &= E[x_1^4x_2^2] - k_{22}E[x_1^4] + 6k_{11}^2k_{22} - 6k_{11}E[x_1^2x_2^2],\\
c_{33} &= E[x_1^3x_2^3] - 3k_{11}E[x_1x_2^3],\\
c_{24} &= E[x_1^2x_2^4] - k_{11}E[x_2^4] - 6k_{22}E[x_1^2x_2^2] + 6k_{11}k_{22}^2,\\
c_{15} &= E[x_1x_2^5] - 10k_{22}E[x_1x_2^3], \quad c_{06} = E[x_2^6] - 15k_{22}E[x_2^4] + 30k_{22}^3,
\end{aligned}
$$

(A.2)
$$
\begin{aligned}
H_{40}(x_1,x_2) &= q_{11}^4x_1^4 - 6q_{11}^3x_1^2 + 3q_{11}^2,\\
H_{31}(x_1,x_2) &= q_{11}^3q_{22}x_1^3x_2 - 3q_{11}^2q_{22}x_1x_2,\\
H_{22}(x_1,x_2) &= q_{11}^2q_{22}^2x_1^2x_2^2 - q_{11}^2q_{22}x_1^2 - q_{11}q_{22}^2x_2^2 + q_{11}q_{22},\\
H_{13}(x_1,x_2) &= q_{11}q_{22}^3x_1x_2^3 - 3q_{22}^2q_{11}x_1x_2^2,\\
H_{04}(x_1,x_2) &= q_{22}^4x_2^4 - 6q_{22}^3x_2^2 + 3q_{22}^2,\\
H_{60}(x_1,x_2) &= q_{11}^6x_1^6 - 15q_{11}^5x_1^4 + 45q_{11}^4x_1^2 - 15q_{11}^3,\\
H_{51}(x_1,x_2) &= q_{11}^5q_{22}x_1^5x_2 - 10q_{11}^4q_{22}x_1^3x_2 + 15q_{11}^3q_{22}x_1x_2,\\
H_{42}(x_1,x_2) &= q_{11}^4q_{22}x_1^4(q_{22}x_2^2 - 1) + 6q_{11}^3q_{22}x_1^2(1 - q_{22}x_2^2)\\
&\qquad + 3q_{11}^2q_{22}(q_{22}x_2^2 - 1),\\
H_{33}(x_1,x_2) &= q_{11}^3q_{22}^3x_1^3x_2^3 + 9q_{11}^2q_{22}^2x_1x_2 - 3q_{11}^2q_{22}^2x_1^3x_2\\
&\qquad - 3q_{11}^2q_{22}^3x_1x_2^3,\\
H_{24}(x_1,x_2) &= q_{11}q_{22}^4x_2^4(q_{11}x_1^2 - 1) + 6q_{11}q_{22}^3x_2^2(1 - q_{11}x_1^2)\\
&\qquad + 3q_{11}q_{22}^2(q_{11}x_1^2 - 1),\\
H_{15}(x_1,x_2) &= q_{11}q_{22}^5x_1x_2^5 - 10q_{11}q_{22}^4x_1x_2^3 + 15q_{11}q_{22}^3x_1x_2,\\
H_{06}(x_1,x_2) &= q_{22}^6x_2^6 - 15q_{22}^5x_2^4 + 45q_{22}^4x_2^2 - 15q_{22}^3.
\end{aligned}
$$

2. Derivation of Eq. (5.23)

We introduce the notation

$$
\alpha_2 = \frac{4hk}{q^2}, \qquad c_{L_2} = \frac{16h^2}{c_L^2q^4}, \qquad g_L^2(x_1,x_2,k) = c_{L_2}k_1\exp(-\alpha_2x^2),
$$

(A.3)
$$
I_{L_2} = \sqrt{\frac{\pi}{\alpha_2}}\int_{-\infty}^{+\infty} c_{L_2}x_2^2\exp\left\{-\frac{4h}{q^2}x_2^2\right\}dx_2.
$$

Using the properties

$$\text{(A.4)}\qquad \int_{-\infty}^{+\infty} x^{2p}\exp(-\alpha x^2)dx = \sqrt{\frac{\pi}{\alpha}}\frac{(2p-1)!!}{(2\alpha)^p},$$

we calculate the integrals appearing in Eq. (5.22)

$$\text{(A.5)}\qquad \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[(\omega_0^2-k_1)x_1+\varepsilon x_1^3\right]^2\left(\frac{1}{2k_1}-\frac{2hx_1^2}{q^2}\right)x_2^2g_L^2(x_1,x_2,k_1)dx_1dx_2$$

$$= I_{L_2}\left[-\frac{2hk_1\varepsilon^2}{q^2}\frac{105}{16\alpha_2^4}+\left(\frac{\varepsilon^2}{2}-\frac{4hk_1\varepsilon(\omega_0^2-k_1)}{q^2}\right)\frac{15}{8\alpha_2^3}\right.$$

$$\left.+\left(\varepsilon(\omega_0^2-k_1)-\frac{2hk_1(\omega_0^2-k_1)^2}{q^2}\right)\frac{3}{4\alpha_2^2}+\frac{(\omega_0^2-k_1)^2}{4\alpha_2}\right],$$

$$\text{(A.6)}\qquad \int_{-\infty}^{+\infty}\left[(\omega_0^2-k_1)x_1+\varepsilon x_1^3\right]x_2^2x_1g_L^2(x_1,x_2,k_1)dx_1dx_2$$

$$= I_{L_2}k_1\left[\varepsilon\frac{3}{4\alpha_2^2}+\frac{(\omega_0^2-k_1)}{2\alpha_2}\right].$$

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