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Stochastic finite element analysis of transient heat transfer in composite materials with interface defects

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The paper presented is devoted to the application of the probabilistic computational analysis based on the stochastic finite element methods in the transient heat transfer problems; the field of application of the method introduced is the mechanics of composite materials.The composite materials considered have randomly defined thermal characteristics and, moreover, the interface discontinuities appearing between constituents have a probabilistic character. The influence of all these parameters on the first two probabilistic moments of temperature are verified on the example of a two-component layered composite with an interphase between the constituents.

1. Introduction

It has been proved by the numerous theoretical considerations and computational experiments that randomness of the material properties as well as geometrical parameters of the structural defects play a crucial role in the overall behaviour of solids. It is especially visible in the field of composite materials where the quality of bonds between the constituents in the context of some micro- or even macro-defects [13] may be decisive for the whole composite structure as it was proved for stochastic elastostatics of fiber-reinforced composite problems [6].

The paper is devoted to the probabilistic computational analysis of composite materials. The field of interest is the transient heat transfer phenomena, expected values of temperature as well as spatial and time cross-covariances. The mathematical model of the interface is based on the "bubble" model introduced in [6 – 8] where the defects have the form of semicircles lying with their diameters on the interface boundary. This strictly theoretical interface is replaced, taking into account the needs of numerical analysis, with the interphase containing all such defects. The interphase has the boundaries parallel to the original

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interface and thermal properties inserted together with structural defects into its region. The variational statement of the problem is formulated on the basis of the second order perturbation, second central probabilistic moment version of the classical transient formulation of the virtual temperatures principle. Starting from such variational equations of the zeroth, first and second order, the respective stochastic finite element equations containing the probabilistic characteristics of component materials thermal characteristics as well as interphase parameters are derived. The numerical procedure built up starting from such a model enables us to perform computational experiments with the transient heat transfer in stochastically defected composite materials. It is important to stress that all the considerations provided within the paper are valid for the stochastic linear potential field problems in electrostatic, magnetic as well as hydraulic fields [1, 15]. The approach proposed is illustrated by the example of two-component stratified composite including the interphase located between the constituents.

Finally, it should be mentioned that further computational studies on these phenomena are to be performed. Especially recommended are the stochastic sensitivity studies of the problem to verify the influence of material parameters and interface defects on overall thermal behaviour of different composite structures (stratified, fiber-reinforced or structures with periodic as well as nonperiodic geometry [6]).

2. Mathematical model

2.1. Transient heat transfer equation in composite materials

Generally, transient heat transfer problem consists in determining the temperature field T governed by the following differential equation [2]:

$$\rho c\dot{T} - (\lambda_{ij}T_{,j})_{,i} - g = 0; \qquad x_i \in \Omega; \qquad \tau \in [0,\infty), \tag{2.1}$$

where $c = c(T)$ is the heat capacity characterizing the region Ω and being a temperature-dependent variable of the problem. Further, $\rho = \rho(T)$ is the density of the material contained in Ω, $\lambda_{ij} = \lambda_{ij}(T)$ is the thermal conductivity tensor while $g = g(T)$ is the rate of heat generated per unit volume; τ denotes time. This equation should fulfill the boundary conditions on $\partial\Omega$ being a continuous and sufficiently smooth contour bounding the Ω region. The boundary conditions discussed for (2.1) are as follows:

1) temperature (essential) boundary conditions

$$T = \hat{T}; \qquad x \in \partial\Omega_T, \tag{2.2}$$

and for $\partial\Omega_q$ part of the total $\partial\Omega$:

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2) heat flux (natural) boundary conditions

$$\frac{\partial T}{\partial n} = \hat{q}; \qquad x \in \partial\Omega_q, \tag{2.3}$$

where $\partial\Omega_T \cup \partial\Omega_q = \partial\Omega$ and $\partial\Omega_T \cap \partial\Omega_q = \{\oslash\}$.

3) Initial conditions have the following form:

$$T^0 = T(x_i;0); \qquad x_i \in \Omega, \quad \tau = 0. \tag{2.4}$$

Further, let Ω contains n coherent and disjoint subregions Ω_a for $a = 1, ..., n$ fulfilling the following conditions

$$\Omega = \bigcup_{a=1}^{n} \Omega_a; \quad \Omega_a \cap \Omega_b = \oslash; \quad a \neq b; \quad 1 \leqslant a, b \leqslant n. \tag{2.5}$$

Thus, all material parameters, denoted by f in the equation presented below, characterizing the composite structure considered (variables ρ, c, g, λ), can be described as

$$f = \chi_a f^{(a)}; \quad 1 \leqslant a \leqslant n, \tag{2.6}$$

where χ_a is a characteristic function given as follows:

$$\chi^{(a)} = \begin{cases} 1; & x \in \Omega_a, \\ 0; & x \notin \Omega_a. \end{cases} \tag{2.7}$$

Including Eqs. (2.6) and (2.7) in the formulation given by the formula (2.1) it is obtained that

$$\chi_a \rho^{(a)} \chi_a c^{(a)} T - \left(\chi_a \lambda_{ij}^{(a)} T_{,j}\right)_{,i} - \chi_a g^{(a)} = 0; \qquad x_i \in \Omega; \quad \tau \in [0, \infty). \tag{2.8}$$

Considering the fact that

$$\chi_a \chi_a = \chi_a^2 = \chi_a, \tag{2.9}$$

we obtain finally that

$$\chi_a \left(\rho^{(a)} c^{(a)} T - \left(\lambda_{ij}^{(a)} T_{,j}\right)_{,i} - g^{(a)}\right) = 0; \qquad x_i \in \Omega; \quad \tau \in [0, \infty). \tag{2.10}$$

The example of the composite structure considered is shown below in Fig. 1.

Let us suppose that all material parameters of the composite considered are uncorrelated, bounded random variables defined uniquely by their first two probabilistic moments as follows:

$$0 < \rho < \infty, \tag{2.11}$$

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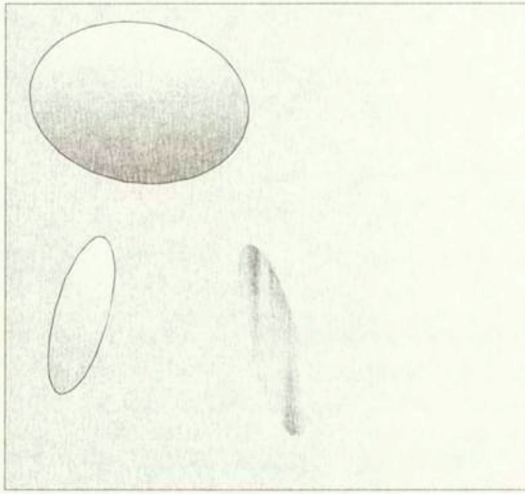

FIG. 1. The n-component composite structure.

$$E[\rho] = \chi_a E[\rho^{(a)}]; \qquad 1 \leqslant a \leqslant n, \tag{2.12}$$

$$\mathrm{Var}(\rho) = \chi_a \mathrm{Var}(\rho^{(a)}); \qquad 1 \leqslant a \leqslant n, \tag{2.13}$$

where $E[\rho]$ and $\mathrm{Var}(\rho)$ are the expected value and the variance of material density, respectively, which can be calculated by means of the following definitions given in Eqs. (2.14) and (2.15):

$$E\left[\rho^{(a)}(x_i)\right] = \int_{-\infty}^{+\infty} \rho^{(a)}(x_i) p(\rho^{(a)}) d\rho, \tag{2.14}$$

$$\mathrm{Var}\left(\rho^{(a)}(x_i)\right) = \int_{-\infty}^{+\infty} \left(\rho^{(a)}(x_i) - E\left[\rho^{(a)}(x_i)\right]\right)^2 p(\rho^{(a)}) d\rho. \tag{2.15}$$

Analogically, we obtain for the random variable of the heat capacity

$$0 < c < \infty, \tag{2.16}$$

$$E[c] = \chi_a E\left[c^{(a)}\right]; \qquad 1 \leqslant a \leqslant n, \tag{2.17}$$

$$\mathrm{Var}(c) = \chi_a \mathrm{Var}\left(c^{(a)}\right); \qquad 1 \leqslant a \leqslant n, \tag{2.18}$$

and the random conductivity tensor components

$$0 < \lambda_{ij} < \infty; \qquad i, j = 1, 2, 3, \tag{2.19}$$

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$$E[\lambda_{ij}] = \chi_a E\left[\lambda_{ij}^{(a)}\right]; \qquad 1 \leqslant a \leqslant n, \tag{2.20}$$

$$\mathrm{Var}\,(\lambda_{ij}) = \chi_a \mathrm{Var}\left(\lambda_{ij}^{(a)}\right); \qquad 1 \leqslant a \leqslant n, \tag{2.21}$$

what completes the stochastic description of physical properties of the composite constituents. The mathematical proof that the solution of Eq. (2.10) exists and is unique may be done on the basis of the considerations provided in [3].

2.2. An idea of the stochastic interface defects

Let the material with indices a contains the stochastic structural interface defects. These defects are be modeled further as semicircles placed with their diameters on the boundary between the composite constituents. Moreover, we assume that the total number of these defects as well as their diameter are Gaussian random variables defined uniquely by their expected values and variances. Spatial averaging or computation of effective characteristics for materials containing voids of some specific shapes may be done in general in different ways [5], however we use the spatial averaging method. Due to that method, we can derive the effective thermal property of the region containing defects as follows [6]:

$$\lambda^{\text{eff}} = \frac{\Omega_d}{\Omega_i}\lambda_d + \frac{\Omega_i - \Omega_d}{\Omega_i}\lambda_i, \tag{2.22}$$

where Ω_i is a region considered for which the effective parameter λ^{eff} is computed, Ω_d denotes the total area of the defects lying in the interior of the region Ω_i, while λ_d and λ_i denote the conductivity coefficients of the regions Ω_d and Ω_i, respectively. The geometrical idealization of the stochastic interface defects for the fiber-reinforced composites is presented in Figs. 2 and 3, while for the laminated structure in Figs. 4 and 5.

Next, let us assume that r and n are random variables of the radii and the total number of the defects considered, thus Ω_d and, at the same time λ^{eff}, can be evaluated as follows:

$$\Omega_d = \frac{1}{2}\Pi n r^2, \tag{2.23}$$

and

$$\lambda^{\text{eff}} = \frac{\Pi}{2\Omega_i} n r^2 \lambda_d + \frac{\Omega_i - \frac{1}{2}\Pi n r^2}{\Omega_i}\lambda_i = \frac{\Pi}{2\Omega_i} n r^2 \lambda_d + \left(1 - \frac{\frac{1}{2}\Pi n r^2}{\Omega_i}\right)\lambda_i. \tag{2.24}$$

Starting from the formula (2.24), the partial derivatives of the first and the second order of the effective conductivity coefficient can be evaluated with respect to the variables r and n. Thus we have

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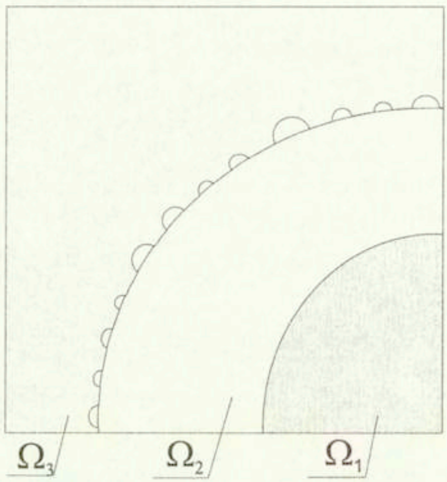

FIG. 2. Interface micro-geometry of fiber-reinforced composite.

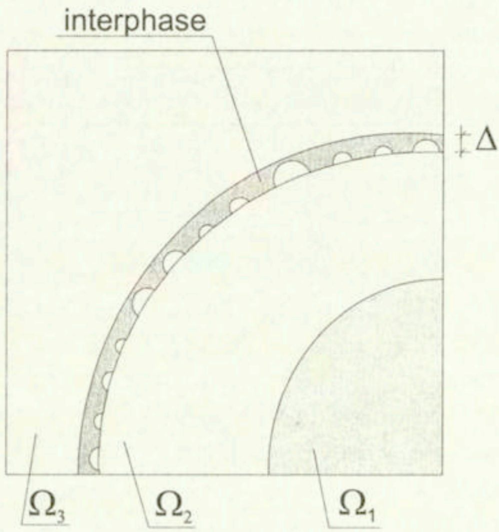

FIG. 3. Interface defects in fiber-reinforced composite.

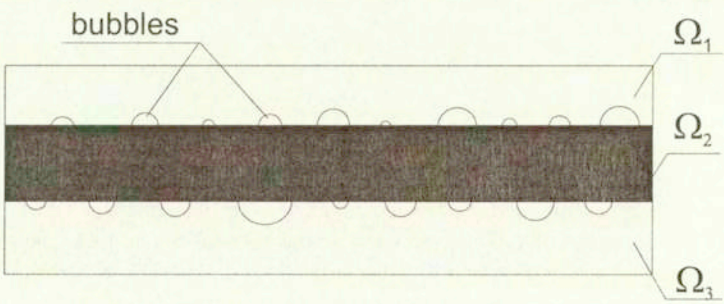

FIG. 4. Outline of the interface for stratified composite.

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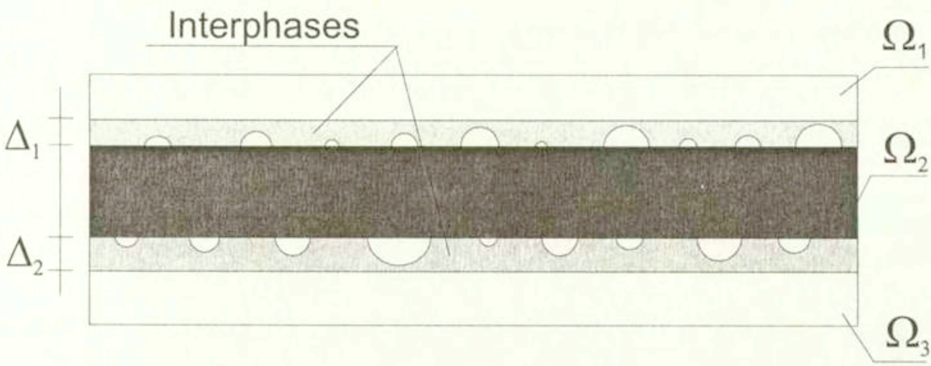

FIG. 5. Interface micro-geometry of stratified composite.

$$\frac{\partial \lambda^{\text{eff}}}{\partial \lambda_d} = \frac{\Pi}{2\Omega_i} n r^2, \tag{2.25}$$

$$\frac{\partial^2 \lambda^{\text{eff}}}{\partial \lambda_d^2} = 0, \tag{2.26}$$

$$\frac{\partial \lambda^{\text{eff}}}{\partial \lambda_i} = 1 - \frac{\Pi n r^2}{2\Omega_i}, \tag{2.27}$$

$$\frac{\partial^2 \lambda^{\text{eff}}}{\partial \lambda_i^2} = 0, \tag{2.28}$$

$$\frac{\partial \lambda^{\text{eff}}}{\partial n} = \frac{\Pi r^2}{2\Omega_i}(\lambda_d - \lambda_i), \tag{2.29}$$

$$\frac{\partial^2 \lambda^{\text{eff}}}{\partial n^2} = 0, \tag{2.30}$$

$$\frac{\partial \lambda^{\text{eff}}}{\partial r} = \frac{\Pi n r}{\Omega_i}(\lambda_d - \lambda_i), \tag{2.31}$$

$$\frac{\partial^2 \lambda^{\text{eff}}}{\partial r^2} = \frac{\Pi n}{\Omega_i}(\lambda_d - \lambda_i). \tag{2.32}$$

By analogous way we derive the effective values of heat capacity as well as material density for the composite components. The derivatives calculated above will be used next in the canonical formulation of the Stochastic Finite Element Method approach to the problem presented below.

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2.3. Variational formulation of the problem

Let us consider any continuous temperature variations $\delta T(x_i)$ defined in the interior of the region Ω and vanishing on $\partial\Omega_T$. Multiplying Eq. (2.10) by the test function specified and integrating over Ω, we obtain

$$\int_{\Omega} \chi_a \left(\rho^{(a)} c^{(a)} T - \left(\lambda_{ij}^{(a)} T_{,j}\right)_{,i} - g^{(a)}\right) \delta T d\Omega = 0; \quad 1 \leqslant a \leqslant n; \quad x_i \in \Omega; \quad \tau \in [0, \infty). \tag{2.33}$$

Taking into account that the derivative defined on the temperature variation is in fact a variation of the respective temperature derivative

$$\frac{\partial(\delta T)}{\partial x_i} = \delta\left(\frac{\partial T}{\partial x_i}\right) \equiv \delta T_{,i}\ , \tag{2.34}$$

we can arrive at

$$\int_{\Omega} \chi_a \left(\rho^{(a)} c^{(a)} T \delta T - \left(\lambda_{ij}^{(a)} T_{,j} \delta T\right)_{,i} - \left(\lambda_{ij}^{(a)} T_{,j}\right) \delta T_{,i} - g^{(a)} \delta T\right) d\Omega = 0; \quad 1 \leqslant a \leqslant n; \quad x_i \in \Omega; \quad \tau \in [0, \infty), . \tag{2.35}$$

Introducing the respective heat transfer boundary conditions

$$\int_{\Omega} \left(\lambda_{ij} T_{,j} \delta T\right)_{,i} d\Omega = \int_{\partial\Omega} \lambda_{ij} T_{,j} n_{,i} \delta T d(\delta\Omega) = \int_{\partial\Omega_q} \hat{q} \delta T d(\partial\Omega) \tag{2.36}$$

and integrating by parts, we obtain

$$\int_{\Omega} \chi_a \left(\rho^{(a)} c^{(a)} \dot{T} \delta T + \lambda_{ij}^{(a)} T_{,j} \delta T_{,i} - g^{(a)} \delta T\right) d\Omega - \int_{\partial\Omega_q} \hat{q} \delta T d(\partial\Omega) = 0; \quad 1 \leqslant a \leqslant n; \quad x_i \in \Omega; \tau \in [0, \infty). \tag{2.37}$$

The equation stated above is the transient formulation of the principle of virtual temperatures. This principle is discretized in the next section by the use of the finite element approach.

2.4. Stochastic perturbation technique

The stochastic variational principle for linear transient heat transfer problems is formulated on the basis of Eq. (2.37) and is employed by the combination of

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the second-order perturbation technique and second-moment stochastic analysis [4, 6, 9].

To provide the formulation let us denote the random variable vector of the problem by $\{b^r(x;\omega)\}$ and the probability densities of its components by $g(b^r)$ and $g(b^r, b^s)$, respectively. Indices r, s are running from 1 to R, where R denotes the total number of random vector components. The expected value of the vector $\{b^r(x;\omega)\}$ can be thus expressed by

$$E[b^r] = \int_{-\infty}^{+\infty} b^r g(b^r) db^r, \tag{2.38}$$

while the covariance is equal to

$$\mathrm{Cov}(b^r, b^s) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (b^r - E[b^r])(b^s - E[b^s]) g(b^r, b^s) db^r db^s. \tag{2.39}$$

The coefficient of variation of the random vector components is derived in the form

$$\alpha[b(x;\omega)] = \sqrt{\frac{\mathrm{Var}[b(x;\omega)]}{E^2[b(x;\omega)]}}. \tag{2.40}$$

Next, let us expand all the random variables into the Taylor series. According to the method, all functions of the problem (heat conductivity, heat capacity, temperature and its gradient as well as the material density) are expressed in the form similar fo the following expansion of function F:

$$F(x) = F^0(x) + \theta F^{,r}(x)\Delta b^r + \frac{1}{2}\theta^2 F^{,rs}(x)\Delta b^r \Delta b^s, \tag{2.41}$$

where θ is a given small perturbation, $\theta\Delta b^r$ denotes the first order variation of b_r from its expected value

$$\theta\Delta b^r = \delta b_r = \theta\left(b_r - b_r^0\right), \tag{2.42}$$

while the second variation is given as follows:

$$\theta^2 \Delta b^r \Delta b^s = \delta b_r \delta b_s = \theta^2 \left(b_r - b_r^0\right)\left(b_s - b_s^0\right). \tag{2.43}$$

Moreover, symbols $(.)^0, (.)^{,r}$ and $(.)^{,rs}$ represent the expected value, the first and the second partial derivatives with respect to the random variables evaluated at the expected values of input random parameters.

According to the second-order perturbation technique [4, 9], the expansion (2.41) is now substituted in the formulation (2.37). As the result, we obtain the three sets of algebraic equations of 0th, 1st and 2nd order. Hence we have:

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• one zeroth-order partial differential equation

$$\int_{\Omega} \chi_a \left(\rho^{(a)0} c^{(a)0} T^0 \delta T + \lambda_{ij}^{(a)0} T^0_{,j} \delta T_{,i} \right) d\Omega = \int_{\partial\Omega_q} \hat{q}^0 \delta T d(\partial\Omega) + \int_{\Omega} \chi_a g^{(a)0} \delta T d\Omega, \tag{2.44}$$

• R first-order partial differential equations, $r = 1, 2, ..., R$:

$$\int_{\Omega} \chi_a \left(\rho^{(a)0} c^{(a)0} T^{,r} \delta T + \lambda_{ij}^{(a)0} T^{,r}_{,j} \delta T_{,i} \right) d\Omega = \int_{\partial\Omega_q} \hat{q}^{,r} \delta T d(\partial\Omega) + \int_{\Omega} \chi_a g^{(a),r} \delta T d\Omega - \int_{\Omega} \chi_a \left(\left(\rho^{(a),r} c^{(a)0} + \rho^{(a)0} c^{(a),r} \right) T^0 \delta T + \lambda_{ij}^{(a),r} T^0_{,j} \delta T_{,i} \right) d\Omega, \tag{2.45}$$

• one second-order partial differential equation:

$$\begin{aligned} \int_{\Omega} \chi_a \left(\rho^{(a)0} c^{(a)0} T^{(2)} \delta T + \lambda_{ij}^{(a)0} T^{(2)}_{,j} \delta T_{,i} \right) d\Omega = \int_{\partial\Omega_q} \hat{q}^{(2)} \delta T d(\partial\Omega) \\ + \int_{\Omega} \chi_a g^{(a)(2)} \delta T d\Omega - \int_{\Omega} \chi_a \left(\left(\rho^{(a),rs} c^{(a)0} + 2\rho^{(a),r} c^{(a),rs} \right) T^0 \right. \\ \left. + \left(\rho^{(a),r} c^{(a)0} + \rho^{(a)0} c^{(a),r} \right) T^{,s} \right) S^{rs} \delta T d\Omega \\ - \int_{\Omega} \left(\lambda_{ij}^{(a),rs} T^0_{,j} + 2\lambda_{ij}^{(a),r} T^{,s}_{,j} \right) S^{rs} \delta T_{,j} d\Omega, \end{aligned} \tag{2.46}$$

where the symbol $(.)^{(2)}$ denotes the double sum $(.)^{,rs} S^{rs}$, $r, s = 1, 2, ..., R$. Having solved these Eqs. (2.44), (2.45) and (2.46) for T^0, $T^{,r}$ and $T^{,rs}$, respectively, we derive the expressions for the expected values and covariances of the temperature field. We obtain [4]:

• the expcted values

$$E\left[T(x_i, \tau)\right] = T^0(x_i, \tau) + \frac{1}{2} T^{(2)}(x_i, \tau); \tag{2.47}$$

• the covariances

$$\mathrm{Cov}\left(T\left(x_i^{(1)}; t_1\right), T\left(x_i^{(2)}; t_2\right) \right) = T^{,r}\left(x_i^{(1)}; t_1\right) T^{,s}\left(x_i^{(2)}; t_2\right) S^{rs}. \tag{2.48}$$

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Symbol S_b^{rs} denotes the matrix of the input random variables of the problem which can be expressed as follows:

$$S_b^{rs} = \mathrm{Cov}\begin{pmatrix} \mathrm{Var}\,\lambda_d & 0 & 0 & 0 \\ & \mathrm{Var}\,\lambda_i & 0 & 0 \\ & & \mathrm{Var}\,n & 0 \\ \mathrm{symm.} & & & \mathrm{Var}\,r \end{pmatrix}, \tag{2.49}$$

where the respective variances are submatrices with different diagonal terms for different components of the composite or different interfaces. Moreover, it should be stressed that the formulation proposed deals with the input random variables which are not stochastic processes, i.e. they are not random in space and in time at the same time.

3. Computational implementation

3.1. Classical finite element technique

Let us assume that the region Ω is discretized by the use of the set of finite elements and that the scalar temperature field T is described by the nodal temperatures vector θ_α

$$T(x_i) = H_\alpha(x_i)\theta_\alpha; \qquad i = 1,2; \qquad \alpha = 1,2,..,N, \tag{3.1}$$

where N is the total number of degrees of freedom introduced. The temperature derivatives can be written in the form

$$T_{,i} = H_{\alpha,i}\theta_\alpha. \tag{3.2}$$

Moreover, let us introduce the capacity matrix $\mathbf{C}_{\alpha\beta}$, the heat conductivity matrix $\mathbf{K}_{\alpha\beta}$ and the vector $\mathbf{P}_\alpha$ as follows [14]:

$$\mathbf{C}_{\alpha\beta} = \int\limits_\Omega \chi_a \rho^{(a)} c^{(a)} H_\alpha H_\beta d\Omega; \qquad 1 \leqslant a \leqslant n, \tag{3.3}$$

$$\mathbf{K}_{\alpha\beta} = \int\limits_\Omega \chi_a \lambda_{ij}^{(a)} H_{\alpha,i} H_{\beta,j} d\Omega; \qquad 1 \leqslant a \leqslant n, \tag{3.4}$$

and

$$\mathbf{P}_\alpha = \int\limits_\Omega g H_\alpha d\Omega + \int\limits_{\partial\Omega} \hat{q} H_\alpha d\Omega. \tag{3.5}$$

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Next, let us introduce these matrixes into the variational formulation (2.37). Hence we must solve the following algebraic equations system [10 – 12]:

$$\mathbf{C}_{\alpha\beta}\dot{\theta}_\beta + \mathbf{K}_{\alpha\beta}\theta_\beta = \mathbf{P}_\alpha. \tag{3.6}$$

Finally, to obtain the solution of the transient heat problem the following algorithm has been applied, cf. Fig. 6.

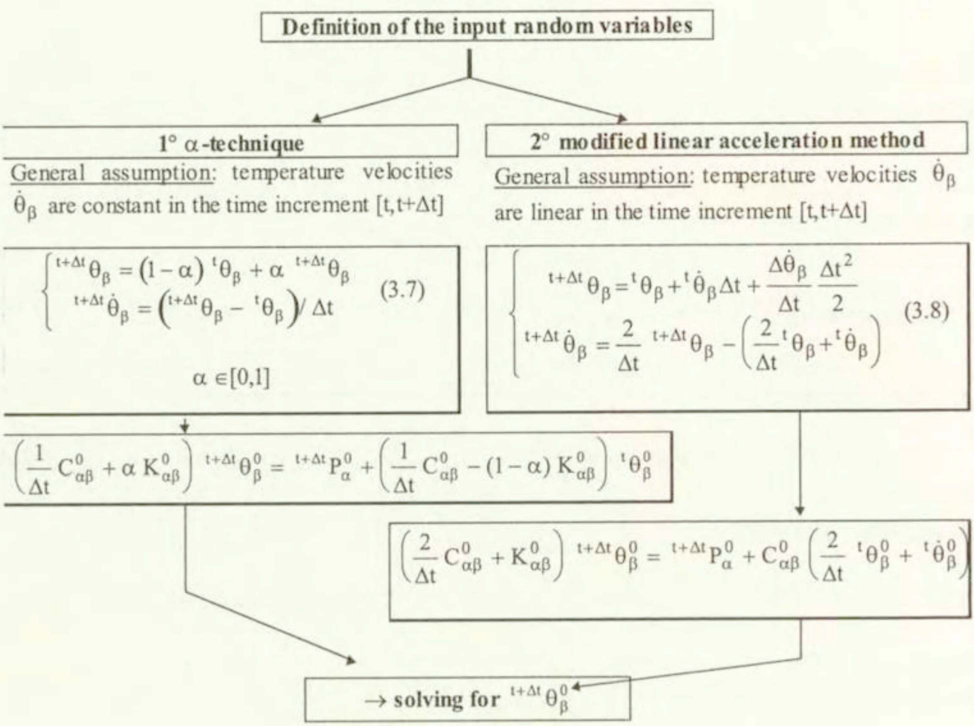

FIG. 6. Transient heat flow solution algorithm.

It should be mentioned that the equations of the 1st and the 2nd order derived by means of the Eqs. (2.45) and (2.46) can be solved by the use of α technique represented by Eq. (3.7), or the modified acceleration method – by Eq. (3.8).

3.2. Stochastic finite element formulation of the problem

Analogically to the previous considerations, we can obtain the following system of algebraic equations describing the second-order stochastic formulation of the transient heat transfer problem [4]:

• zeroth-order, one system of N ordinary differential equations

$$C^0_{\alpha\beta}\dot{\theta}^0_\beta + K^0_{\alpha\beta}\theta^0_\beta = P^0_\alpha; \tag{3.7}$$

• first-order, R systems of N ordinary differential equations

$$C^0_{\alpha\beta}\dot{\theta}^{,r}_\beta + K^0_{\alpha\beta}\theta^{,r}_\beta = P^{,r}_\alpha - \left(C^{,r}_{\alpha\beta}\dot{\theta}^0_\beta + K^{,r}_{\alpha\beta}\theta^0_\beta\right); \tag{3.8}$$

• second-order, one system of N ordinary differential equations

$$C^0_{\alpha\beta}\dot{\theta}^{(2)}_\beta + K^0_{\alpha\beta}\theta^{(2)}_\beta = \Big[P^{,rs}_\alpha - 2\left(C^{,r}_{\alpha\beta}\dot{\theta}^{,s}_\beta + K^{,r}_{\alpha\beta}\theta^{,s}_\beta\right) - \left(C^{,rs}_{\alpha\beta}\dot{\theta}^0_\beta + K^{,rs}_{\alpha\beta}\theta^0_\beta\right)\Big] S^{rs}_b. \tag{3.9}$$

In the equations stated above we have introduced the following matrix notation:

• the heat capacity matrix and their derivatives

$$C^0_{\alpha\beta} = \int_\Omega \rho^0 c^0 H_\alpha H_\beta d\Omega, \tag{3.10}$$

$$C^{,r}_{\alpha\beta} = \int_\Omega \left(\rho^{,r} c^0 + \rho^0 c^{,r}\right) H_\alpha H_\beta d\Omega, \tag{3.11}$$

$$C^{,rs}_{\alpha\beta} = \int_\Omega \left(\rho^{,rs} c^0 + 2\rho^{,r} c^{,s} + \rho^0 c^{,rs}\right) H_\alpha H_\beta d\Omega; \tag{3.12}$$

• the heat conductivity matrix and their derivatives

$$K^0_{\alpha\beta} = \int_\Omega \lambda^0_{ij} H_{\alpha,i} H_{\beta,j} d\Omega, \tag{3.13}$$

$$K^{,r}_{\alpha\beta} = \int_\Omega \lambda^{,r}_{ij} H_{\alpha,i} H_{\beta,j} d\Omega, \tag{3.14}$$

$$K^{,rs}_{\alpha\beta} = \int_\Omega \lambda^{,rs}_{ij} H_{\alpha,i} H_{\beta,j} d\Omega; \tag{3.15}$$

• the Right-Hand Side (RHS) vector and their derivatives

$$P^0_\alpha = \int_\Omega g^0 H_\alpha d\Omega + \int_{\partial\Omega_q} \hat{q}^0 H_\alpha d\Omega, \tag{3.16}$$

$$P^{,r}_\alpha = \int_\Omega g^{,r} H_\alpha d\Omega + \int_{\partial\Omega_q} \hat{q}^{,r} H_\alpha d\Omega, \tag{3.17}$$

$$P^{,rs}_\alpha = \int_\Omega g^{,rs} H_\alpha d\Omega + \int_{\partial\Omega_q} \hat{q}^{,rs} H_\alpha d\Omega, \tag{3.18}$$

where all expressions are evaluated at the expectated values of the input random variables vector components.

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4. Computational experiments

The example deals with the computational modeling of the two-component layered composite with randomly defined heat conductivity coefficient (test 1) and heat capacity (test 2) and stochastic interface defects. The finite element discretization as well as boundary conditions of the problem are shown in Fig. 7.

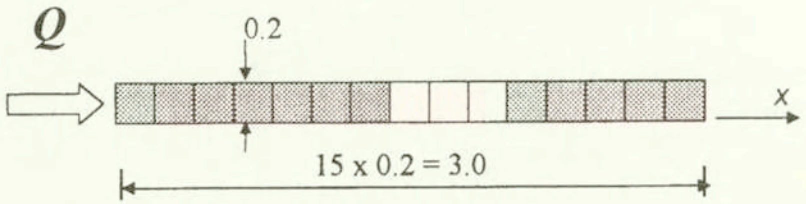

FIG. 7. Semi-infinite composite subjected to surface heat flux.

Heat conductivity has been taken as $E[\lambda] = \{1.2; 0.3; 0.5\}$ and $\mathrm{Cov}(\lambda^r, \lambda^s) = \alpha \exp[-\mathrm{abs}(x^r - x^s)]$, while the heat capacity as $E[c] = 1.0$ and its covariance matrix has the same form. The coefficient of variation α has been taken as 0.14 for all tests. The results of the computational experiments in the sense of SFEM analysis, are presented in Figs. 8–13.

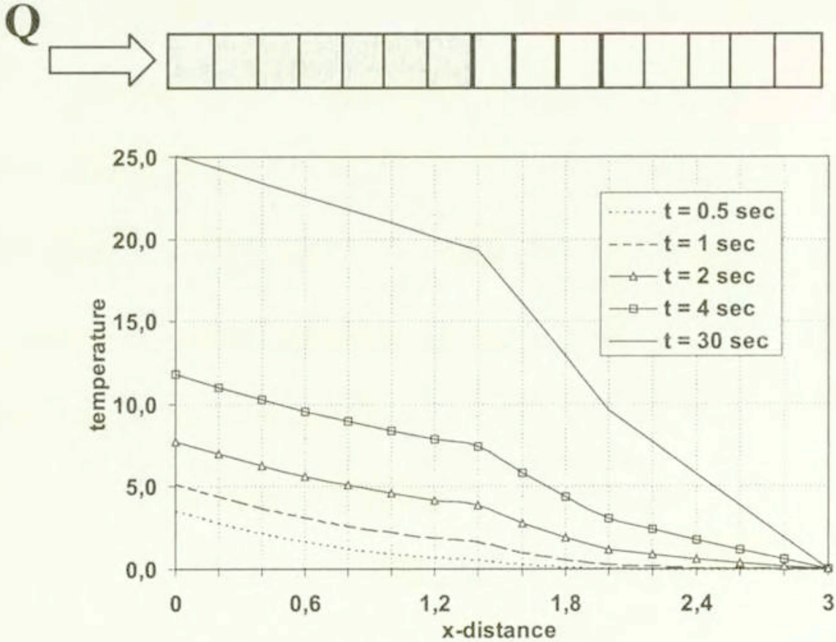

FIG. 8. Spatial expected values for test 1.

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Figure 8 shows spatial expectations of the temperature field in the function of the composite thickness. It is visible that the expected values of temperature decrease quasi-linearly from the left boundary to the right one for the composite structure. Next, we observe that temperature expectations are not smooth at the interfaces of the composite being modeled and, moreover, that the expected values of temperature generally increase with time, what agrees with engineering intuition very well.

Figure 9 illustrates the cross-covariances of temperature field in function of the composite thickness. We observe that these cross-covariances increase together with time and non-smoothness appears at the composite interfaces analogically to the case of expected values. Moreover, it can be seen that decrease of cross-covariances together with the increase of the distance x has a quasi-linear character. Finally, it should be underlined that some negative values of probabilistic characteristics computed are obtained near the initial state of the structure (caused by the SFEM procedure instabilities only).

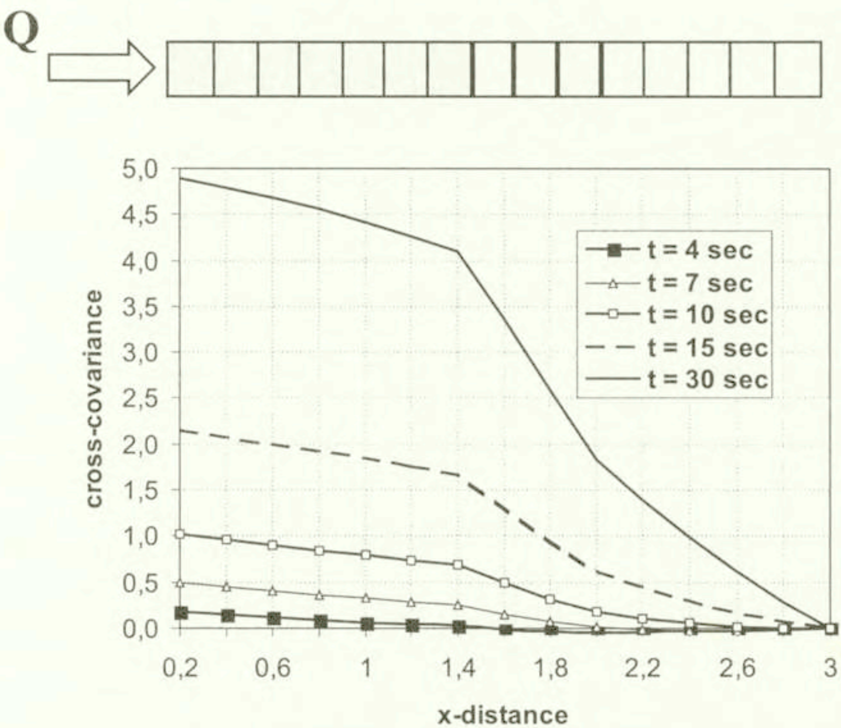

FIG. 9. Spatial cross-covariances for test 1.

Next figure (Fig. 10) shows us the expected values of temperature histories in the composite considered. It can be seen that temperature expectations increase together with time and that the greatest temperatures appear at the left-hand

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boundary of the composite while the smallest – at the right-hand one. Contrary to the previous figures, these expectations change very smoothly at all interfaces.

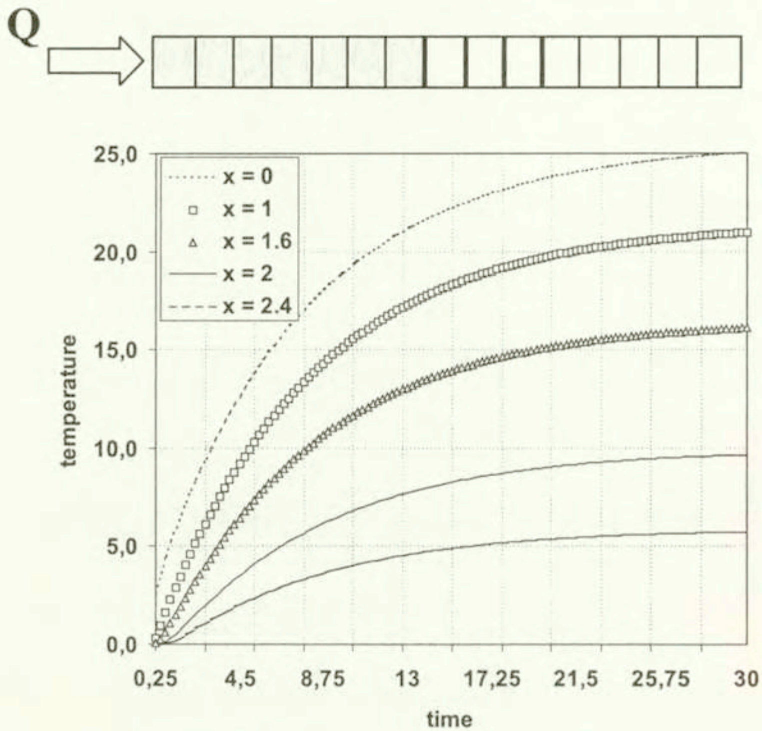

FIG. 10. Time expected values for test 1.

The cross-covariances of nodal temperatures are presented in Fig. 11 – these covariances increase together with time. Analogically to the cross-covariances in the function of composite thickness, we observe some negative values at the beginning of the heat transfer process. The greatest values are observed at the left-hand surface while the smallest – near the right-hand one.

Figures 12 and 13 illustrate the spatial and time cross-covariances for the test 2. It should be mentioned that the expected values in the function of time and composite thickness for random specific heat capacity have the same character as previously. Analogically to the case 1, cross-covariances are greater near the left-hand boundary of the structure where the heat flux is applied and seem to be smaller at the other boundary. Moreover, we can see that cross-covariances tend to 0 when the solution tends to a stationary state. Finally, it can be observed that there is some characteristic time when the cross-covariances reach their maximum (8 seconds for the example being analyzed). Analogically to the previous figure, the time cross-covariances for nodal temperatures have smooth and strongly nonlinear character.

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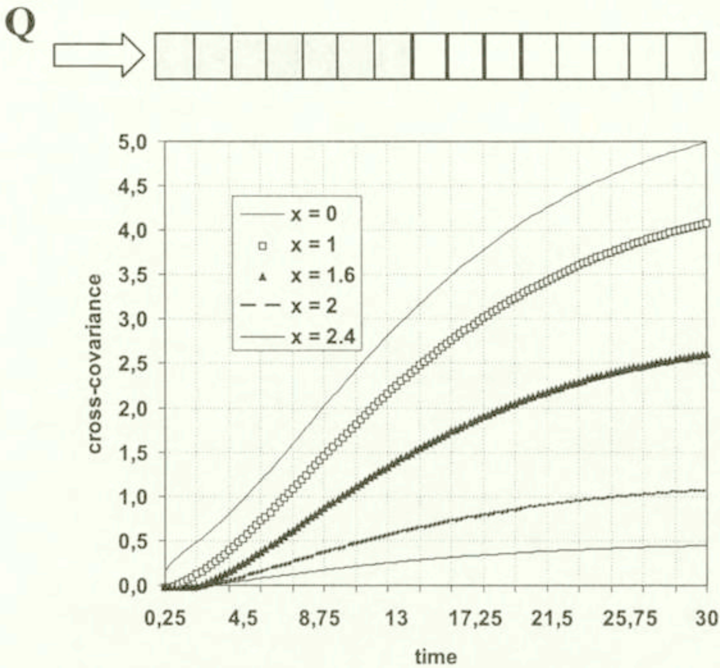

FIG. 11. Time cross-covariances for test 1.

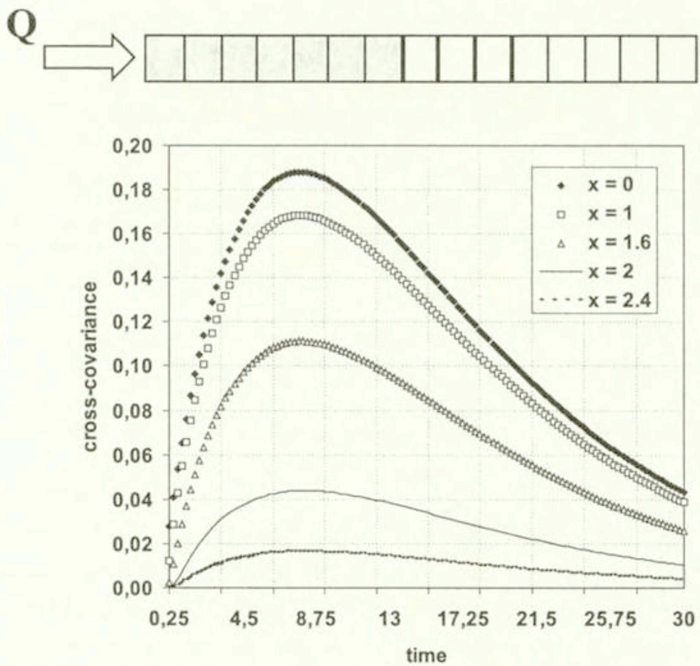

FIG. 12. Time cross-covariances for test 2.

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The relation between the cross-covariances and composite thickness is presented in Fig. 13. We may observe that the covariance changes are of a quasi-linear character and non-smoothness appears at the composite interfaces. Moreover, it is visible that for time equal to some characteristic value, the cross-covariances reach the extremum values. These values are equal to 0 near the right-hand surface of the composite structure and, for time equal to approximately 30 seconds we obtain a quasi-stationary state. Analogically to the cross-covariances in case 1, the second probabilistic moments change quasi-linearly and show non-smoothness at interfaces between the constituents.

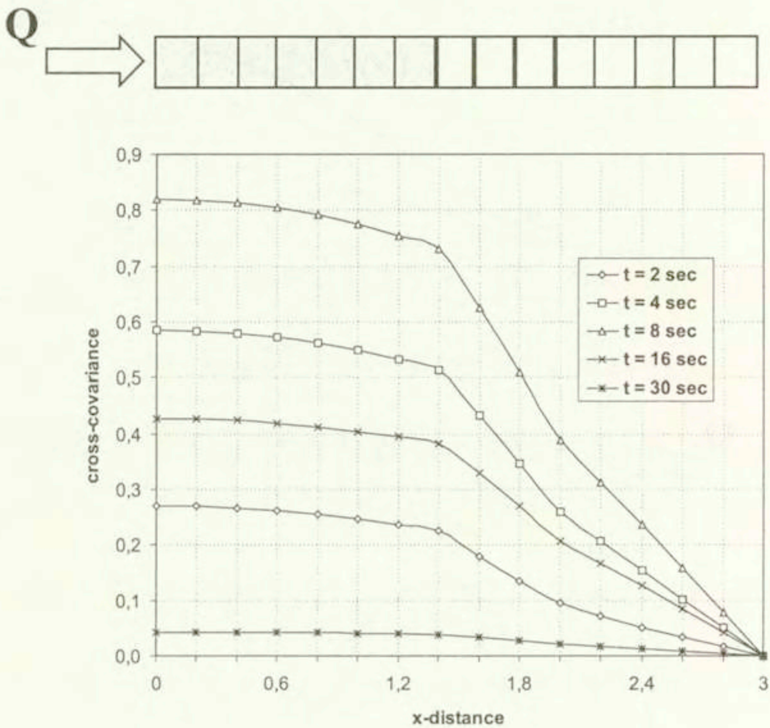

FIG. 13. Spatial cross-covariances for test 2.

5. Conclusions

1. Numerical analysis carried out in the paper proved that transient heat transfer problems in layered composites with spatially random parameters and stochastic interface defects can be efficiently modeled by the use of the stochastic finite element methodology based on the second order perturbation, stochastic second central moment analysis. Moreover, the methodology improved can be applied to stochastic modeling of general composites as well as to simulation

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of the composite materials with stochastic interface defects due to the model presented in [6].

2. Computational experiments provided in the paper show that the heat transfer in composite materials is very sensitive to random changes of material parameters of their constituents. The procedure involved may be applied for seepage, torsion, irrotational and incompressible flow, film lubrication, acoustic vibration as well as for electric conduction, electrostatic field, electromagnetic waves and all field problems with stochastically defined physical or geometrical characteristics [1, 15].

3. Further computational studies on the phenomena are recommended in the context of the stochastic sensitivity of the problem. The influence of material parameters and interface defects on overall thermal behavior of different composite structures (stratified, fiber-reinforced or structures with periodic as well as non-periodic geometry) should be numerically approximated to find out the crucial parameters for the composite structure thermal behavior.

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