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Observable plastic spin and comparison with other approaches

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THE OBJECT OF THIS PAPER is to formulate a relation for plastic spin, basing on the work of RANIECKI and MRÓZ [17] and the suggestion of HILL [5].

1. Introduction

IN DESCRIPTION of strain-induced anisotropy during finite (large) straining of metals and alloys the use of the concept of plastic spin was found to be instrumental.

As we know, the plastic spin represents the mean relative spin of all material fibers measured with respect to some chosen triad which can be thought as being attached to the substructures. The spin being the difference between the plastic spin and the material spin, takes part in the substructure corotational rates. For completing the constitutive equations, one has to specify three additional equations for plastic spin. As the plastic spin is not measurable explicitly, the representation theorems for isotropic functions have been used in conjunction with the concept of tensorial structure variables to provide explicit forms for it ONAT [12, 13], LORET [9], DAFALIAS [2], PAULUN and PĘCHERSKI [15]. Another possibility was proposed in the work of RANIECKI and MRÓZ [17] for a model of rigid-plastic solids. They indicate that in certain circumstances the plastic spin can be regarded, at least conceptually, as a measurable quantity. Supposing that the measurable texture orientation is specified by a rigidly rotating triad during consecutive steps of plastic deformation, the plastic spin is defined as the difference of material and texture spin. Such an approach closely follows the ideas of MANDEL [10, 11].

Tensors will be denoted by boldface characters. With the summation over repeated indices implied, the following symbolic operations apply : $\mathbf{AB} \to A_{ij}B_j$, $\mathbf{A}\cdot\mathbf{B} \to A_{ij}B_{ij}$, $\mathbf{A}\otimes\mathbf{B} \to A_{ij}B_{kl}$ with proper extension to different orders tensor. The prefix tr indicates the trace, a superscript T the transpose and a superposed dot the material time derivative or rate. By $\mathbf{1}$ we denote the identity tensor and by a superscript -1 the inverse.

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2. Basic relations

2.1. Basic kinematic quantities

Following RANIECKI and MRÓZ [17], consider a uniform deformation of the rigid-plastic material element. Assume that the orientation of the texture can be specified at each instant of the process by three unit mutually orthogonal vectors triad. Let $\mathbf{t}_i^0$ ($i = 1, 2, 3$) be the triad representing the initial texture orientation in the initial stress-free configuration (k_0), see Fig. 1. Assuming that at each subsequent instant t the orientation of texture is specified by the triad $\mathbf{t}_i(t)$, there is $\mathbf{t}_i(t_0) = \mathbf{t}_i^0$ obviously. Let $\mathbf{Q}(t)$ be the proper orthogonal tensor ($\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{1}$) which transforms initial triad $\mathbf{t}_i^0$ into instantaneous $\mathbf{t}_i(t)$, that is $\mathbf{t}_i(t) = \mathbf{Q}(t)\mathbf{t}_i^0$. The *instantaneous texture spin* $\boldsymbol{\omega}^t$ is defined as:

$$\dot{\mathbf{t}}_i(t) = \boldsymbol{\omega}^t(t)\,\mathbf{t}_i(t)\,, \tag{2.1}$$

where

$$\boldsymbol{\omega}^t(t) = \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}} = \sum_{i=1}^{3} \dot{\mathbf{t}}_i(t) \otimes \mathbf{t}_i(t) \equiv \dot{\mathbf{t}}_i \otimes \mathbf{t}_i\,. \tag{2.2}$$

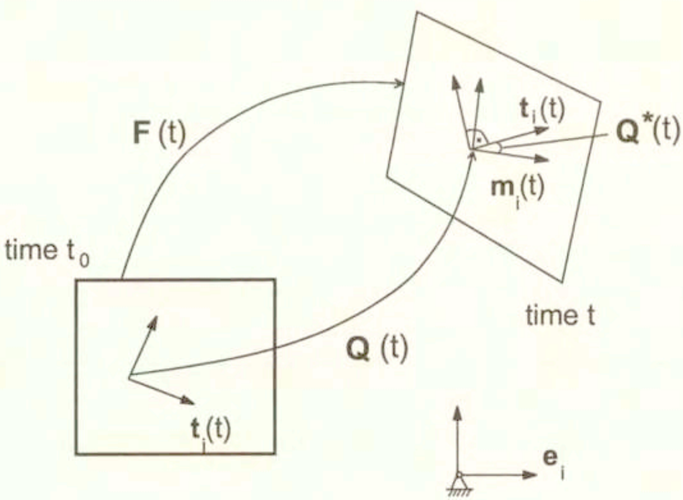

FIG. 1.

Introduce the notion of *texture reference frame* $\mathbf{m}_i$ selected in the following way: vector $\mathbf{m}_1$ represents a material fiber lying in a chosen material plane with normal $\mathbf{m}_2$ and $\mathbf{m}_3 = \mathbf{m}_1 \wedge \mathbf{m}_2$ (the vector product), see MANDEL [10, 11].

The spin $\boldsymbol{\omega}^m(m, t)$ of the reference triad $\mathbf{m}_i$ will be defined as

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(2.3) $$\boldsymbol{\omega}^m(m,t) = \dot{\mathbf{m}}_i \otimes \mathbf{m}_i\,, \qquad \dot{\mathbf{m}}_i = \boldsymbol{\omega}^m(m,t)\,\mathbf{m}_i\,.$$

Here the symbol m in brackets emphasize the fact that $\boldsymbol{\omega}^m(m,t)$ depend on the selection of texture reference frame.

Denote by $\mathbf{V}$ the spatial velocity gradient $\mathbf{V} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ where $\mathbf{F}$ is the deformation gradient. The symmetric and antisymmetric part of $\mathbf{V}$ are, respectively, the rate of permanent strain $\mathbf{D}$ and the material spin $\boldsymbol{\omega}$. This idea follows the concept of director triad proposed by MANDEL [10, 11] defined on macrolevel. Some experiments of plastic deformation of the metal sheet subjected to the tension test produced the fiber-like textures (see, for example [1]).

To express the spin $\boldsymbol{\omega}^m$ in terms of $\mathbf{V}$ and $\mathbf{m}_i$ we find first the material derivative of the reference triad. The material derivative of vector $\mathbf{m}_1$ can be found by differentiating the relation $\mathbf{m}_1 = \dfrac{\mathrm{d}\mathbf{x}}{|\mathrm{d}\mathbf{x}|}$ to obtain:

(2.4) $$\dot{\mathbf{m}}_1 = \mathbf{V}\mathbf{m}_1 - (\mathbf{m}_1 \cdot \mathbf{V}\mathbf{m}_1)\,\mathbf{m}_1\,.$$

In a similar way one can find the material derivative of vector $\mathbf{m}_2 = \dfrac{\operatorname{grad} f}{|\operatorname{grad} f|}$, where $f(\mathbf{x},t) = 0$ is the material surface in actual configuration in the following form:

(2.5) $$\dot{\mathbf{m}}_2 = (\mathbf{m}_2 \cdot \mathbf{V}\mathbf{m}_2)\,\mathbf{m}_2 - \mathbf{V}^{\mathrm{T}}\mathbf{m}_2\,.$$

From the relation $\mathbf{m}_3 = \mathbf{m}_1 \wedge \mathbf{m}_2$ we have

(2.6) $$\dot{\mathbf{m}}_3 = \dot{\mathbf{m}}_1 \wedge \mathbf{m}_2 + \mathbf{m}_1 \wedge \dot{\mathbf{m}}_2\,.$$

Introducing (2.4), (2.5) and (2.6) into $(2.3)_1$ it follows that the spin $\boldsymbol{\omega}^m$ of the triad $\mathbf{m}_i$ is a very simple function of the strain rate tensor $\mathbf{D}$ and the chosen triad (see, [7, 18]):

(2.7) $$\begin{gathered} \boldsymbol{\omega}^m(m,t) = \boldsymbol{\omega}(t) - \hat{\boldsymbol{\omega}}(m,t)\,, \\ 2\hat{\boldsymbol{\omega}}(m,t) = \hat{\omega}_{ij}(\mathbf{m}_i \otimes \mathbf{m}_j - \mathbf{m}_j \otimes \mathbf{m}_i)\,, \\ \hat{\omega}_{12} = \mathbf{m}_1 \cdot \mathbf{D}\mathbf{m}_2\,, \qquad \hat{\omega}_{13} = \mathbf{m}_1 \cdot \mathbf{D}\mathbf{m}_3\,, \qquad \hat{\omega}_{32} = \mathbf{m}_2 \cdot \mathbf{D}\mathbf{m}_3\,. \end{gathered}$$

If the transformation from $\mathbf{m}_i$ to $\mathbf{t}_i$ is specified by an orthogonal tensor $\mathbf{Q}^\star$, then

(2.8) $$\mathbf{t}_i = \mathbf{Q}^\star(t)\,\mathbf{m}_i\,, \qquad \mathbf{Q}^\star(t) = \mathbf{t}_i \otimes \mathbf{m}_i\,.$$

The *relative texture spin* $\boldsymbol{\omega}^r$ is defined as follows:

(2.9) $$\boldsymbol{\omega}^r(m,t) = \dot{\mathbf{Q}}^\star\mathbf{Q}^{\star\mathrm{T}}\,.$$

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We can write also

$$2\boldsymbol{\omega}^r(m,t) = \dot{Q}^\star_{ik}(t)Q^\star_{jk}(t)\,(\mathbf{m}_i\otimes\mathbf{m}_j - \mathbf{m}_j\otimes\mathbf{m}_i)\,, \tag{2.10}$$

where Q^r_{ij} is the matrix of direction cosines of angles between the vectors $\mathbf{t}_i$ and $\mathbf{m}_j$:

$$Q^\star_{ij} = \mathbf{m}_i(t)\cdot\mathbf{t}_j(t) = \mathbf{m}_i\cdot\mathbf{Q}^\star\mathbf{m}_j\,, \qquad \mathbf{Q}^\star = Q^\star_{ij}\,\mathbf{m}_i\otimes\mathbf{m}_j\,. \tag{2.11}$$

Differentiating $(2.11)_2$ with respect to time and using $(2.3)_2$, we obtain

$$\dot{\mathbf{Q}}^\star = \dot{Q}^\star_{ij}\,\mathbf{m}_i\otimes\mathbf{m}_j + \boldsymbol{\omega}^m\,\mathbf{Q}^\star - \mathbf{Q}^\star\,\boldsymbol{\omega}^m\;;$$

then on accounting (2.10) we have

$$\dot{\mathbf{Q}}^\star\mathbf{Q}^{\star\mathrm{T}} = \boldsymbol{\omega}^r + \boldsymbol{\omega}^m - \mathbf{Q}^\star\boldsymbol{\omega}^m\mathbf{Q}^{\star\mathrm{T}}\,. \tag{2.12}$$

On the other hand, differentiation $(2.8)_2$ with respect to t, accounting for (2.1) and $(2.3)_2$, then multiplying two sides by $\mathbf{Q}^{\star\mathrm{T}}$, gives

$$\dot{\mathbf{Q}}^\star\mathbf{Q}^{\star\mathrm{T}} = \boldsymbol{\omega}^t - \mathbf{Q}^\star\boldsymbol{\omega}^m\mathbf{Q}^{\star\mathrm{T}}\,. \tag{2.13}$$

From (2.12), (2.13) and $(2.7)_2$ it follows

$$\boldsymbol{\omega}^t = \boldsymbol{\omega}^m + \boldsymbol{\omega}^r = \boldsymbol{\omega}(t) - (\hat{\boldsymbol{\omega}}(m,t) - \boldsymbol{\omega}^r(m,t))\,. \tag{2.14}$$

Defining *the plastic texture spin* $\boldsymbol{\omega}^p(t)$ as the difference of $\boldsymbol{\omega}(t)$ and $\boldsymbol{\omega}^t$, the followings relation between these spins take place [17]

$$\boldsymbol{\omega}^p(t) = \boldsymbol{\omega} - \boldsymbol{\omega}^t = \hat{\boldsymbol{\omega}}(m,t) - \boldsymbol{\omega}^r(m,t)\,. \tag{2.15}$$

The symbol m in brackets emphasizes the fact that $\hat{\boldsymbol{\omega}}(m,t)$ depends on the selection of the texture reference frame. Both $\boldsymbol{\omega}^t$ and $\boldsymbol{\omega}$ are independent of the selection of the particular texture reference frame. Therefore, from (2.15) it follows that the plastic texture spin $\boldsymbol{\omega}^p(t)$ is also independent of the choice of $\mathbf{m}_i$, and equation $(2.15)_2$ shows the possibility for measurement of $\boldsymbol{\omega}^p$ by metallografic methods.

If at generic instant t the texture reference triad $\mathbf{m}_i$ is assumed to coincide with $\mathbf{t}_i$, then the corresponding tensors are denoted by $\hat{\boldsymbol{\omega}}$ and $\boldsymbol{\omega}^r$, dropping symbol "m" in brackets, and are called respectively, *the texture R-spin* and *plastic R-spin*, see Fig. 2. In general, at the subsequent instant $\mathbf{m}_i(t+\delta t) \neq \mathbf{t}_i(t+\delta t)$. Setting $\mathbf{m}_i(t) = \mathbf{t}_i(t)$ in (2.7), (2.9) leads to the expression for R-spins:

$$2\hat{\boldsymbol{\omega}} = \hat{\omega}_{ij}(\mathbf{t}_i\otimes\mathbf{t}_j - \mathbf{t}_j\otimes\mathbf{t}_i)\,,$$

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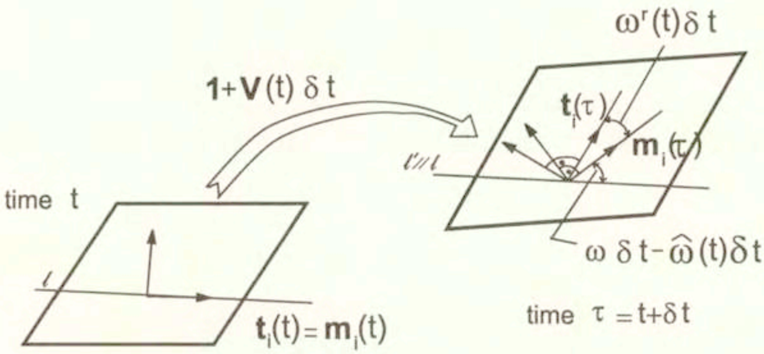

FIG. 2.

$$\hat{\omega}_{12} = \mathbf{t}_1 \cdot \mathbf{D}\mathbf{t}_2\,, \qquad \hat{\omega}_{13} = \mathbf{t}_1 \cdot \mathbf{D}\mathbf{t}_3\,, \qquad \hat{\omega}_{32} = \mathbf{t}_2 \cdot \mathbf{D}\mathbf{t}_3\,, \tag{2.16}$$

and

$$2\boldsymbol{\omega}^r = \omega^r_{ij}\left(\mathbf{t}_i \otimes \mathbf{t}_j - \mathbf{t}_j \otimes \mathbf{t}_i\right). \tag{2.17}$$

Tensor $\boldsymbol{\omega}^r \delta t$ represents the angles between vectors $\mathbf{t}_i(t+\delta t)$ and $\mathbf{m}_i(t+\delta t)$ while tensor $(\boldsymbol{\omega} - \hat{\boldsymbol{\omega}})\delta t$ specifies the angles between $\mathbf{m}_i(t)$ and $\mathbf{m}_i(t+\delta t)$. The relation (2.15) remains valid, so that:

$$\boldsymbol{\omega}^p = \boldsymbol{\omega} - \boldsymbol{\omega}^t = \hat{\boldsymbol{\omega}} - \boldsymbol{\omega}^r\,. \tag{2.18}$$

The plastic spin is thus divided into two parts. From (2.16), it follows that $\hat{\boldsymbol{\omega}}$ is a known function of $\mathbf{t}_i$ and $\mathbf{D}$. Both the relative texture spin and plastic reference spin (including R-spins) are objective tensors. To complete the description, a constitutive equation for $\boldsymbol{\omega}^r$ should be formulated. This could be verified by metallografic measurements of the texture orientation changes during deformation. Due to the lack of appropriate experimental data, some theoretical assumptions may be proposed to find meaningful constitutive equations. For example, RANIECKI and SAMANTA supposed $\boldsymbol{\omega}^r = 0$ what means that the variation of texture orientation and the variation of its reference frame is the same during the deformation process [18]. First, we consider the case of simple shear to explain the notions of this section. The more general case of generalized plane strain state was studied in the same paper of RANIECKI and MRÓZ [17].

2.2. General structure of evolution rules for rigid-plastic solids

Consider the texture anisotropy assuming that the yield surface is specified by the reduced stress $\boldsymbol{\sigma} - \boldsymbol{\alpha}$, where $\boldsymbol{\sigma}$ is the Cauchy stress and $\boldsymbol{\alpha}$ is the texture internal stress defining the shift of the yield surface. Let $\mathcal{A}$ be the set of variable

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$$\mathcal{A} = (\mathbf{D}, \boldsymbol{\sigma}, \boldsymbol{\alpha}, \mathbf{t}_i, \alpha)\,, \tag{2.19}$$

where $\mathbf{t}_i$ is the texture triad and α is scalar internal variable. Assume that we know the initial orientation $\mathbf{t}_i^0$ of the texture triad and the initial texture stress $\boldsymbol{\alpha}(t_0) = \boldsymbol{\alpha}^0$. A general form of evolution equations for rigid-plastic solids is

$$\boldsymbol{\omega}^r = \mathcal{P}(\mathcal{A})\,, \qquad \overset{\circ}{\boldsymbol{\alpha}} = \mathcal{Q}(\mathcal{A})\,, \qquad \dot{\alpha} = \mathcal{K}(\mathcal{A})\,; \tag{2.20}$$

here $\overset{\circ}{\boldsymbol{\alpha}}$ is the corotational rate with texture

$$\overset{\circ}{\boldsymbol{\alpha}} = \dot{\boldsymbol{\alpha}} + \boldsymbol{\alpha}\boldsymbol{\omega}^t - \boldsymbol{\omega}^t\boldsymbol{\alpha} \tag{2.21}$$

and $\mathcal{P}$ is antisymmetric, $\mathcal{Q}$ is symmetric and $\mathcal{K}$ is a scalar isotropic functions of all tensor arguments. They are homogeneous functions of degree one of $\mathbf{D}$ for rate-independent materials. We will use some theoretical assumptions for the form of $\mathcal{P}$. For linear kinematic hardening the evolution equation for $\boldsymbol{\alpha}$ is as follows:

$$\overset{\circ}{\boldsymbol{\alpha}} = c\,\mathbf{D} \tag{2.22}$$

where c =const. For this rule, the equation $(2.19)_2$ can be integrated to obtain interesting integral form of the equation for the internal variable [17]:

$$\boldsymbol{\alpha}(t) = c\left(\int\limits_{t_0}^{t} \mathbf{t}_i(\tau)\cdot\mathbf{D}(\tau)\,\mathbf{t}_j(\tau)d\tau\right)\mathbf{t}_i(t)\otimes\mathbf{t}_j(t) + \alpha_{ij}^0\,\mathbf{t}_i(t)\otimes\mathbf{t}_j(t)\,; \tag{2.23}$$

here α_{ij}^0 are the initial texture stress components in the initial texture triad

$$\alpha_{ij}^0 = \mathbf{t}_i^0\cdot\boldsymbol{\alpha}^0\,\mathbf{t}_j^0\,. \tag{2.24}$$

The first term in (2.23) describes the change of internal stress due to plastic deformation while the second term specifies the rotation of the initial internal stress with the texture.

3. Simple shear analysis

3.1. Expressions for the spins of previous section

The motion in simple shear is given by

$$x_1 = X_1 + \operatorname{ctg}\chi X_2 = X_1 + \gamma X_2\,, \qquad x_2 = X_2\,, \qquad x_3 = X_3\,, \tag{3.1}$$

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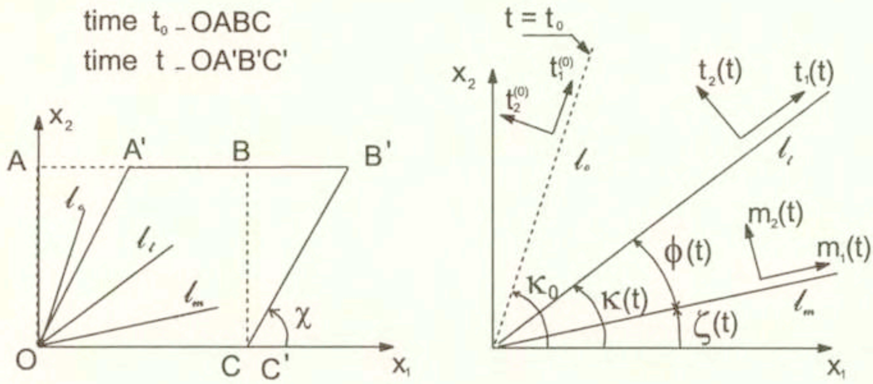

FIG. 3.

where x_i and X_i, $i = 1, 2, 3$, are the Cartesian coordinates of the current and initial position of a material point, $\gamma = \operatorname{ctg} \chi$ is called the engineering shear strain. The deformation gradient $\mathbf{F}$ in polar decomposition reads $\mathbf{F} = \mathbf{RU} = \mathbf{SR}$. After a straightforward computation we obtain the following relations

$$\mathbf{S}^2 = \begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.2}$$

The rate of plastic deformation, the material spin and the rate $\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}$ in truncated form (all other components that have an index equal to 3 being identically zero) are

$$\begin{gathered} \mathbf{D} = \frac{\dot{\gamma}}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \boldsymbol{\omega} = \frac{\dot{\gamma}}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} = \frac{2\dot{\gamma}}{\gamma^2+4} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{gathered} \tag{3.3}$$

Let the rectangle OABC be deformed into a parallelogram OA'B'C' at the instant t, see Fig. 3. The texture orientation at the initial instant t_0 is specified by the line l_0 (the vector $\mathbf{t}_1^{(0)}$ is parallel to l_0). At the instant t the texture orientation is presented by the position of the line l_t (vector $\mathbf{t}_1(t)$ lies on this direction). The material line coinciding before deformation with the segment l_0 at instant t occupies the position l_m (vector $\mathbf{m}_1(t)$ is parallel to this line). Denote by $\zeta(t)$ the angle between the axis x_1 and l_m. Let the angle $\kappa(t)$ specify the instantaneous orientation of texture l_t and introduce the rotation $\phi = \zeta(t) - \kappa(t)$.

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We have at $t = t_0$:

$$\zeta(t_0) = \kappa(t_0) = \kappa_0\,, \qquad \phi(t_0) = 0\,. \tag{3.4}$$

From (2.7) and (3.3) it follows that the orientation of l_m does not depend on the stretch and shear strain histories:

$$\operatorname{ctg}\zeta(t) = \gamma + \operatorname{ctg}\kappa_0 \tag{3.5}$$

and all the spins defined in previous section have now only one independent component with index $_{12}$ which will be dropped:

$$\omega(t) = \dot{\gamma}/2 \qquad \text{for material spin}\,, \tag{3.6}$$

$$\omega^t(t) = -\,\dot{\kappa} \qquad \text{for instantaneous texture spin}\,, \tag{3.7}$$

$$\omega^r(m,t) = \dot{\phi} \qquad \text{for relative texture spin}\,, \tag{3.8}$$

$$\hat{\omega}(m,t) = \frac{\cos 2\zeta}{2}\dot{\gamma} \qquad \text{for relative plastic spin}\,, \tag{3.9}$$

$$\hat{\omega} = \frac{\cos 2\kappa}{2}\dot{\gamma} \qquad \text{for plastic } R\text{-spin}\,, \tag{3.10}$$

$$\omega^r(t) = \dot{\Omega}(t) \qquad \text{for texture } R\text{-spin}\,, \tag{3.11}$$

$$\begin{aligned}\omega^p(t) &= \hat{\omega}(m,t) - \dot{\phi}(t) \qquad \text{for total plastic spin}\,. \\ &= \hat{\omega} - \dot{\Omega}(t)\end{aligned} \tag{3.12}$$

The equation (2.1) specifying the texture orientation leads to one differential equation for $\kappa(t)$. Supposing that the texture R-spin $\dot{\Omega}$ is known, this equation reads

$$\dot{\kappa} + \dot{\Omega} + \dot{\gamma}\sin^2\kappa = 0\,. \tag{3.13}$$

and in the case where the total plastic spin is known:

$$\dot{\kappa} = \omega^p(t) - \dot{\gamma}/2\,. \tag{3.14}$$

Equations (2.23) give, after some transformation, the expressions for internal stresses

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$$\alpha_{22}-\alpha_{11} = -c\left[\cos 2\kappa(\gamma)\int_0^\gamma \sin 2\kappa(g)dg - \sin 2\kappa(\gamma)\int_0^\gamma \cos 2\kappa(g)dg\right]$$

$$+ (\alpha_{22}^0 - \alpha_{11}^0)\cos 2\left(\kappa(\gamma)-\kappa_0\right) + 2\alpha_{12}^0 \sin 2\left(\kappa(\gamma)-\kappa_0\right),$$

(3.15)
$$\alpha_{22}+\alpha_{11} = (\alpha_{22}^0+\alpha_{11}^0),$$

$$\alpha_{12} = \frac{c}{2}\left[\cos 2\kappa(\gamma)\int_0^\gamma \cos 2\kappa(g)dg + \sin 2\kappa(\gamma)\int_0^\gamma \sin 2\kappa(g)dg\right]$$

$$+ (\alpha_{11}^0 - \alpha_{22}^0)\sin 2\left(\kappa(\gamma)-\kappa_0\right) + \alpha_{12}^0 \cos 2\left(\kappa(\gamma)-\kappa_0\right),$$

$$\alpha_{33} = \alpha_{33}^0 .$$

Here the function $\kappa(\gamma)$ is found from the functions $\kappa(t)$ and $\gamma(t)$, $\kappa|_{\gamma=0} = \kappa_0$. For incompressible materials $-\alpha_{33}^0 = \alpha_{11}^0 + \alpha_{22}^0$.

When a Mises-type yield criterion is adopted

$$f = \frac{3}{2}(\bar{\boldsymbol{\sigma}} - \boldsymbol{\alpha})\cdot(\bar{\boldsymbol{\sigma}} - \boldsymbol{\alpha}) - \sigma_0^2 = 0 \tag{3.16}$$

where $\bar{\boldsymbol{\sigma}}$ is the deviator of $\boldsymbol{\sigma}$, one has

$$\bar{\sigma}_{11} = \alpha_{11}, \qquad \bar{\sigma}_{22} = \alpha_{22}, \qquad \bar{\sigma}_{12} = \frac{\sigma_0}{\sqrt{3}} + \alpha_{12}, \tag{3.17}$$

so, for zero initial value of $\boldsymbol{\alpha}$, the relations (3.15 - 3.17) lead to the following normalized stresses (with respect to yield):

$$\sigma_{11}' \equiv \frac{\bar{\sigma}_{11}}{\sigma_0} = \frac{c}{2\sigma_0}\left[\cos 2\kappa(\gamma)\int_0^\gamma \sin 2\kappa(g)dg\right.$$

$$\left. - \sin 2\kappa(\gamma)\int_0^\gamma \cos 2\kappa(g)dg\right] = -\sigma_{22}',$$

(3.18)

$$\sigma_{12}' \equiv \frac{\bar{\sigma}_{12}}{\sigma_0} = \frac{1}{\sqrt{3}} + \frac{c}{2\sigma_0}\left[\cos 2\kappa(\gamma)\int_0^\gamma \cos 2\kappa(g)dg\right.$$

$$\left. + \sin 2\kappa(\gamma)\int_0^\gamma \sin 2\kappa(g)dg\right].$$

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3.2. Different assumptions for texture orientation

1. When total plastic spin is neglected, from (3.14) one has $\dot{\kappa} = -\dot{\gamma}/2$ then $\kappa(\gamma) = \kappa_0 - \gamma/2$. Introducing this relation to (3.18), the known oscillatory results are obtained:

$$\sigma'_{11} = \frac{c}{2\sigma_0}(1-\cos\gamma), \qquad \sigma'_{12} = \frac{1}{\sqrt{3}} + \frac{c}{2\sigma_0}\sin\gamma. \tag{3.19}$$

2. With the assumption of RANIECKI and SAMANTA $\boldsymbol{\omega}^r = 0$ [18], from (2.18) one has $\boldsymbol{\omega} - \boldsymbol{\omega}^t = \hat{\boldsymbol{\omega}}$, and the last relation leads to the differential equation

$$\dot{\kappa} + \dot{\gamma}\sin^2\kappa = 0, \tag{3.20}$$

which gives $\gamma = \operatorname{cotg}\kappa - \operatorname{cotg}\kappa_0$. After a long but straightforward calculations, one obtains from (3.18):

$$\begin{aligned} \sigma'_{11} &= \frac{c}{2\sigma_0(\gamma^2+1)}\left[(\gamma^2-1)\ln(\gamma^2+1) - 2\gamma(\gamma - 2\operatorname{arctg}\gamma)\right], \\ \sigma'_{12} &= \frac{1}{\sqrt{3}} + \frac{c}{2\sigma_0(\gamma^2+1)}\left[(\gamma^2-1)(\gamma - 2\operatorname{arctg}\gamma) + 2\gamma\ln(\gamma^2+1)\right]. \end{aligned} \tag{3.21}$$

3. In paper [2] DAFALIAS considered the corotational rates with spin $\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}$. This rate was also studied by DIENES [3] for hypoelasticity. This case leads to

$$\dot{\kappa} = -\frac{2\dot{\gamma}}{\gamma^2+4} \qquad \text{or} \qquad \kappa = -\operatorname{arctg}\frac{\gamma}{2} + \kappa_0$$

and from (3.18) we obtain

$$\begin{aligned} \sigma'_{11} &= \frac{c}{2\sigma_0}\Big[4\cos\Big(2\operatorname{arctg}\frac{\gamma}{2}\Big)\ln\Big(\cos\operatorname{arctg}\frac{\gamma}{2}\Big) \\ &\qquad + \sin\Big(2\operatorname{arctg}\frac{\gamma}{2}\Big)\Big(4\operatorname{arctg}\frac{\gamma}{2} - \gamma\Big)\Big], \\ \sigma'_{12} &= \frac{1}{\sqrt{3}} + \frac{c}{2\sigma_0}\Big[\cos\Big(2\operatorname{arctg}\frac{\gamma}{2}\Big)\Big(4\operatorname{arctg}\frac{\gamma}{2} - \gamma\Big) \\ &\qquad - 4\sin\Big(2\operatorname{arctg}\frac{\gamma}{2}\Big)\ln\Big(\cos\operatorname{arctg}\frac{\gamma}{2}\Big)\Big], \end{aligned} \tag{3.22}$$

which can be shown to be the same as in [2].

4. The simple shear process was considered by PAULUN and PĘCHERSKI [14, 16] by introducing an influence function which produces a retardation of the material spin. In this case, the instantaneous texture spin (3.7) is in the form

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$$\dot{\kappa} = -\frac{\dot{\gamma}}{2\,(1+\gamma^2)}\,; \tag{3.23}$$

then we find $\kappa = \kappa_0 - 1/2\,\mathrm{arctg}\gamma$ and equations (3.18) give the followings relations for stresses as in the papers mentioned above:

$$\begin{aligned} \sigma'_{11} &= \frac{c}{2\sigma_0\sqrt{\gamma^2+1}}\left[1-\sqrt{\gamma^2+1}+\gamma\ln\left(\sqrt{\gamma^2+1}+\gamma\right)\right], \\ \sigma'_{12} &= \frac{1}{\sqrt{3}}+\frac{c}{2\sigma_0\sqrt{\gamma^2+1}}\left[\ln\left(\sqrt{\gamma^2+1}+\gamma\right)+\gamma\,(\sqrt{\gamma^2+1}-1)\right]. \end{aligned} \tag{3.24}$$

5. The assumption that every material remains isotropic becomes less good as the deformation continues. Individual crystal grains are elongated in the direction of the greatest tensile strain and the texture of the specimen becomes fibrous (see HILL [5]). Assume that the texture triad coincides with that of the principal axes of the left stretch tensor **S** during the deformation process [19]. In this case, from (3.2) we can find the principal axes of **S** and the instant texture triad is defined as follows:

$$\begin{aligned} \mathbf{t}_1 &= \left(\frac{\gamma^2+\sqrt{\gamma^2\,(\gamma^2+4)}}{\sqrt{4\gamma^2+\left[\gamma^2+\sqrt{\gamma^2(\gamma^2+4)}\right]^2}};\frac{2\gamma}{\sqrt{4\gamma^2+\left[\gamma^2+\sqrt{\gamma^2(\gamma^2+4)}\right]^2}}\right), \\ \mathbf{t}_2 &= \left(\frac{\gamma^2-\sqrt{\gamma^2\,(\gamma^2+4)}}{\sqrt{4\gamma^2+\left[\gamma^2-\sqrt{\gamma^2(\gamma^2+4)}\right]^2}};\frac{2\gamma}{\sqrt{4\gamma^2+\left[\gamma^2+\sqrt{\gamma^2(\gamma^2+4)}\right]^2}}\right), \end{aligned} \tag{3.25}$$

and the instant orientation of the texture triad is

$$\kappa = \mathrm{arctg}\left(\frac{2}{\gamma+\sqrt{\gamma^2+4}}\right).$$

Now the stresses are

$$\begin{aligned} \sigma'_{11} &= \frac{c}{\sigma_0\sqrt{\gamma^2+4}}\left[\gamma\ln\left(\frac{\gamma+\sqrt{\gamma^2+4}}{2}\right)-\sqrt{\gamma^2+4}+2\right], \\ \sigma'_{12} &= \frac{1}{\sqrt{3}}+\frac{c}{2\sigma_0\sqrt{\gamma^2+4}}\left[\gamma(\sqrt{\gamma^2+4}-2)+4\ln\left(\frac{\gamma+\sqrt{\gamma^2+4}}{2}\right)\right]. \end{aligned} \tag{3.26}$$

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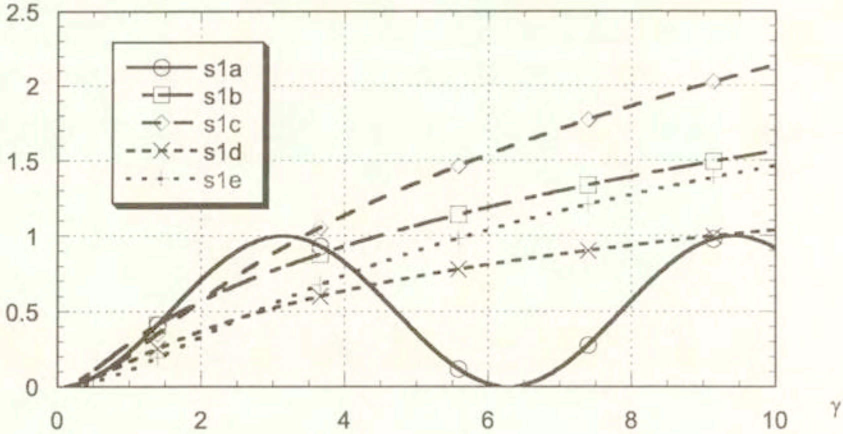

FIG. 4. Normalized stress σ_{11}/σ_0 vs. strain; s1a – Jaumann; s1b – Raniecki and Samanta; s1c – Dafalias; s1d – Paulun and Pęcherski; s1e – this paper.

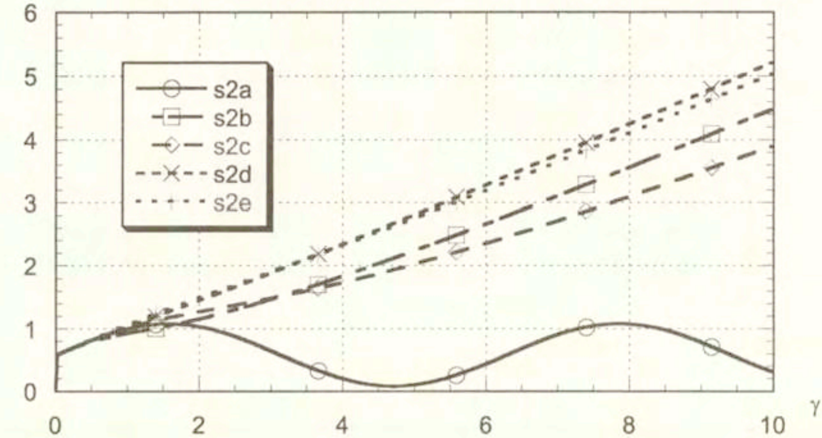

FIG. 5. Normalized stress σ_{12}/σ_0 vs. strain; s2a – Jaumann; s2b – Raniecki and Samanta; s2c – Dafalias; s2d – Paulun and Pęcherski; s2e – this paper.

We take the material constants for an aluminum alloy from the paper of LEE *et al.* [8]: initial yield stress $\sigma_0 = 207\text{MPa}$ and modulus of linear harderning $c = 206.6\text{MPa}$, to plot the normal σ'_{11} and shear stress σ'_{12} vs. strain γ for all the case considered in this section. They are shown in Fig. 4 and Fig. 5 where curve (a) for Jaumann rate, curve (b) for Raniecki and Samanta's assumption, curve (c) for the case of Dafalias, curve (d) for the case of Paulun and Pęcherski, and curve (e) for the assumption of this paper.

4. The expression for the proposed spin in absolute representation

Decompose the deformation gradient **F** in the polar form

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$$\text{(4.1)} \qquad \mathbf{F} = \mathbf{RU} = \mathbf{SR},$$

where $\mathbf{U}$, $\mathbf{S}$ are, respectively, the right and the left strech tensors and $\mathbf{R}$ is proper orthogonal. Let $\mathbf{N}_i$ and λ_i, $(i = 1, 2, 3)$ be respectively, the principal directions and eigenvalues of the stretch tensor $\mathbf{U}$ (which are also those of $\mathbf{S}$)

$$\text{(4.2)} \qquad \mathbf{U} = \lambda_i \, \mathbf{N}_i \otimes \mathbf{N}_i .$$

Denote by $\mathbf{n}_i$ the principal directions of $\mathbf{S}$, $\mathbf{n}_i = \mathbf{RN}_i$, then

$$\text{(4.3)} \qquad \mathbf{S} = \lambda_i \, \mathbf{n}_i \otimes \mathbf{n}_i .$$

By differentiating this spectral decomposition with respect to time we obtain

$$\text{(4.4)} \qquad \mathbf{S}\boldsymbol{\Omega}^S - \boldsymbol{\Omega}^S\mathbf{S} = \dot{\lambda}_i \, \mathbf{n}_i \otimes \mathbf{n}_i - \dot{\mathbf{S}} .$$

Supposing $\mathbf{S}$ known function of t we can find $\boldsymbol{\Omega}^S$ from the previous relation. The equation of this form was studied by GUO and others in the paper [4]. Denote by I, II and III the principal invariants of $\mathbf{S}$:

$$\text{(4.5)} \qquad \begin{aligned} \mathrm{I} &= \lambda_1 + \lambda_2 + \lambda_3 , \\ \mathrm{II} &= \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2 , \\ \mathrm{III} &= \lambda_1\lambda_2\lambda_3 . \end{aligned}$$

In case of distinct eigenvalues, the solution of (4.4) is

$$\text{(4.6)} \qquad \begin{aligned} \boldsymbol{\Omega}^S = \Delta^{-2}[&(6\mathrm{I}\cdot\mathrm{III} - 5\mathrm{I}^2\cdot\mathrm{II} + \mathrm{I}^4 + 4\mathrm{II}^2)(\dot{\mathbf{S}}\mathbf{S} - \mathbf{S}\dot{\mathbf{S}}) \\ &+(4\mathrm{I}\cdot\mathrm{II} - \mathrm{I}^3 - 9\mathrm{III})(\dot{\mathbf{S}}\mathbf{S}^2 - \mathbf{S}^2\dot{\mathbf{S}}) + (\mathrm{I}^2 - 3\mathrm{II})(\mathbf{S}\dot{\mathbf{S}}\mathbf{S}^2 - \mathbf{S}^2\dot{\mathbf{S}}\mathbf{S})] \end{aligned}$$

here

$$\text{(4.7)} \qquad \begin{aligned} \Delta^2 &= (\lambda_2 - \lambda_3)^2(\lambda_3 - \lambda_1)^2(\lambda_1 - \lambda_2)^2 \\ &= 18\mathrm{I}\cdot\mathrm{II}\cdot\mathrm{III} + \mathrm{I}^2\cdot\mathrm{II}^2 - 4\mathrm{I}^3\cdot\mathrm{III} - 4\mathrm{II}^3 - 27\mathrm{III}^2. \end{aligned}$$

Note that from (4.1) $\mathbf{S} = \mathbf{RUR}^{\mathrm{T}}$ we can express $\dot{\mathbf{S}}$ in terms of the velocity gradient using the following relations derived by HOGER [6] (valid in case of distinct eigenvalues)

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$$\dot{\mathbf{U}} = \frac{1}{\mathrm{I}\cdot\mathrm{II} - \mathrm{III}}\mathbf{R}^{\mathrm{T}}\left[\mathbf{S}^2\mathbf{D}\mathbf{S}^2 - \mathrm{I}(\mathbf{S}^2\mathbf{D}\mathbf{S} + \mathbf{S}\mathbf{D}\mathbf{S}^2) + (I^2 + \mathrm{II})\mathbf{S}\mathbf{D}\mathbf{S} - \mathrm{III}(\mathbf{S}\mathbf{D} + \mathbf{D}\mathbf{S}) + (\mathrm{I}\cdot\mathrm{III})\mathbf{D}\right]\mathbf{R} \tag{4.8}$$

and

$$\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} = \omega + \frac{1}{\mathrm{I}\cdot\mathrm{II} - \mathrm{III}}[\mathrm{I}^2(\mathbf{D}\mathbf{S} - \mathbf{S}\mathbf{D}) + \mathrm{I}(\mathbf{S}^2\mathbf{D} - \mathbf{D}\mathbf{S}^2) + (\mathbf{S}\mathbf{D}\mathbf{S}^2 - \mathbf{S}^2\mathbf{D}\mathbf{S})]. \tag{4.9}$$

In the particular case, where $\mathbf{DS} = \mathbf{SD}$ (HOGER [6] have proved that this is the necessary and sufficient condition for the corotational Jaumann derivative of $\ln\mathbf{S}$ to be equal to the stretching tensor $\mathbf{D}$), the relation (4.6) takes the form

$$\begin{aligned}\boldsymbol{\Omega}^S = \Delta^{-2}\Big\{&(6\mathrm{I}\cdot\mathrm{III} - 5\mathrm{I}^2\cdot\mathrm{II} + \mathrm{I}^4 + 4\mathrm{II}^2)\left[(\omega\mathbf{S} - \mathbf{S}\omega)\mathbf{S} - \mathbf{S}(\omega\mathbf{S} - \mathbf{S}\omega)\right]\\ &+(4\mathrm{I}\cdot\mathrm{II} - \mathrm{I}^3 - 9\mathrm{III})\left[(\omega\mathbf{S} - \mathbf{S}\omega)\mathbf{S}^2 - \mathbf{S}^2(\omega\mathbf{S} - \mathbf{S}\omega)\right]\\ &+(\mathrm{I}^2 - 3\mathrm{II})\left[\mathbf{S}(\omega\mathbf{S} - \mathbf{S}\omega)\mathbf{S}^2 - \mathbf{S}^2(\omega\mathbf{S} - \mathbf{S}\omega)\mathbf{S}\right]\Big\},\end{aligned} \tag{4.10}$$

and the relation for plastic spin (2.18) now reads

$$\omega^p = \omega - \boldsymbol{\Omega}^S. \tag{4.11}$$

5. Conclusions

In the paper we use the suggestion of HILL to propose the orientation of instant texture triad for accomplishing the description of rigid-plastic material in finite deformation. The paper shows the relation between the two approaches for defining the plastic spin. The considerations presented in the work of Raniecki and Mróz permit to understand the physical meaning and, in principle, shows the possible way to find experimentally the evolution law for plastic spin.

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