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Localisation of deformation as a local quasi-static/dynamic transition (¹)

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THE PROBLEM OF THE MODELLING of the strain localisation in elasto-visco-nonlinear materials and structures submitted to so-called "quasistatic" loadings is here considered. Unlike the usual approaches, which suppose that the localisation band remains in (quasi)static equilibrium, it is here assumed that **localisation is essentially a dynamic phenomenon**, even if the external loadings are "quasistatic". This means that the localisation criterion proposed is also a "local loss of (quasi)staticity" criterion. As soon as the criterion is verified, the dynamic problem is treated (at least locally) instead of the (quasi)static one.

1. Introduction

LOCALISATION OF DEFORMATION is an instability process, accompanying inelastic deformation, widely observed under quasi-static as well as dynamic loading conditions. This mechanism is characterised by the transition from a diffuse mode of deformation to a localised mode associated with the formation of narrow zones in which strains quickly and highly concentrate. Besides, once such bands appear, they persist and under favourable circumstances become immediate precursor to failure. This phenomenon is followed either by the emergence of a macroscopic crack leading to fracture (brittle materials like rocks and concretes) or by a softening regime ending also by fracture events (ductile materials like steels).

Properly modelling the initiation and development of such a material instability could constitute one of the missing links between continuum mechanics framework and fracture mechanics framework.

For a non-viscous material, modelled as rate-independent (²), considered to

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(²)Referred to in this paper as elasto-nonlinear materials or ENL.

deform quasi-statically, the usual mathematical method employed to predict the onset of strain localisation is the bifurcation analysis. This approach, within a theoretical framework due to HADAMARD [11], is based on the requirement of traction continuity and a jump of the velocity gradient across a singularity surface limiting the band. These conditions lead to the well-known condition for the singularity of the associated acoustic tensor resulting from a double contraction of the dissipative branch of the tangent operator with the normal to the band (RICE [20]). The vanishing of the determinant of the acoustic tensor at the onset of localisation is connected with the loss of ellipticity of the local rate equilibrium equations and with the occurrence of stationary acceleration waves in the solid.

As a consequence of the loss of uniqueness of the mechanical response, a fundamental difficulty in numerical simulations appears. For grid based methods such as finite elements or finite differences, the width of the band of localisation depends on the size of the elements meshing the zone where the instability is detected. This pathological dependence induces the use of regularisation techniques based either on the kinematics of localisation phenomenon at the elementary level (finite element regularisation method (ORTIZ *et al.* [17])) or by incorporating an internal length scale or a higher order continuum structure in constitutive relations (PIJAUDIER-CABOT and BAZĂNT [19], ZBIB and AIFANTIS [21]).

For a viscous material, modelled as rate-dependent ([3]), the application of the bifurcation analysis provides a criterion never fulfilled. The tangent operator coincides with the elastic stiffness tensor which remains positive definite. The uniqueness of the solution of the local rate (or incremental) constitutive equations is guaranteed and the initial boundary-value problem remains well posed.

Yet, experimental tests establish the existence of localisation patterns for viscous materials. Moreover, when numerical simulations are performed, even if bifurcation is precluded, by localisation instability is observed; mesh sensitivity is either reduced for slightly viscous materials or suppressed with higher viscosity (FOREST and CAILLETAUD [13]). This apparent paradox may be understood by the fact that finite element method constraints require the use of a pseudo-tangent operator, instead of the real one, allowing the bifurcation of the numerical (incremental) problem in a localised mode. Besides, it is just numerical interpretation that does not provide any answer to defining a localisation criterion for viscous materials.

To remove this difficulty, changes in the mode of deformation may be detected by employing a linear perturbation stability analysis. At any stage of the postulated deformation process, an infinitesimal exponential disturbance is superimposed onto the regular solution. The homogeneous solution is said to be unstable if the analysis reveals the growth of an admissible perturbation. This method has been employed first in one-dimensional problems (CLIFTON [7] and

([3])Referred to in this paper as elasto-visco-nonlinear materials or EVNL.

BAI [2]), extended to three-dimensional problems neglecting elasticity (ANAND *et al.* [1]), generalised to three dimensional problems with the effect (among others) of elasticity (DOBOVŠEK and MORAN [9], CANO [4]).

This kind of analysis, as compared to a full non linear study, because of linearisation, predicts only the necessary conditions for the onset of an instability and fails to forecast the evolution of localisation during an extended period of time. The rate of growth value appears in a characteristic stability equation and localisation is detected when this value is sufficiently large compared with the variation of the homogeneous solution (MOLINARI [16]). As noticed by CANO [4], a question remains for qualifying this rate of growth: the highest value admissible seems to be an upper bound to localisation.

In this paper, the aim is to establish an alternative criterion for EVNL materials and structures submitted to quasi-static loadings, also based on a perturbation stability analysis, but grounded on the hypothesis that even for so-called quasi-static loadings localisation is an intrinsically local dynamic phenomenon.

A large class of EVNL materials is first presented in Sec. 2. The initial-boundary value problem associated is then considered in Sec. 3. Next in Sec. 4, we propose to quantify the notion of quasi-staticity: as soon as the inertia terms of the dynamic problem exceed a critical value, the problem to be considered is no longer the quasi-static problem but, at least locally, the dynamic one. This value is then coupled to a critical kinetic power reached at the incipience of localisation. To let the viscosity effects appear, a perturbation stability analysis of the acceleration boundary-value problem is carried out in Sec. 5, which provides a characteristic stability equation function of the rate of growth of the perturbation. At last in Sec. 6, by considering the instability of the acceleration problem corresponding to the reach of the critical value of inertia terms, i.e. by choosing the lowest rate of growth value violating the quasistaticity condition, a loss of quasistaticity/localisation criterion is established.

2. Constitutive equations

Attention is focused hereafter on a large class of constitutive equations, established under the small strain framework, within the first gradient theory and under isothermal conditions. We admit the reversible behaviour of the rate-dependent materials considered in this paper (instantaneous elasticity) to be determined by a potential free energy $w(\boldsymbol{\epsilon}, \boldsymbol{\alpha})$, function of the linearised strain $\boldsymbol{\epsilon}$, at most quadratic to preserve linear elasticity, and a given number of internal variables $\boldsymbol{\alpha}_p$. These internal variables may be scalars, vectors or tensors; for simplicity, they will be denoted as $\boldsymbol{\alpha}$.

Thereafter, the stress tensor $\boldsymbol{\sigma}$ (connected with $\boldsymbol{\epsilon}$) and the thermodynamic forces $\mathbf{A}$ connected with the internal variables $\boldsymbol{\alpha}$, are given by the state laws:

$$\boldsymbol{\sigma} = \frac{\partial w}{\partial \boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \boldsymbol{\alpha}), \qquad \mathbf{A} = -\frac{\partial w}{\partial \boldsymbol{\alpha}}(\boldsymbol{\epsilon}, \boldsymbol{\alpha}), \tag{2.1}$$

As far as the material irreversible behaviour is concerned, we define a convex reversibility domain limited by the criterion $f = 0$ (in the space of forces) where f (the yield function) is a function of $\mathbf{A}$, eventually parameterised by $\boldsymbol{\alpha}$, i.e. $f = f(\mathbf{A}; \boldsymbol{\alpha})$; inside this domain no irreversibility is possible.

We also assume the existence of a pseudo-potential $g(\mathbf{A}; \boldsymbol{\alpha})$ from which the evolution laws, assuming $\dot{\boldsymbol{\alpha}}$-normality, follow:

$$\dot{\boldsymbol{\alpha}} = \Lambda \frac{\partial g}{\partial \mathbf{A}}, \tag{2.2}$$

where Λ is called pseudo-visco-nonlinear multiplier. The elasto-visco-nonlinear class of models considered here is an extension of the elasto-viscoplastic materials first proposed by PERZYNA [18]. The multiplier is given by:

$$\Lambda = \frac{1}{\eta} < \Phi(f) >^N, \tag{2.3}$$

where $<>$ are the Macauley brackets ($< x >= \mathrm{Max}(x, 0)$), η is a relaxation time. Common choices for the function Φ are:

$$\underset{\text{linear form}}{\Phi(f) = \frac{f}{K}} \quad \underset{\text{power law}}{\Phi(f) = \left(\frac{f}{K}\right)^N}, \tag{2.4}$$

where N is a dimensionless viscosity exponent (Norton's coefficient) and K a resistance coefficient depending on the material mechanical state, say $K = K(\boldsymbol{\epsilon}, \boldsymbol{\alpha})$. In the sequel, the power law form of (2.4) is assumed.

By differentiating (2.1) with respect to time, one obtains the rate constitutive equations:

$$\dot{\boldsymbol{\sigma}} = \mathbf{E}^w : \dot{\boldsymbol{\epsilon}} + \mathbf{B}(\boldsymbol{\epsilon}, \boldsymbol{\alpha}). \tag{2.5}$$

$\mathbf{E}^w = \dfrac{\partial^2 w}{\partial \boldsymbol{\epsilon}\, \partial \boldsymbol{\epsilon}}$ is the elastic stiffness tensor and $\mathbf{B}$ is a function of non rate terms:

$$\mathbf{B}(\boldsymbol{\epsilon}, \boldsymbol{\alpha}) = -\frac{1}{\eta} < \frac{f}{K} >^N \boldsymbol{\chi}^w(\boldsymbol{\epsilon}, \boldsymbol{\alpha}) \bullet \frac{\partial g}{\partial \mathbf{A}}(\mathbf{A}(\boldsymbol{\epsilon}, \boldsymbol{\alpha}), \boldsymbol{\alpha}). \tag{2.6}$$

The tensor $\boldsymbol{\chi}^w$ is given by:

$$\boldsymbol{\chi}^w = -\frac{\partial^2 w}{\partial \boldsymbol{\epsilon} \partial \boldsymbol{\alpha}}. \tag{2.7}$$

3. Static, quasi-static and dynamic considerations

Let us consider a structure taking up an arbitrary opened domain Ω in $\mathbf{R}^3$ which constitutive material belongs to the general class previously described. This structure, during the time interval $[0, T]$, where $T > 0$, is subjected to body forces $\mathbf{f}^g(\mathbf{x}, t)$ and its boundary $\partial\Omega$ subjected to surface tractions $\mathbf{F}^g(\mathbf{x}, t)$ on the part $\partial\Omega_1$, and to given displacements $\mathbf{u}^g(\mathbf{x}, t)$ on $\partial\Omega_2$, so that $\partial\Omega_1 \cup \partial\Omega_2 = \partial\Omega$ and $\partial\Omega_1 \cap \partial\Omega_2 = \emptyset$.

One has to solve the nonlinear dynamical initial and boundary value problem (P_g):

Find $\mathbf{u}(\mathbf{x}, t), \boldsymbol{\sigma}(\mathbf{x}, t)$, and $\boldsymbol{\alpha}(\mathbf{x}, t)$ defined for $\mathbf{x} \in \Omega$ and $0 \leq t \leq T$ satisfying:

(3.1) (P_g)
$$\begin{cases} \bullet \text{ motion equations: } \operatorname{div} \boldsymbol{\sigma} + \mathbf{f}^g = \rho\, \ddot{\mathbf{u}} \quad \text{in } \Omega \times [0, T], \\ \bullet \text{ constitutive equations given in Sec. 2}, \\ \bullet \text{ compatibility conditions:} \\ \qquad \boldsymbol{\epsilon}(\mathbf{u}) = \dfrac{1}{2}(\operatorname{grad} \mathbf{u} + (\operatorname{grad} \mathbf{u})^T) \quad \text{in } \Omega \times [0, T], \\ \bullet \text{ boundary conditions: } \begin{cases} \mathbf{u} = \mathbf{u}^g & \text{on } \partial\Omega_2 \times [0, T] \\ \sigma \cdot \mathbf{n} = F^g & \text{on } \partial\Omega_1 \times [0, T] \end{cases}, \\ \bullet \text{ initial conditions: } \begin{cases} \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^0 & \text{in } \Omega \\ \boldsymbol{\alpha}(\mathbf{x}, 0) = \boldsymbol{\alpha}^0 & \text{in } \Omega \end{cases}, \end{cases}$$

where $\mathbf{n}$ is the outward unit normal to $\partial\Omega$.

The presence in $(3.1)_1$ of the inertia terms, say $\rho\ddot{\mathbf{u}}$ qualifies (P_g) to be the dynamic problem.

When the structure Ω undergoes **quasi-static loadings**, i.e. when the loadings are slow with respect to the time dimension, it is usually assumed that inertia terms in Eq.$(3.1)_1$ may be neglected. Then a new problem, say (P), is to be treated, in which $(3.1)_1$ becomes:

(3.2) $$\operatorname{div} \boldsymbol{\sigma} + \mathbf{f}^g = 0, \qquad \text{quasi-static equilibrium conditions}.$$

Actually, (P) is the **static problem** and coincides with the quasi-static problem as an approximation of (P_g).

In the same way, for problems of higher orders undergoing quasi-static loadings, equilibrium equations for both the rate problem $(\dot{P})$ and the acceleration problem $(\ddot{P})$ are:

$$\begin{aligned} &\operatorname{div}\dot{\boldsymbol{\sigma}} + \dot{\mathbf{f}}^g = 0\,, \\ &\operatorname{div}\ddot{\boldsymbol{\sigma}} + \ddot{\mathbf{f}}^g = 0\,. \end{aligned} \tag{3.3}$$

4. A quantitative definition of the quasi-staticity notion linked to an energetic measure at the onset of localisation

4.1. A condition for the loss of quasi-staticity

Following Sec. 3, one must say that no quantitative definition of the quasi-staticity is usually defined, i.e. a fundamental question is addressed:

When does the problem (P) cease to be quasi-static?

or equivalently, when should the general problem (P_g) be considered instead of (P)? Let us point out that external loadings are considered to stay quasi-static.

Definition: *The solution* $\mathbf{u}(\mathbf{x},t)$ *of* (P_g) *will be described as quasi-static when the r.h.s. of the balance of linear momentum equation* $(3.1)_1$ *is lower than a critical norm, say* γ_{crit}:

$$\rho\,|\ddot{\mathbf{u}}(\mathbf{x},t)| < \gamma_{\text{crit}} \quad \forall\mathbf{x}\,\forall t\,. \tag{4.1}$$

If not, the solution of (P_g) *is qualified as dynamic.*

By opposition to the usual hypothesis for qualifying an initial-boundary problem as quasi-static (practically the latter is treated as static, i.e. $\operatorname{div}\boldsymbol{\sigma} + \mathbf{f}^g = 0$), no hypothesis is formulated *a priori* for acceleration, qualified *a posteriori*.

As soon as the inequality (4.1) is not satisfied, the general problem should be solved. But, in fact, we may imagine that in a substructure Ω_i of Ω the solution of the boundary value subproblem violates the condition of quasi-staticity while in the other subdomains composing Ω the inequality (4.1) is still satisfied. Then, the problem should be, at least in the substructure Ω_i, solved as dynamic.

4.2. How to connect γ_{crit} to the occurence of localisation?

Localisation is considered as being an intrinsically dynamic local phenomenon: this idea is akin to observations of Leroy [14] in finite-element simulations of a Von Mises viscous solid under plane strain tensile loading: "*shear-band failure mode can be defined as a continuous increase in strain rate in a band of decreasing thickness. This observation also indicates that localization should be interpreted as a dynamic process even in a kinematically controlled experimental set-up*".

One suggests, at a given displacement rate $|\dot{\mathbf{u}}|$, that γ_{crit} may be linked to a critical kinetic power at localisation (κ_{crit}), experimentally accessible. This

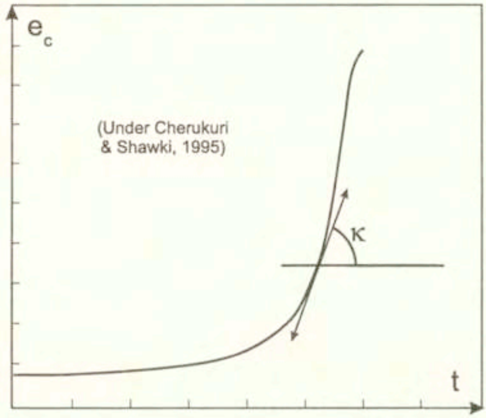

FIG. 1. Evolution of the kinetic energy.

suggestion coincides with the hypothesis of CHERUKURI and SHAWKI [5] for their localisation work under dynamic loadings.

A "pseudo-local" kinetic energy e_c ("pseudo" because linked to ρ_0, initial density) and its associated kinetic power $\dot{e_c}$ are defined first:

$$
\begin{aligned}
e_c &= \frac{1}{2}\,\rho_0\,\dot{\mathbf{u}}\cdot\dot{\mathbf{u}}\,,\\
\dot{e_c} &= \rho_0\,\dot{\mathbf{u}}\cdot\ddot{\mathbf{u}} \le \rho_0|\dot{\mathbf{u}}||\ddot{\mathbf{u}}|\,.
\end{aligned}
\tag{4.2}
$$

In order to link the parameter γ_{crit} with experimental evidences, let us consider a sample. Whenever no localisation, its response for its gauge length is characteristic of the material behaviour. The local kinetic energy could be estimated by experimental procedures (e.g. by infrared and speckle image processing techniques, see CHRYSOCHOOS *et al.* [6]) because:

$$
e_c \approx \frac{\rho_0\,l_0}{2V_0}\int\limits_S \dot{\mathbf{u}}\cdot\dot{\mathbf{u}}\,dS\,,
\tag{4.3}
$$

where S is the gauge length section, where the measures are made. An eventual critical kinetic power κ_{crit} at the localisation onset follows (see Fig. 1):

$$
\dot{e}_c = \kappa_{\text{crit}}\,.
\tag{4.4}
$$

If κ_{crit} is admitted being a material characteristic, independent of the loading path at a given $|\dot{\mathbf{u}}|$, linking loss of quasi-staticity (violation of Eq.(4.1)) and localisation (Eq.(4.4)) one may write:

$$
\gamma_{\text{crit}} = \rho\,(\rho_0|\dot{\mathbf{u}}|)^{-1}\kappa_{\text{crit}} \approx \kappa_{\text{crit}}|\dot{\mathbf{u}}|^{-1}\,,
\tag{4.5}
$$

this latter approximation being valid in the small strain framework.

5. Stability analysis

5.1. Description of the linear perturbation method

The essence of the linear perturbation stability analysis is to determine under which conditions, if the homogeneous solution of the boundary value problem is disturbed by a small perturbation, this perturbation is likely to decay or grow while the constitutive and momentum balance equations are still satisfied.

The characteristic stability equation obtained is commonly a function of the rate of growth of the disturbance. A zero root corresponds to a loss of stability of the homogeneous solution. The onset of instability corresponds to a zero value (absolute instability), but such "*an unstable mode can sometimes grow very slowly and therefore be overcome by another mode appearing later with a much higher rate of growth*" (DUDZINSKI and MOLINARI [10]).

In the context of localisation instability process, its utilisation requires the following assumptions:

• in its current configuration (t_0), the structure Ω is supposed to be homogeneous, homogeneously deformed and evolving slowly.

• the perturbation superimposed is a displacement rate one (or a displacement one).

• the disturbance chosen has a form of an exponential wave that may lead to a localised deformation mode:

$$\dot{\mathbf{u}} = \dot{\mathbf{u}}^0 + \Delta\dot{\mathbf{u}}(\mathbf{x}, t)\,,$$

with $\Delta\dot{\mathbf{u}}(\mathbf{x}, t) = \delta\dot{\mathbf{u}}\exp(ik\mathbf{x}\cdot\mathbf{n} + \omega(t - t^0))$; thus $\Delta\dot{\boldsymbol{\epsilon}} = ik\,(\Delta\dot{\mathbf{u}}\otimes\mathbf{n})_s = (\mathbf{g}\otimes\mathbf{n})_s$, where $\delta\dot{\mathbf{u}}$ is the initial perturbation amplitude, $\mathbf{n}$ the propagation direction (unit) that determines the orientation of the localisation surface, ω the rate of growth, $\mathbf{k} = k\,\mathbf{n}$ the wave vector. The relative orientation of $\mathbf{n}$ and $\mathbf{g}$ informs about the type of localisation.

• the perturbation amplitude is sufficiently small to allow the study of the "first-order problem" (linearisation).

• the results of the linear stability analysis are valid only for perturbations whose evolutions are rapid compared to variations of the homogeneous solution. MOLINARI [16] suggests to study the rate of growth of a relative perturbation defined as the disturbance divided by the corresponding homogeneous solution. Implicitly, two time scales are introduced, the former linked to the perturbation (i.e. "small"), the latter macroscopic (i.e. "large") linked to the regular solution (BATAILLE and KESTIN [3]). These hypotheses allow to consider the coefficients in the linearised perturbed equations to be constant.

Furthermore, it is usually assumed that the perturbation for the other field quantities takes the same exponential form (e.g. [9], [4]), that is to say: $\dot{\boldsymbol{\alpha}} = \dot{\boldsymbol{\alpha}}^0 + \Delta\dot{\boldsymbol{\alpha}}(\mathbf{x}, t)$ with $\Delta\dot{\boldsymbol{\alpha}}(\mathbf{x}, t) = \delta\dot{\boldsymbol{\alpha}} \exp(ik\mathbf{x}\cdot\mathbf{n} + \omega(t - t^0))$. As a consequence, all the disturbated variables present the same rate of growth and stability conditions for the problem $(\dot{P})$ are explored.

5.2. Stability of the local acceleration problem

As previously recalled, in the classical linear stability analysis, internal variables are supposed to be perturbed in the same exponential form and with the same rate of growth as the displacement rate (or displacement) perturbation. This assumption is quite disconcerting: since these variables are internal and consequently not measurable by direct observations (LEMAITRE and CHABOCHE [12]), how to justify the way they are perturbed?

In this paper, we do not use the classical linear stability analysis: instead of postulating the way the internal variables are perturbed, we let them evolve freely so that they could take into account the change in displacement rate. Following this assumption, the rate (or incremental) problem $(\dot{P})$, reflecting the instantaneous material response, is considered as stable. Suggesting (as DÉSOYER *et al.* [8]) that the response to a loading of a rate-dependent material is delayed compared to the response obtained for a rate-independent material, one naturally considers the local acceleration problem $(\ddot{P})$.

5.2.1. Momentum balance acceleration equation. The acceleration problem $(\ddot{P})$ is written, in the absence of body forces by $(3.3)_2$, as:

$$\operatorname{div} \ddot{\boldsymbol{\sigma}} = 0 . \tag{5.1}$$

Eq.(5.1) requires the expression of $\ddot{\boldsymbol{\sigma}}$ to be formulated, which is done by time derivation of Eq.(2.5):

$$\ddot{\boldsymbol{\sigma}} = \mathbf{E}^w : \ddot{\boldsymbol{\epsilon}} - V(\boldsymbol{\epsilon}, \boldsymbol{\alpha}) : \dot{\boldsymbol{\epsilon}} + \mathbf{C}(\boldsymbol{\epsilon}, \boldsymbol{\alpha}) , \tag{5.2}$$

where $\mathbf{V}$ and $\mathbf{C}$ are non-rate terms:

$$\begin{gathered} \mathbf{C}(\boldsymbol{\epsilon}, \boldsymbol{\alpha}) = \frac{1}{\eta} < \frac{f}{K} >^N \frac{\partial \mathbf{B}}{\partial \boldsymbol{\alpha}} \bullet \frac{\partial g}{\partial \mathbf{A}} \qquad \text{and} \\ \mathbf{V}(\boldsymbol{\epsilon}, \boldsymbol{\alpha}) = -\frac{1}{\eta} < \frac{f}{K} >^N \frac{\partial \mathbf{E}^w}{\partial \boldsymbol{\alpha}} \bullet \frac{\partial g}{\partial \mathbf{A}} - \frac{\partial \mathbf{B}}{\partial \boldsymbol{\epsilon}} . \end{gathered} \tag{5.3}$$

Using Eq.(2.6), expression of $\mathbf{V}$ is more precisely given by:

$$
(5.4)\qquad \eta\mathbf{V}(\boldsymbol{\epsilon},\boldsymbol{\alpha}) = \begin{cases} \dfrac{N}{K} < \dfrac{f}{K} >^{N-1} \Big[\Big(\chi^w \bullet \dfrac{\partial g}{\partial \mathbf{A}}\Big) \otimes \Big(\dfrac{\partial f}{\partial \mathbf{A}} \bullet \chi^w \\ \qquad\qquad - \dfrac{f}{K}\dfrac{\partial K}{\partial \boldsymbol{\epsilon}} \Big) \Big] , \\ - < \dfrac{f}{K} >^{N} \Big[\dfrac{\partial \mathbf{E}^w}{\partial \boldsymbol{\alpha}} \bullet \dfrac{\partial g}{\partial \mathbf{A}} - \chi^w \bullet \dfrac{\partial^2 g}{\partial \mathbf{A}\,\partial \mathbf{A}} \bullet \chi^w \\ \qquad\qquad - \Big(\dfrac{\partial \chi^w}{\partial \boldsymbol{\epsilon}}\Big)^T \bullet \dfrac{\partial g}{\partial \mathbf{A}} \Big] , \end{cases}
$$

where $\bullet$ denotes the inner product on tensors of required orders.

5.2.2. Perturbation stability analysis. The problem is considered to be initially quasi-static, with a structure undergoing quasi-static loadings, initially homogeneous, homogeneously deformed. The homogeneous or regular solution is denoted by $(\mathbf{u}^0, \boldsymbol{\epsilon}^0, \boldsymbol{\alpha}^0)$ at time t^0.

A perturbation stability analysis is then conducted involving a rate displacement perturbation $\Delta\dot{\mathbf{u}}$ of the solution $\dot{\mathbf{u}}^0$ of $(\ddot{P})$:

$$
(5.5)\qquad \forall t \geq t^0 \quad \begin{cases} \dot{\mathbf{u}} = \dot{\mathbf{u}}^0 + \Delta\dot{\mathbf{u}} , \\ \Delta\dot{\mathbf{u}}(\mathbf{x},t) = \delta\dot{\mathbf{u}} \exp\big(ik\mathbf{x}\cdot\mathbf{n} + \omega(t-t^0)\big) . \end{cases}
$$

In the perturbed state equation Eq.(5.2) becomes:

$$
(5.6)\qquad \ddot{\boldsymbol{\sigma}}(\boldsymbol{\epsilon}^0, \boldsymbol{\alpha}^0, \dot{\boldsymbol{\epsilon}}^0 + \Delta\dot{\boldsymbol{\epsilon}}, \ddot{\boldsymbol{\epsilon}}^0 + \Delta\ddot{\boldsymbol{\epsilon}}) = \mathbf{E}^w(\boldsymbol{\epsilon}^0,\boldsymbol{\alpha}^0) : (\ddot{\boldsymbol{\epsilon}}^0 + \Delta\ddot{\boldsymbol{\epsilon}}) - \mathbf{V}(\boldsymbol{\epsilon}^0,\boldsymbol{\alpha}^0) : (\dot{\boldsymbol{\epsilon}}^0 + \Delta\dot{\boldsymbol{\epsilon}}) + \mathbf{C}(\boldsymbol{\epsilon}^0,\boldsymbol{\alpha}^0) .
$$

With the regular solution $(\boldsymbol{\epsilon}^0, \boldsymbol{\alpha}^0)$ assumed to be homogeneous, Eq.$(3.3)_2$ in the perturbed state, combining Eq.(5.1) and Eq.(5.6), $\operatorname{div}(\ddot{\boldsymbol{\sigma}}(\boldsymbol{\epsilon}^0, \boldsymbol{\alpha}^0, \dot{\boldsymbol{\epsilon}}^0 + \Delta\dot{\boldsymbol{\epsilon}}, \ddot{\boldsymbol{\epsilon}}^0 + \Delta\ddot{\boldsymbol{\epsilon}})) = 0$ in Ω becomes:

$$
(5.7)\qquad \mathbf{E}^w(\boldsymbol{\epsilon}^0,\boldsymbol{\alpha}^0) \bullet \Big(\nabla\cdot\big(\nabla_s(\Delta\ddot{\mathbf{u}})\big)\Big) - \mathbf{V}(\boldsymbol{\epsilon}^0,\boldsymbol{\alpha}^0) \bullet \Big(\nabla\cdot\big(\nabla_s(\Delta\dot{\mathbf{u}})\big)\Big) = 0 \qquad \text{in } \Omega ,
$$

where ∇ is the gradient operator. Substituting Eq.(5.5) into (5.7) one obtains:

$$
(5.8)\qquad \Big\{ \mathbf{n}\cdot\Big[\mathbf{E}^w - \frac{1}{\omega}\mathbf{V}\Big]\cdot\mathbf{n}\Big\}\cdot\delta\dot{\mathbf{u}} = 0 .
$$

A non-trivial solution of (5.8) is obtained as soon as

$$\exists\,(\mathbf{n},\boldsymbol{\epsilon}^0,\boldsymbol{\alpha}^0) \quad \text{and} \quad \exists\,\omega>0 \quad / \quad \det\left(\mathbf{n}\cdot\left\{\mathbf{E}^w-\frac{1}{\omega}\mathbf{V}\right\}\cdot\mathbf{n}\right)=0\,. \tag{5.9}$$

Owing to the fact that $\mathbf{n}$ and ω are solutions of Eq.(5.9) which is time-dependent, it should be noticed that only necessary conditions for an instability onset could be obtained.

5.3. Connection with the occurence of dynamic effects at localisation

If there exists $\omega<0$ satisfying Eq.(5.9), the regular solution of $(\ddot{P})$ is stable; if $\omega>0$ satisfying Eq.(5.9) exists, the regular solution is unstable, as existence of a growing-in-time perturbed solution is possible.

In Eq.(5.9), it is possible to find ω and $\mathbf{n}$ as solutions; but by conjecturing the localisation to be an intrinsically dynamic phenomenom, one retains only perturbations violating the quantitative criterion for quasi-staticity (4.1), i.e.:

$$\begin{aligned} &\rho|\Delta\ddot{\mathbf{u}}|\geq\gamma_{\text{crit}}\,,\\ &|\Delta\ddot{\mathbf{u}}|=\omega|\delta\dot{\mathbf{u}}|\,. \end{aligned} \tag{5.10}$$

The first perturbation fulfilling Eq.(5.10) verifies the condition:

$$\rho\,\omega|\delta\dot{\mathbf{u}}|=\gamma_{\text{crit}}\approx\kappa_{\text{crit}}|\dot{\mathbf{u}}^0|^{-1}\,. \tag{5.11}$$

Replacing ω from Eq.(5.11) in the instability criterion (5.9), one obtains as a criterion for a localised static/dynamic transition:

$$\boxed{\begin{aligned} &\exists?(\mathbf{n},\boldsymbol{\epsilon}^0,\boldsymbol{\alpha}^0,\delta\dot{\mathbf{u}}) \quad / \\ &\qquad \det\left(\mathbf{n}\cdot\left[\mathbf{E}^w(\boldsymbol{\epsilon}^0,\boldsymbol{\alpha}^0)-\frac{\rho|\delta\dot{\mathbf{u}}||\dot{\mathbf{u}}^0|}{\kappa_{\text{crit}}}\mathbf{V}(\boldsymbol{\epsilon}^0,\boldsymbol{\alpha}^0)\right]\cdot\mathbf{n}\right)=0 \end{aligned}} \tag{5.12}$$

6. Discussion

It appears that the criterion proposed depends on material characteristics (viscous parameters η, N, K) and current mechanical state via, $(\boldsymbol{\epsilon}^0,\boldsymbol{\alpha}^0)$ but on $\rho, |\delta\dot{\mathbf{u}}|, |\dot{\mathbf{u}}^0|$, κ_{crit} as well.

- The κ_{crit}-dependence and the ρ-dependence reflect a material feature and a loading rate feature (via $\dot{\mathbf{u}}^0$). This rate-dependence would then exclude a material classification based only on their behaviour at the onset of the localisation: only a classification based on the material and the loading rate would be relevant. This remark follows the conclusions of BATAILLE and KESTIN [3].

• The $|\delta\dot{\mathbf{u}}|$-dependence could also be connected to the loading rate conditions and suggests that the material will not localise if a too small perturbation is introduced compared to the loading displacement rate $\dot{\mathbf{u}}^0$.

A new approach of modelling the physical phenomenon of localisation of deformation is thus presented. It is viewed as a local transition from a (quasi)-static behaviour to a dynamic one. The boundary-value problem is to be treated, at least locally, as dynamic after the onset of localisation.

Apart from this new definition, a localisation criterion is proposed, function of parameters usually not encountered: the amplitude of the perturbation, the density of the material, the loading rate displacement and a critical "pseudo-kinetic" energy. This approach, insofar as a critical kinetic power could be experimentally measured, also permits to remove the difficulty of determining the rate of growth of the perturbation.

Considering the dynamic boundary-value problem for the rate-dependent material linked to the post-localisation behaviour, some difficulties stemming from the rate-independent models may arise. These are a null width of the localisation band and a loss of objectivity of the Finite Elements response. It is known that for rate-dependent materials in a dynamic context LOREST and PREVOST [15], a band width naturally appears. Also the mesh sensitivity is suppressed [15] whereas in the quasi-static context it may be only reduced as observed by FOREST and CAILLETAUD [13].

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