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Transverse Stokes flow through regular arrays of cylinders

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THE PAPER PRESENTS RESULTS of calculations of the Stokes flow trough square and triangular arrays of parallel cylinders. The results were derived using the method of singular force distributions [1]. The results are obtained to $O(f^7)$ for a square and to $O(f^8)$ for a triangular array, where f is the reduced volume fraction. They are compared with the results of previous authors. The new expressions for the drag force and the permeability coefficient valid in the whole range of f are also derived, using Padé approximation technique.

1. Introduction

THE STOKES FLOW through a square array of parallel cylinders was recently the subject of the paper [1] where the problem was studied by means the method of singular force distribution proposed by HASIMOTO [2]. In the present paper we extend the study to the array of triangular geometry and derive new results of higher order approximation than in the previous papers, for both the square and triangular arrays.

We treat here the array of cylinders as a porous medium where mean velocity U and mean pressure gradient ∇p are related by the linear Darcy's law [3]

$$U = -K \cdot \tau_0 \cdot \frac{l^2}{\mu} \cdot \nabla p \,. \tag{1.1}$$

Here μ denotes dynamic viscosity, τ_0 is a non-dimensional cross-sectional area of a unit cell and l is the distance between axes of cylinders within the unit cell. The symbol K denotes a non-dimensional permeability coefficient which may be presented by the following general expression:

$$K(a) = \frac{1}{8\pi} \cdot \left[\ln 1/a^2 - C_0 + 2\beta(a)\right] \,, \tag{1.2}$$

where

$$\beta(a) = \sum_{i=1}^{N} C_i \cdot a^{2i} \,, \tag{1.3}$$

a being the cylinder radius non-dimensionalized with the distance l and C_i are coefficients of expansion. The radius a and the volume fraction of cylinders φ are related as follows

$$\varphi = \pi a^2/\tau_0 , \tag{1.4}$$

where τ_0 is a non-dimensional cross-sectional area of a unit cell.

HASIMOTO [2] obtained the result (1.3) to $N = 1$ for square arrays. The method of Hasimoto was then developed by SANGANI and ACRIVOS [4] who got expansions to $N = 3$ for the both square and triangular arrays. DRUMMOND and TAHIR [5] made calculations applying the method of singularities using a different analytical technique than that of Hasimoto. They obtained expansions (1.3) to $N = 4$ for square arrays and to $N = 6$ for triangular arrays. All these results have been collected in a monograph by ADLER [3]. Recently the present author the extended calculations to $N = 5$ for square arrays [1].

The theoretical research of the Stokes flow through spatially periodic systems of particles found several technological applications. The approach of HASIMOTO [2] inspired DAVIS and JAMES [6] to calculate the Stokes flow through a periodic array of thin rings. These results appeared to be important in mathematical modelling of paper formation processes. Another application of the Stokes flow calculations for periodic systems in industrial practice is mathematical modelling of hydrodynamic processes in man-made fiber formation. This problem was first studied by SZANIAWSKI and ZACHARA [7] who treated a bundle of fibers as a porous medium using the transverse and parallel permeability coefficients according to HAPPEL and BRENNER [8]. This approach was then used to study fiber formation in various conditions of manufacturing (see ZACHARA [9]). Recently this method was modified and applied by OCKENDON and TERRILL [10] to mathematical modelling of various aspects of wet-spinning processes. They used permeability coefficients according to DRUMMOND and TAHIR [5].

The present paper follows the approach of HASIMOTO [2] and SANGANI and ACRIVOS [4]. This approach was modified by the present author [1] where a new functional basis has been derived. It allowed to obtain explicit expressions for matrix elements of the system of linear algebraic equations. The solution of the truncated system could be then derived using the symbolic computations of *Mathematica* [11] and the analytical expression for the permeability coefficient (1.2) was obtained. This expression has been expanded to $N = 7$ for square and to $N = 8$ for a triangular arrays. It covered a wide range of the volume fraction φ with the exception of φ close to $\varphi_{\max}$, corresponding to densely packed cylinders. However, making use of the asymptotic solution of KELLER [12], valid in the range $\varphi \to \varphi_{\max}$, and the solutions obtained in this paper, we have derived the new expressions for the permeability coefficient K valid in the full range of the volume fraction. These expressions have been obtained using the multipoint Padé approximants technique [13]. The results of calculations derived in various orders of approximation have been compared with the numerical results of SANGANI and ACRIVOS [14].

2. Calculation of the permeability coefficient

The calculation method, based on the approach of HASIMOTO [2] and SANGANI and ACRIVOS [4], has been derived in the previous paper [1]. We do not repeat here details of the analytical procedure which can be found in [1] but instead, we present new results of calculations which enable higher order approximation of the permeability coefficient $K(a)$ (1.2) for square and triangular arrays of cylinders. The quantity β which appears in (1.2) is defined as

$$\beta = \frac{1}{Y_1}, \tag{2.1}$$

with Y_1 being the component of the vector $\mathbf{Y}$ which fulfils the algebraic system of equations

$$W_{ij}Y_j = \delta_{i1}. \tag{2.2}$$

The matrix elements W_{ij} are presented by the expressions (2.4)–(2.6). These expressions are different for odd and even subscripts j corresponding to the matrix columns. The elements of the first column W_{i1} are here excluded to simplify the odd columns expression. Thus the matrix elements may be presented as follows:

– for $j = 1$:

$$\begin{aligned} W_{i1} = \frac{\pi a^2}{2\tau_o} \cdot \delta_{i1} + \frac{1}{2} \cdot \left(1 - \frac{\pi a^2}{2\tau_o}\right)(\delta_{i3} - \delta_{i2}) + A_{i-1} \cdot a^{i-1} \\ -(i+1) \cdot \left[\frac{1}{2} A_{i+1} \cdot a^2 + i \cdot B_{i+1}\right] \cdot a^{i-1} \\ +(i+2) \cdot \left[\frac{1}{2} A_{i+2} \cdot a^2 + (i+1) \cdot B_{i+2}\right] \cdot a^{i}, \end{aligned} \tag{2.3}$$

– for other odd subscripts $j = 3, 5, 7, \ldots$:

$$\begin{aligned} W_{ij} = \frac{\pi}{2\tau_o} \cdot \delta_{i1}\delta_{j3} + \frac{(j-1)!}{2a^{j-1}} \cdot (\delta_{i,j+2} - \delta_{i,j+1} + \delta_{i,,j-1}) \\ -\frac{(j-1)(j-2)!}{2 \cdot a^{j-1}} \cdot \delta_{ij} - \frac{(j-3)(i+j-2)!}{2 \cdot (i-1)!} A_{i+j-2} \cdot a^{i-1} \end{aligned} \tag{2.4}$$

$$-\frac{(j-1)(i+j-1)!}{2i!}\cdot A_{i+j-1}\cdot a^{i}-\frac{(i+j)!}{i!}\cdot\left[\frac{1}{2}\cdot A_{i+j}\cdot a^{2}+i\cdot B_{i+j}\right]\cdot a^{i-1}$$
$$-\frac{(i+j+1)!}{(i+1)!}\cdot\left[\frac{1}{2}\cdot A_{i+j+1}\cdot a^{2}+(i+1)\cdot B_{i+j+1}\right]\cdot a^{i},$$

– for even subscripts $j=2,4,6,\ldots$:

$$W_{ij}=-\frac{2\pi}{\tau_0}\cdot\delta_{i1}\delta_{j2}+\frac{2\cdot(j-1)!}{a^{j}}\cdot(\delta_{i,j+1}-\delta_{ij})$$
$$+\frac{2\cdot(i+j-1)!}{(i-1)!}\cdot A_{i+j-1}\cdot a^{i-1}+\frac{2\cdot(i+1)!}{i!}\cdot A_{i+j}\cdot a^{i}. \tag{2.5}$$

The system (2.2) has been truncated to the size 9×9 and solved using the symbolic computations *Mathematica*. As a result, the expansions of $\beta(a)$ (1.3) were obtained with $N=7$ for a square array and $N=8$ for a triangular array. The calculated coefficients C_i from (1.3) are as follows:

– for a square array:

$$C_1=\pi/\tau_0,\quad C_2=-\left[(\pi/\tau_0)^2/4+576\cdot B_4^2\right],\quad C_3=-768\cdot A_4B_4,$$

$$C_4=288\cdot\pi/\tau_0\cdot A_4B_4-260\cdot A_4^2,\qquad C_5=192\cdot\pi/\tau_0\cdot A_4^2,$$

$$C_6=-12\cdot(3\cdot(\pi/\tau_0)^2\cdot A_4^2+6912\cdot A_4^2B_4^2-26880\cdot A_4B_4B_8+3136\cdot B_8^2),$$

$$C_7=768\cdot(-288\cdot A_4^3B_4+224\cdot A_4B_4A_8+560\cdot A_4^2B_8-105\cdot A_8B_8),$$

– for a triangular array:

$$C_1=\pi/\tau_0,\qquad C_2=-(\pi/\tau_0)^2/4,\qquad C_3=0,$$

$$C_4=-7200\cdot B_6^2,\qquad C_5=-17280\cdot A_6B_6,$$

$$C_6=6A_6\cdot(600\cdot\pi/\tau_0\cdot B_6-433\cdot A_6),$$

$$C_7=2160\cdot\pi/\tau_o\cdot A_6^2,\qquad C_8=-450\cdot(\pi/\tau_o)^2\cdot A_6^2.$$

The coefficients C_1, C_2, C_3 for a square and a triangular array are fully equivalent to those obtained by SANGANI and ACRIVOS [4]. The coefficient C_4 for a square array and the coefficients C_4, C_5, C_6 for a triangular array were derived by DRUMMOND and TAHIR [5] with the calculation technique different from ours hence the corresponding expressions cannot be directly compared. Instead we can compare their numerical values. To this aim we take the numerical values of A_4, B_4, A_8, B_8, C_0 evaluated for a square array [1] and calculate the numerical values of A_6, B_6, C_0 for a triangular array using procedures given by SANGANI and ACRIVOS [4] or ZACHARA [1].

Hence for a square array we have:

$$A_4 = 7.878030005 \cdot 10^{-1}, \qquad B_4 = -1.044856181 \cdot 10^{-1},$$
$$A_8 = 5.319716294 \cdot 10^{-1}, \qquad B_8 = -4.031710210 \cdot 10^{-2},$$
$$C_0 = 2.621065852,$$

and for a triangular array:

$$A_6 = 9.771719489 \cdot 10^{-1}, \qquad B_6 = -9.428004796 \cdot 10^{-2},$$
$$C_0 = 2.786075894.$$

In the case of a square array $\tau_0 = 1$, and of a triangular array $\tau_0 = \sqrt{3}/2$.

Table 1. Expansion coefficients in the expression (1.2) **for a square and a triangular array.**

i	C_i (square)	C_i (triangle)
0	1.310532926	1.393037947
1	π	3.627598728
2	-8.755733869	-3.289868134
3	$6.321721609 \cdot 10^1$	0
4	$-2.358407557 \cdot 10^2$	$-6.399883760 \cdot 10^1$
5	$3.743573485 \cdot 10^2$	$7.959843494 \cdot 10^2$
6	$2.267883043 \cdot 10^2$	$-3.683869238 \cdot 10^3$
7	$-2.632730210 \cdot 10^3$	$7.481952988 \cdot 10^3$
8		$-5.654483988 \cdot 10^3$

Inserting all these data to the expressions for C_i we obtain their numerical values which are collected in Table 1. The data of DRUMMOND and TAHIR [5] correspond to our data for $i = 0 \div 4$ (square array) and for $i = 0 \div 6$ (triangular array). The literature data for higher i are not known to the author. If we compare the results collected in Table 1 with the results of DRUMMOND and TAHIR we can see that they are equal at least to eight decimal places. The agreement is then excellent although these results were derived with the use of two different techniques.

It is convenient to present the permeability coefficient or the drag force as a function of a reduced volume fraction f which varies from 0 to 1

$$f = \varphi/\varphi_{\max},$$

where φ is the volume fraction of cylinders and $\varphi_{\max}$ is the limiting volume fraction corresponding to the case of touching cylinders. In the case of a square array $\varphi_{\max} = \pi/4$, and in the case of triangular array $\varphi_{\max} = \pi\sqrt{3}/6$. It can be easily shown that the variable f is related to the radius of a cylinder a for both the square and triagular array as

$$f = 4a^2. \tag{2.6}$$

Substituting (2.6) to (1.2) we may present the coefficient K in the form

$$K(f) = \frac{1}{8\pi}\left[\ln 1/f + \sum_{i=0}^{N} T_i \cdot f^i\right], \tag{2.7}$$

where the coefficients T_i are related to the coefficients C_i as follows:

$$T_0 = \ln 4 - C_0, \tag{2.8}$$

$$T_i = \frac{C_i}{4^i}, \qquad \text{for } i > 0. \tag{2.9}$$

We obtain the new expressions for the permeability coefficient K (2.7). They are:

– for a square array

$$K(f) = \frac{1}{8\pi}\Big[\ln 1/f - 1.234771491 + \frac{\pi}{2}\cdot f - 1.094466733\cdot f^2 + 1.975538003\cdot f^3 - 1.842505904\cdot f^4 + 0.7311666962\cdot f^5 + 0.1107364767\cdot f^6 - 0.3213781994\cdot f^7\Big], \tag{2.10}$$

– for a triangular array

$$K(f) = \frac{1}{8\pi}\Big[\ln 1/f - 1.399781533 + 1.813799364\cdot f - 0.4112335167\cdot f^2 - 0.4999909187\cdot f^4 + 1.554656932\cdot f^5 - 1.798764276\cdot f^6 + 0.9133243394\cdot f^7 - 0.1725611569\cdot f^8\Big]. \tag{2.11}$$

The results obtained are presented in Fig. 1 for a square array and in Fig. 2 for a triangular array, in a form $F(f) = K^{-1}$, where F is a drag force per unit

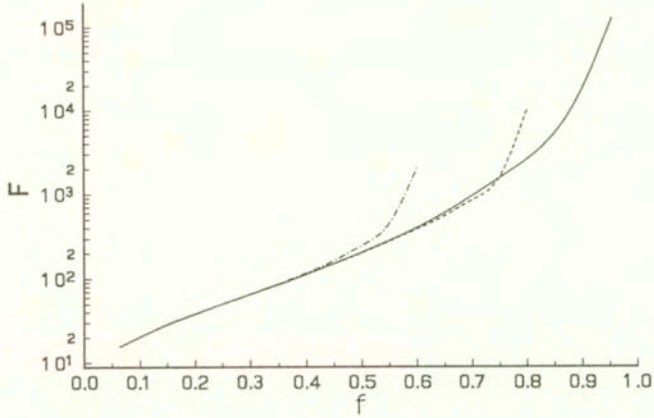

FIG. 1. Drag force $F(f) = K^{-1}(f)$ for a square array calculated from (2.10) at various truncations: $- \cdot - \cdot -$ $O(f^4)$ [5]; $- - -$ $O(f^7)$ present results; ——— reference data [14].

length of the cylinder (see [1]). They are compared with the numerical results of SANGANI and ACRIVOS [14] derived in a wide range of the volume fraction from $f = 0$ to f close to 1. It is seen that the present results evidently allowed to increase the accuracy of calculations. Thus the results are in agreement with the reference data [14] in a wide range of the volume fraction f up to about $f = 0.8$.

In the range of f close to 1 the coefficient $K(f)$ is well approximated by the asymptotic relation of KELLER [12] obtained by means the theory of lubrication. For a square array this relation reads

$$K(f) = \frac{2\sqrt{2}}{9\pi} \cdot \left(1 - f^{1/2}\right)^{5/2}, \tag{2.12}$$

and for a triangular array

$$K(f) = \frac{4\sqrt{2}}{27\pi} \cdot \left(1 - f^{1/2}\right)^{5/2}. \tag{2.13}$$

Thus we have two pairs of relations (2.10), (2.12) and (2.11), (2.13) for a square and triangular arrays, respectively. It would be however convenient to have only one expression $K(f)$ for each geometry, which could cover the complete range of f with a good accuracy. To this aim we use the technique of multipoint Padé approximants [13]. This type of approximation often appears to be more convenient than polynomials since Padé approximants need in general much less terms to achieve a good accuracy. We seek the expression for the permeability coefficient in the following form:

$$K(f) = \frac{1}{8\pi}\left[\ln 1/f - T_0 + [L/M]\right], \tag{2.14}$$

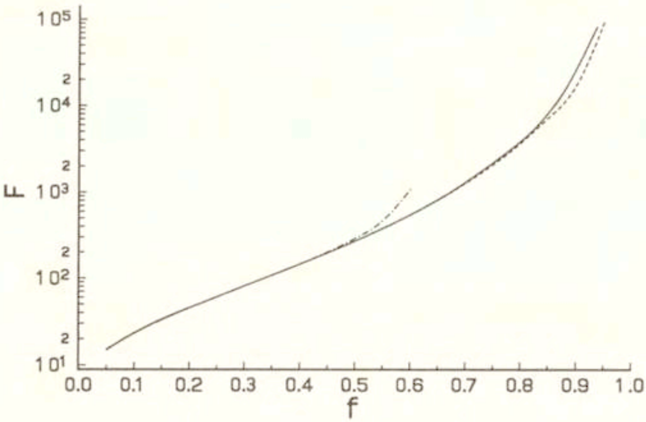

FIG. 2. Drag force $F(f) = K^{-1}(f)$ for a triangular array calculated from (2.11) at various truncations.: $- \cdot - \cdot -$ $O(f^6)$ [5]; $- - -$ $O(f^8)$ present results; ——— reference data [14].

where

$$(2.15) \qquad [L/M] = \frac{\sum\limits_{i=1}^{L} a_i \cdot f^i}{1 + \sum\limits_{i=1}^{M} b_i \cdot f^i},$$

is a Padé approximant. The coefficients a_i, b_i can be evaluated so that the expression (2.14) could take the values determined by (2.10) and (2.12) or by (2.11) and (2.13) for several selected values of f. We chose $L = 3$ and $M = 2$ and evaluated coefficients a_i, b_i which are presented in Table 2.

Table 2. Coefficients of the Padé approximant (2.15) for a square and a triangular array.

	Square array	Triangular array
a_1	1.556322044	1.823817915
a_2	4.462879026	−1.723614004
a_3	−3.117657159	0.174433692
b_1	3.689159025	−0.681877655
b_2	−2.339295968	−0.121922012

The coefficient $K(f)$ (2.14) has been calculated and the results are presented in Fig. 3 for a square, and in Fig. 4 – for a triangular array. The results obtained from the power expansions (2.10), (2.11) and the Keller solutions (2.12), (2.13) are also included. The Padé approximants technique was also applied by DRUMMOND

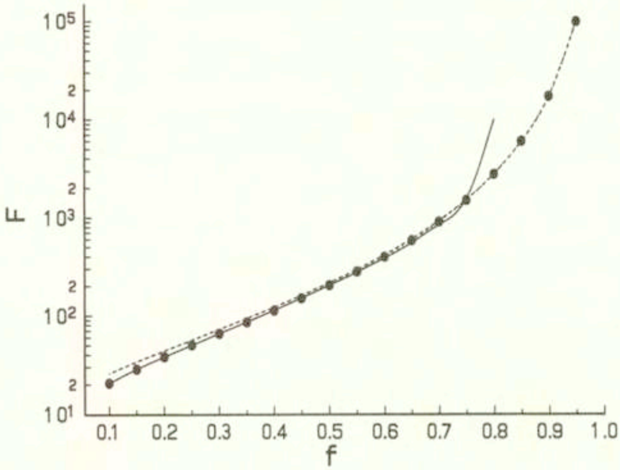

FIG. 3. Drag force $F(f) = K^{-1}(f)$ for a square array; ——— $O(f^7)$, present results; – – – Keller approximation calculated from (2.12); • • • Padé approximant calculated from (2.14) and Table 2.

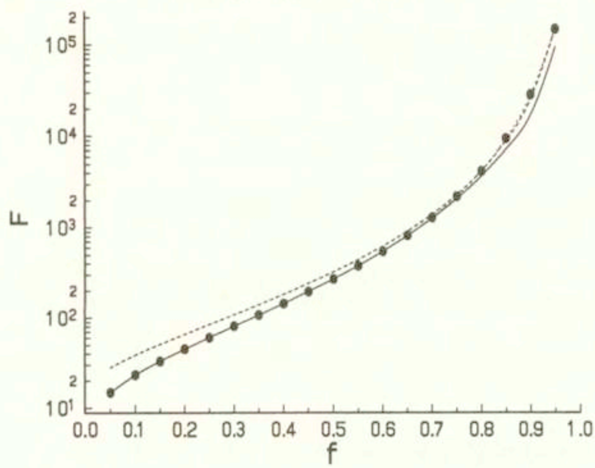

FIG. 4. Drag force $F(f) = K^{-1}(f)$ for a triangular array; ——— $O(f^8)$, present results; – – – Keller approximation calculated from (2.13); • • • Padé approximant calculated from (2.14) and Table 2.

and TAHIR [5], however their expressions in the Padé version were merely some correction of the expressions with power series expansions and did not cover the complete range of the volume fraction.

3. Conclusions

The Stokes flow through a square and triangular arrays of parallel cylinders was studied using the method based on the approach of HASIMOTO [2]. This approach was modified and a new analytical technique was developed (see [1]). The solution of the governing equations has been derived with the aid of the functional basis which allowed to transform these equations into the system of linear algebraic equations with the matrix elements given in an explicit analytical form. It made possible to obtain the analytical expression for the permeability coefficient K or the drag force F using symbolic computations of *Mathematica* [11]. It is a meaningful advantage of the method since derivation of drag force expressions without computer assistance is very tedious even for quite a moderate order of approximation. The expressions for F derived in this paper are of a higher order of approximation than those previously obtained, i.e. to $O(f^7)$ for a square and to $O(f^8)$ for a triangular array.

It was of course possible to continue calculations and derive new expressions (2.7) of higher order. However these expressions, being more and more extended, would never cover the complete range of the volume fraction. For this reason the new expressions have been derived using the multipoint Padé approximants technique. These expressions, based on the results of the present paper and the asymptotic results of KELLER [12], allowed us to evaluate with a good accuracy the drag force F or, all the same, the permeability coefficient K in the full range of the reduced volume fraction f from 0 to 1.

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