Arch. Mech., 51, 1, pp. 87–104, Warszawa 1999

Friction relations for the Oseen hydrodynamic interactions of spheres at large separations

I. PIEŃKOWSKA

Polish Academy of Sciences
Institute of Fundamental Technological Research
Świętokrzyska 21, 00-049 Warszawa, Poland

We consider the hydrodynamic forces, exerted by the incompressible fluid, having a uniform velocity $\mathbf{U}$ at infinity, on a finite number of fixed rigid spheres. The convective inertia effects of the fluid are described by the Oseen equations. The hydrodynamic interactions between the spheres are treated as multiple scattering events of the perturbations of the uniform velocity $\mathbf{U}$. The hydrodynamic forces are considered under the assumption that the characteristic Reynolds number is small.

Notations

a	radius of the sphere
A	used to denote a certain estimate
B_1, B_2	coefficients, defined in the Appendix A
d_l^m	coefficients, introduced in the expression (3.1)
$(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$	Cartesian coordinate system
E_i	exponential integral
$\mathbf{f}_k(\mathbf{r}_k)$	forces induced on the surface of the k-th sphere
$\mathbf{f}_{k,lm}$	(l, m) component of the induced forces
$\mathbf{F}_k$	hydrodynamic forces, exerted by the fluid on the k-th sphere
F_4	hypergeometric series of two variables
${}_pF_q$	generalized hypergeometric function
$G_{p,q}^{m,n}\left(x \middle\vert {a_p \atop b_q}\right)$	Meijer's G-function
i	imaginary unit
$I_\wedge$	modified Bessel function of the first kind
j_l	spherical Bessel function
$\mathbf{k}(k, \chi, \eta)$	wave-vector (spherical polar coordinates), $\hat{\mathbf{k}} = \mathbf{k}/\lvert\mathbf{k}\rvert$
$\mathbf{K}_m$	vectors, defined by the relations (3.2)
K_λ	modified Bessel function of the second kind
N	number of rigid spheres
$p(\mathbf{r})$	pressure field of the fluid
$\mathbf{P}(\mathbf{r}_j)$	the stress tensor inside the volumes of the spheres
P_l^m	associated Legendre function of the first kind
$\mathbf{r}$	position vector, specified in a Cartesian coordinate system $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$

$\mathbf{R}_j$	position of the surface of j-th sphere
$\mathbf{R}_j^0$	position of the centre of j-th sphere
$\mathbf{r}_j = \mathbf{R} - \mathbf{R}_j^0, \mathbf{r}_j(a, \Omega_j)$	in spherical polar coordinates
$\mathbf{R}_{jk} = \mathbf{R}_k^0 - \mathbf{R}_j^0, \mathbf{R}_{jk}(R_{jk}, \Omega_{jk})$	in spherical polar coordinates
$\mathbf{R}_j$	position of the surface of j-th sphere
$\mathbf{R}_j^0$	position of the centre of j-th sphere
$\mathbf{r}_j = \mathbf{R} - \mathbf{R}_j^0, \mathbf{r}_j(a, \Omega_j)$	in spherical polar coordinates
$\dot{\mathbf{R}}_j$	velocity of the j-th sphere
$\mathbf{R}$	typical distance between the centres of two spheres
Re	Reynolds number, $\mathrm{Re} = a\lvert\mathbf{U}\rvert/\nu$
$\mathbf{T}(\mathbf{r}\text{-}\mathbf{r}')$	fundamental tensor, defined by (2.3)
$\mathbf{T}_{l_1m_1}^{l_2m_2}(\mathbf{O}_j)$	self-interaction tensor
$\tilde{\mathbf{T}}_{l_1m_1}^{l_2m_2}(\mathbf{O}_j)$	inverse self-interaction tensor
$\tilde{\mathbf{T}}_j, \tilde{\mathbf{T}}_j^1$	contributions to the inverse self-interaction tensor $\tilde{\mathbf{T}}_{l_1m_1}^{l_2m_2}(\mathbf{O}_j)$
$\mathbf{T}_{l_1m_1}^{l_2m_2}(\mathbf{R}_{jk})$	mutual interaction tensor
$\mathbf{T}_{l_1m_1,l_3m_3}^{l_2m_2}(\mathbf{R}_{jk})$	(l_3, m_3) component of $\mathbf{T}_{l_1m_1}^{l_2m_2}(\mathbf{R}_{jk})$
$\mathbf{T}_{jl} = \mathbf{T}_{00}^{00}(\mathbf{R}_{jl})$	
$\mathbf{U}$	uniform velocity of the fluid at infinity, $\hat{\mathbf{U}} = \mathbf{U}/\lvert\mathbf{U}\rvert$
$\mathbf{v}(\mathbf{r})$	velocity field of the fluid
$\mathbf{V}_j$	relative velocity of the j-th sphere with respect to the fluid
$\mathbf{V}_{j,lm}$	(l, m) component of the relative velocity $\mathbf{V}_j$
$x = (R\alpha)^2$	
Y_l^m	normalized spherical harmonics
z	quantity, defined by (4.1)
	Greek letters
$\alpha = \lvert\mathbf{U}\rvert/\nu$	
β_m	quantity, defined in the expressions (4.1)
Γ	gamma function
γ	Euler's constant
δ	Kronecker delta, Dirac delta function
η	meridional angle, appearing in $\mathbf{k}(k, \chi, \eta)$
θ_{jk}	polar angle, appearing in $\mathbf{R}_{jk}(R_{jk}, \theta_{jk}, \phi_{jk})$
λ, Λ	quantities, introduced in the formulae (3.1)
μ	dynamic viscosity
ν	kinematic viscosity
ξ_{jj}^{TV}	self-friction tensor
ξ_{jk}^{TV}	mutual-friction tensor
ξ	$= \cos(\hat{U}, \hat{k})$
ρ	quantity, introduced in the formulae (4.1)
$\tilde{\rho}$	density of the fluid
σ	$= a/R$
ϕ_{jk}	meridional angle, appearing in $\mathbf{R}_{jk}(R_{jk}, \theta_{jk}, \phi_{jk})$
χ	polar angle, appearing in $\mathbf{k}(k, \chi, \eta)$
Ω_j	angular variables, describing the vector $\mathbf{r}_j$
Ω_{jk}	angular variables, describing the vector $\mathbf{R}_{jk}$

1. Introduction

THE HYDRODYNAMIC FORCES exerted on small particles, interacting through the ambient fluid, are important, for example, in the small Reynolds number hydrodynamics and in the examination of properties of suspensions or of porous media. We will regard the hydrodynamics forces, exerted on a finite number N of rigid spheres, immersed in an incompressible, unbounded fluid, under the condition of the Reynolds number less than unity. The complexity of the forces, generated by the hydrodynamic interactions, is due to [1]:

(i) long range of the perturbation of the velocity of the fluid, due to the presence of a body,

(ii) non-additivity of the interactions.

The known solutions to the problems of the many-sphere hydrodynamic interactions concern mainly the stationary and the transient Stokes interactions [2]. However, even in the range of the Reynolds number less than unity, the hydrodynamic interactions can exhibit the strong dependence on the convective inertia of the fluid [3]. The previous investigations of the inertia effects have been reported, for example, in the author's earlier paper [4].

The present paper is devoted to the examination of the convective effects, appearing in the hydrodynamic interactions of N spheres, held fixed in the flow $\mathbf{U}$ uniform at infinity, of the viscous fluid. The range of the interactions considered is characterized by the following conditions:

$$\text{(1.1)} \quad \begin{aligned} &\text{(i)} && \mathrm{Re} = a\alpha < 1, \qquad \alpha = U/\nu, \qquad U = |\mathbf{U}|, \\ &\text{(ii)} && \sigma = a/R, \qquad \frac{1}{2} < \sigma < \infty, \\ &\text{(iii)} && R\alpha > 1, \end{aligned}$$

where Re is the Reynolds number, a – the radius of the sphere, ν – the kinematic viscosity of the fluid, R – the distance between the centres of the spheres. The hydrodynamic interactions are considered in the framework of the Oseen equations, giving an approximate description of the convective inertia of the fluid. In this paper we analyse the so-called friction relations, expressing the dependence of the hydrodynamic forces, exerted by the fluid on the spheres, on the spatial distribution of the spheres and on the Reynolds number Re. To this purpose we use the boundary integral approach, involving the Green tensor, depending on the uniform velocity of the fluid at infinity. The hydrodynamic interactions are treated as the multiple scattering events, conditioned by the configuration of the spheres and the inertia effects. They are characterized by the so-called hydrodynamic interactions tensors, describing the propagation of the perturbations by the fluid. The friction relations are expressed in terms of the series expansion with respect to the characteristic parameters σ and Re. The series represents different types of the admissible sequences of the hydrodynamic interactions between the spheres. The paper is a continuation of the earlier publication of the author [4], presenting the analysis of the friction relations at the regime:

$$
\text{(1.2)} \qquad \begin{aligned} &\text{(i)} \qquad a\alpha < 1, \qquad \frac{1}{2} < \sigma < \infty, \\ &\text{(ii)} \qquad R\alpha < 1. \end{aligned}
$$

Finally, as an illustration, we consider the particular case of the hydrodynamic interactions of three spheres, at large separations, fixed in line with the flow direction.

2. Formulation of the problem

The influence of the spheres on the unperturbed uniform flow is taken into account through the so-called induced forces $\mathbf{f}_k$, distributed on the surfaces of the spheres $\mathbf{R}_k, k = 1, ..., N$. We impose the no-slip boundary conditions. The basic relations of the induced forces $\mathbf{f}_k$ to the relative velocities of the spheres with respect to the fluid $\mathbf{V}_k$ can be presented in the form of the set of the boundary integrals over the surfaces of the spheres [4]:

$$
\text{(2.1)} \qquad \dot{\mathbf{R}}_j(\Omega_j) = \mathbf{U} + \int d\Omega_j' \mathbf{T}\left[\mathbf{R}_j(\Omega_j) - \mathbf{R}_j'(\Omega_j')\right] \cdot \mathbf{f}_j'(\Omega_j') + \sum_{k \neq j}^{N} \int d\Omega_k \mathbf{T}\left[\mathbf{R}_j(\Omega_j) - \mathbf{R}_k(\Omega_k)\right] \cdot \mathbf{f}_k(\Omega_k),
$$

$$
\text{(2.2)} \qquad \mathbf{V}_j(\Omega_j) = \dot{\mathbf{R}}_j(\Omega_j) - \mathbf{U}, \qquad j = 1, ..., N,
$$

where the velocities of the spheres $\dot{\mathbf{R}}_j$ are assumed to be equal to zero, the induced forces are treated as unknown quantities. The convolution character of the integrals expresses the non-local properties of the interactions.

For further convenience, the fundamental tensor $\mathbf{T}(\mathbf{R}_j - \mathbf{R}_k)$ is expressed in the form of the spatial Fourier transform (presented, for example, in the paper [12]):

$$
\text{(2.3)} \qquad \mathbf{T}(\mathbf{r} - \mathbf{r}') = \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{\exp(i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}'))}{\mu(k^2 + i\nu^{-1}\mathbf{U} \cdot \mathbf{k})} \left(1 - \hat{\mathbf{k}}\hat{\mathbf{k}}\right),
$$

where the wave-vector $\mathbf{k} = (k, \chi, \eta)$ in spherical polar coordinates, $\hat{\mathbf{k}} = \mathbf{k}/|\underline{\mathbf{k}}|$, μ is the dynamic viscosity of the fluid.

Using the technique of the expansions in terms of the normalized spherical harmonics Y_l^m [5], the set of convolution integrals (2.1) can be transformed to the following set of algebraic equations:

$$(2.4)\qquad \mathbf{V}_{j,l_1m_1} = \sum_{l_2m_2} \mathbf{T}^{l_2m_2}_{l_1m_1}(\mathbf{O}_j)\cdot \mathbf{f}_{j,l_2m_2} + \sum_{k\neq j}^{N}\sum_{l_2m_2} \mathbf{T}^{l_2m_2}_{l_1m_1}(\mathbf{R}_{jk})\cdot \mathbf{f}_{k,l_2m_2},$$

where $\mathbf{V}_{j,lm}$ and $\mathbf{f}_{k,lm}$ are, respectively, the (l,m) components of the relative velocities $\mathbf{V}_j$ and the induced forces $\mathbf{f}_k$. The first term on the right-hand side of (2.4), involving the so-called self-interaction tensors $\mathbf{T}^{l_2m_2}_{l_1m_1}(\mathbf{O}_j)$, accounts for the hydrodynamic interactions of a single sphere; the second term, involving the so-called mutual interaction tensors $\mathbf{T}^{l_2m_2}_{l_1m_1}(\mathbf{R}_{jk})$, represents interactions between different spheres through the surrounding fluid. It follows from (2.3) that the self-interaction tensors $\mathbf{T}^{l_2m_2}_{l_1m_1}(\mathbf{O}_j)$, corresponding to the relations between the (l_2m_2) component of the induced force and the (l_1m_1) component of the relative velocity on the surface of the j-th sphere, are equal to:

$$(2.5)\qquad \mathbf{T}^{l_2m_2}_{l_1m_1}(\mathbf{O}_j) = \frac{i^{l_1-l_2}}{2\pi^2\mu}\int d\mathbf{k}\frac{(1-\hat{\mathbf{k}}\hat{\mathbf{k}})}{k^2+i\nu^{-1}\mathbf{U}\cdot\mathbf{k}}Y^{-m_1}_{l_1}Y^{m_2}_{l_2}j_{l_1}(ak)j_{l_2}(ak),$$

where j_l are the spherical Bessel functions.

The mutual-interaction tensors $\mathbf{T}^{l_2m_2}_{l_1m_1}(\mathbf{R}_{jk})$, describing the relations between the respective quantities on the surfaces of the j-th and k-th sphere, are given by:

$$(2.6)\qquad \mathbf{T}^{l_2m_2}_{l_1m_1}(\mathbf{R}_{jk}) = \sum_{l_3m_3}^{\infty}\mathbf{T}^{l_2m_2}_{l_1m_1,l_3m_3}(R_{jk})Y^{m_3}_{l_3}(\Omega_{jk}),$$

where

$$(2.7)\qquad \mathbf{T}^{l_2m_2}_{l_1m_1,l_3m_3} = \frac{2}{\pi\mu}i^{l_1-l_2-l_3}\int d\mathbf{k}\frac{(1-\hat{\mathbf{k}}\hat{\mathbf{k}})}{k^2+i\nu^{-1}\mathbf{U}\cdot\mathbf{k}} \times Y^{-m_1}_{l_1}Y^{m_2}_{l_2}Y^{-m_3}_{l_3}j_{l_1}(ak)j_{l_2}(ak)j_{l_3}(R_{jk}k),$$

$\mathbf{R}_{jk} = \mathbf{R}^0_k - \mathbf{R}^0_j$ is the distance between the centres of relevant spheres, $\mathbf{R}_{jk}(R_{jk},\Omega_{jk})$ in spherical polar coordinates. To determine the friction relations, we present the (l_1m_1) components of the induced forces in terms of the (l_2m_2) components of the relative velocity $\mathbf{V}_{j,l_2m_2}$:

$$(2.8)\qquad \mathbf{f}_{j,l_1m_1} = \sum_{l_2m_2}\tilde{\mathbf{T}}^{l_2m_2}_{l_1m_1}(\mathbf{O}_j)\cdot\left[\mathbf{V}_{j,l_2m_2} - \sum_{k\neq j}\sum_{l_3m_3}\sum_{l_4m_4}\mathbf{T}^{l_3m_3}_{l_2m_2}(\mathbf{R}_{jk}) \cdot\tilde{\mathbf{T}}^{l_4m_4}_{l_3m_3}(\mathbf{O}_k)\cdot\mathbf{V}_{k,l_4m_4+\ldots}\right],$$

where the iterative series describes the multiple scattering character of the interactions between the spheres [9]. The inverse self-interaction tensors $\tilde{\mathbf{T}}^{l_2m_2}_{l_1m_1}(\mathbf{O}_j)$, appearing in the above series, fulfill the relations:

$$\sum_{l_3 m_3} \tilde{\mathbf{T}}^{l_3 m_3}_{l_1 m_1}(\mathbf{O}_j) \cdot \mathbf{T}^{l_2 m_2}_{l_3 m_3}(\mathbf{O}_j) = \mathbf{1}\delta_{m_1 m_2}\delta_{l_1 l_2}. \tag{2.9}$$

In what follows, the properties of the hydrodynamic interaction tensors will be analysed under the assumption of Re < 1. Next, the $(l_0 m_0)$ components of the induced forces, related to the respective vector forces, will be determined within the assumed approximation with respect to σ and Re [4].

3. Properties of the self-interaction tensors

As it follows from Eq.(2.5), the self-interaction tensors are introduced to investigate the consequences of inertia of the fluid in the hydrodynamic interactions. We will consider these consequences in the regime of small values of Re. To examine these effects, the tensors are presented in the following form (for the sake of simplicity, we assume $\mathbf{U}(0,0,U)$):

$$\mathbf{T}^{l_2 m_2}_{l_1 m_1}(\mathbf{O}_j) = \pm \frac{i^{l_1 - l_2 + |l_1 - l_2|}}{\mu a} d_{l_1}^{-m_1} d_{l_2}^{m_2} \int_0^1 d\xi \left[\frac{2}{3}\delta_{m_1, m_2} - \sqrt{\frac{2\pi}{15}} \cdot \sum_{m_6} \mathbf{K}_{m_6} \delta_{m_1, m_2 + m_6} d_2^{m_6} P_2^{m_6}(\xi) \right] P_{l_1}^{-m_1}(\xi) P_{l_2}^{m_2}(\xi) I_\Lambda(\operatorname{Re}\xi) K_\lambda(\operatorname{Re}\xi), \tag{3.1}$$

where I_Λ, K_λ denote the modified Bessel function, P_l^m are the associated Legendre functions [6],

$$\Lambda = \max(l_1 + 1/2, l_2 + 1/2), \qquad \lambda = \min(l_1 + 1/2, l_2 + 1/2),$$

$$\hat{U} = \mathbf{U}/U, \qquad \xi = \cos(\hat{\mathbf{U}}, \hat{\mathbf{k}}),$$

$$d_l^m = (-1)^{(m - |m|)/2} \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}.$$

The signs $\{\pm\}$ refer to the cases $l_1 + l_2 = 2n$ and $l_1 + l_2 = 2n + 1$, respectively. The second order tensors $\mathbf{K}_m$ are cited after [5]:

$$\begin{aligned} \mathbf{K}_0 &= \sqrt{\frac{2}{3}}(-\mathbf{e}_x\mathbf{e}_x - \mathbf{e}_y\mathbf{e}_y + 2\mathbf{e}_z\mathbf{e}_z), \\ \mathbf{K}_{\pm 1} &= \mathbf{e}_x\mathbf{e}_z + \mathbf{e}_z\mathbf{e}_x \mp i\mathbf{e}_y\mathbf{e}_z \mp i\mathbf{e}_z\mathbf{e}_y, \\ \mathbf{K}_{\pm 2} &= \mathbf{e}_x\mathbf{e}_x - \mathbf{e}_y\mathbf{e}_y \mp i\mathbf{e}_x\mathbf{e}_y \mp i\mathbf{e}_y\mathbf{e}_x. \end{aligned} \tag{3.2}$$

Replacing the Bessel functions by their small argument asymptotic approximations, we obtain the Stokes self-interaction tensors under the condition $l_1 = l_2$:

$$\text{(3.3)} \qquad \mathbf{T}^{l_1 m_2}_{l_1 m_1}(\mathbf{O}_j) = \frac{1}{6\pi\mu a}\left[\delta_{m_1,m_2}\frac{1}{(2l_1+1)} - \sqrt{\frac{3}{8}}\begin{pmatrix} 2 & l_1 & l_1 \\ 0 & 0 & 0 \end{pmatrix}\right.$$

$$\left.\cdot \sum_{m_6} \delta_{m_6+m_2,m_1}\begin{pmatrix} 2 & l_1 & l_1 \\ m_6 & -m_1 & m_2 \end{pmatrix}\cdot(-1)^{(|m_1|+|m_6|+|m_1-m_6|)/2}\mathbf{K}_{m_6}\right],$$

where $\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ are the Wigner 3-j symbols [6]. The Stokes self-interaction tensors have been introduced, for the first time, byYOSHIZAKI and YAMAKAWA [5], in their paper devoted to an application of the modified Oseen tensor to rigid polymers. The tensors (3.3), being diagonal with respect to l_i, lead to the limitations of the admissible sequences of the hydrodynamic interactions (comp. Eq. (2.8)).

The case of low but nonzero Reynolds number is described by the contributions to the self-interaction tensors, being of the order of 0(Re). Firstly, the respective leading order contributions are obtained from Eq. (3.1), under the condition $|l_1 - l_2| = 1$. It implies that the 0(Re) tensors are off-diagonal with respect to l_i. They can be presented in the following form:

(i) the case $l_1 - l_2 = -1$;

$$\text{(3.4)} \qquad \mathbf{T}^{l_1+1 m_2}_{l_1 m_1} = -\frac{\mathrm{Re}(-1)^{(-m_1-|m_1|+m_2-|m_2|)/2}}{6\pi\mu a\sqrt{(2l_1+1)(2l_1+3)}}\left\{\delta_{m_1,m_2}\begin{pmatrix} l_1 & l_1+1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right.$$

$$\begin{pmatrix} l_1 & l_1+1 & 1 \\ -m_1 & m_2 & 0 \end{pmatrix} - \frac{\sqrt{3}}{10\sqrt{2}}\sum_{m_6}\mathbf{K}_{m_6}\delta_{m_1,m_2+m_6}(-1)^{(m_6-|m_6|)/2}$$

$$\cdot\left[\sqrt{9-m_6^2}\begin{pmatrix} l_1 & l_1+1 & 3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l_1 & l_1+1 & 3 \\ -m_1 & m_2 & m_6 \end{pmatrix}\right.$$

$$\left.\left. + \sqrt{4-m_6^2}\cdot\begin{pmatrix} l_1 & l_1+1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l_1 & l_1+1 & 1 \\ -m_1 & m_2 & m_6 \end{pmatrix}\right]\right\},$$

(ii) the case $l_1 - l_2 = 1$;

the tensors $\mathbf{T}^{l_2 m_2}_{l_2+1 m_1}$ can be obtained from the expression (3.4) by interchanging the following indices:

$$\text{(3.5)} \qquad l_1 \to l_2 + 1; \qquad l_1 + 1 \to l_2.$$

Secondly, the 0(Re) contributions appear in the series expansion of the tensor $\mathbf{T}_{00}^{00}(\mathbf{O}_j)$:

$$\mathbf{T}_{00}^{00}(\mathbf{O}_j) = \frac{1}{6\pi\mu a}\left[\mathbf{1} - \frac{3}{16}\mathrm{Re}(\mathbf{3} - \hat{\mathbf{U}}\hat{\mathbf{U}})\right] + \cdots. \tag{3.6}$$

The respective inverse self-interaction tensor $\tilde{\mathbf{T}}_{00}^{00}(\mathbf{O}_j)$ reads:

$$\tilde{\mathbf{T}}_{00}^{00}(\mathbf{O}_j) = 6\pi\mu a\left[\mathbf{1} + \frac{3}{16}\mathrm{Re}(\mathbf{3} - \hat{\mathbf{U}}\hat{\mathbf{U}})\right] + \cdots. \tag{3.7}$$

The 0(Re) contributions to the self- and inverse self-interaction tensors give rise to the particular types of the hydrodynamic interactions, absent at the Stokes regime.

4. The properties of the mutual-interaction tensors

The mutual-interaction tensors, $\mathbf{T}_{l_1m_1}^{l_2m_2}$, given by (2.6) and (2.7), account for the dependence of the fluid propagated forces on the inertia of the fluid and on the spatial distribution of the spheres. To examine these dependences, the mutual-interaction tensors are presented in the following form:

$$\mathbf{T}_{l_1m_1,l_3m_3}^{l_2m_2} = \pm\frac{2(\pi)^{3/2} i^{l_1-l_2-l_3}}{a\mu\Gamma(l_1+3/2)\Gamma(l_2+3/2)} d_{l_1}^{-m_1} d_{l_2}^{m_2} d_{l_3}^{-m_3}\left(\frac{a}{R}\right)^{l_1+l_2+1}$$
$$\sum_m \beta_m(l_1,l_2)\cdot i^{|l_1+l_2+2m-l_3|}\int_0^1 d\xi\left[\frac{2}{3}\delta_{m_1+m_3,m_2} - \sqrt{\frac{2\pi}{15}}\right.$$
$$\left.\cdot\sum_{m_6}\mathbf{K}_{m_6} d_2^{m_6} P_2^{m_6}\delta_{m_6+m_2,m_1+m_3}\right]\cdot P_{l_1}^{-m_1} P_{l_2}^{m_2} P_{l_3}^{-m_3} I_z(R\alpha\xi)K_\rho(R\alpha\xi), \tag{4.1}$$

where the signs $\{\pm\}$ refer to the cases $l_1+l_2+l_3 = 2n$ and $l_1+l_2+l_3 = 2n+1$, respectively, $z = \max\,(l_1+l_2+2m+1/2, l_3+1/2)$, $\rho = \min\,(l_1+l_2+2m+1/2, l_3+1/2)$,

$$\beta_m = \frac{(l_1+l_2+2m+1/2)\Gamma(l_1+l_2+m+1/2)}{m!}F_4\left[-m, l_1+l_2+m+1/2;\right.$$
$$\left. l_1+3/2, l_2+3/2; \left(\frac{a}{R}\right)^2, \left(\frac{a}{R}\right)^2\right],$$

F_4 is the hypergeometric series of two variables.

Similarly, as in the Section 3, we assume $\hat{U}(0,0,1)$. We note that the effects of the inertia of the fluid are described in terms of the mutual-interaction parameter $R\alpha$ (analogous of the self-interaction parameter $\alpha\alpha$). Taking into account the properties of the associated Legendre functions P_l^m, expressed by the formulae (A1) and (A2), the mutual interaction tensors can be rewritten in the more convenient form:

$$\text{(4.2)}\qquad \mathbf{T}_{l_1m_1,l_3m_3}^{l_2m_2} = \pm\frac{2i^{(l_1-l_2-l_3)}(\pi)^{3/2}}{a\mu\Gamma(l_1+3/2)\Gamma(l_2+3/2)} d_{l_1}^{-m_1} d_{l_2}^{m_2} d_{l_3}^{-m_3}\left(\frac{a}{R}\right)^{l_1+l_2+1}$$
$$\sum_{m=0}\beta_m(l_1,l_2)i^{|l_1+l_2+2m-l_3|}\int\limits_0^1 d\xi\left[\frac{2}{3}\sum_{l_4m_4}\sum_{l_5}B_1P_{l_5}^0-\sqrt{\frac{2\pi}{15}}\sum_{m_6}\mathbf{K}_{m_6}d_2^{m_6}\right.$$
$$\left.\cdot\sum_{l_4m_4}\sum_{l_7m_7}\sum_{l_8}B_2P_{l_8}^0\right]\cdot I_z(R\alpha\xi)K_\rho(R\alpha\xi),$$

where the general formulae for the coefficients B_1 and B_2 are derived in the Appendix A. From the expression (4.2) we obtain, in particular, the uniform estimate:

$$\text{(4.3)}\qquad \mathbf{T}_{l_1m_1,l_3m_3}^{l_2m_2}\sim A(R\alpha)\left(\frac{a}{R}\right)^{l_1+l_2+1}.$$

The estimation of the type of $\left(\frac{a}{R}\right)^{l_1+l_2+1}$ is common to all hydrodynamic interactions at Re < 1, both the quasi-stationary and the time-dependent ones [10]. The integral with respect to ξ can be calculated with the help of the formula (2.24.6.1) from [7]:

$$\text{(4.4)}\qquad \int\limits_0^1 d\xi P_l^0(\xi)I_z(R\alpha\xi)K_\rho(R\alpha\xi)=\frac{1}{4\sqrt{\pi}}G_{4,6}^{2,4}$$
$$\cdot\left((R\alpha)^2\left|\begin{matrix}0,0,1/2,1/2\\ \frac{z+\rho}{2},\frac{z-\rho}{2},\frac{-z+\rho}{2},\frac{-z-\rho}{2},\frac{-1-l}{2},\frac{l}{2}\end{matrix}\right.\right),$$

where $G_{p,q}^{m,n}$ is Meijer's function. Meijer's function has been applied to obtain the compact form of the integral (4.4). We assume that the parameters of the $G_{p,q}^{m,n}\left(\begin{matrix}a_p\\ b_q\end{matrix}\right)$ functions fulfill the conditions [8]:

$$
\begin{aligned}
&a_k - b_j \qquad \text{is not a positive integer,}\\
&k = 1, ..., n; \qquad j = 1, ..., m.
\end{aligned}
\tag{4.5}
$$

The properties of the mutual-interaction tensors in the range of small values of $R\alpha$ are governed by the following estimate [8]:

$$
\begin{aligned}
&G^{m,n}_{p,q}\left((R\alpha)^2 \left| \begin{matrix} a_p \\ b_q \end{matrix} \right.\right) \simeq (R\alpha)^{2\gamma},\ p \leqslant q, \qquad \gamma = \min(b_h),\\
&h = 1, ..., m, \qquad R\alpha < 1.
\end{aligned}
\tag{4.6}
$$

Hence it can be readily verified that for $R\alpha \ll 1$, the Stokes hydrodynamic interaction tensors are recovered [4].

As an example, using the described procedure we have calculated the tensors $\mathbf{T}^{00}_{00,l_30}$. Due to the properties of the tensors $\mathbf{K}_m$ (3.2), the Cartesian components of the tensors $\mathbf{T}^{00}_{00,l_30}$ fulfill the relations:

$$
\begin{aligned}
&\mathbf{T}^{00}_{00,l_30}|_{xy} = \mathbf{T}^{00}_{00,l_30}|_{xz} = \mathbf{T}^{00}_{00,l_30}|_{yz} = 0,\\
&\mathbf{T}^{00}_{00,l_30}|_{xx} = \mathbf{T}^{00}_{00,l_30}|_{yy}\,.
\end{aligned}
\tag{4.7}
$$

The respective diagonal components are equal to:

$$
\begin{aligned}
\mathbf{T}^{00}_{00,l_30}|_{zz} = \pm\frac{\sqrt{2l_3+1}(i)^{-l_3}}{4\pi\mu R\sqrt{\pi}} \sum_{m=0}^{\infty} \beta_m(0,0) i^{|2m-l_3|}\\
\left[G^{2,4}_{4,6}\left((R\alpha)^2 \left| \begin{matrix} -1, 1/2, \quad 0, 1/2 \\ \frac{z+\rho}{2}, \frac{z-\rho}{2}, \frac{-z+\rho}{2}, \frac{-z-\rho}{2}, \frac{-1-l_3}{2}, \frac{l_3}{2} \end{matrix} \right.\right),\right.\\
\left.-G^{2,4}_{4,6}\left((R\alpha)^2 \left| \begin{matrix} 0, 1/2, \quad 0, 1/2 \\ \frac{z+\rho}{2}, \frac{z-\rho}{2}, \frac{-z+\rho}{2}, \frac{-z-\rho}{2}, \frac{-3-l_3}{2}, \frac{-2+l_3}{2} \end{matrix} \right.\right)\right],
\end{aligned}
\tag{4.8}
$$

$$
\begin{aligned}
\mathbf{T}^{00}_{00,l_30}\Big|_{\substack{xx\\yy}} = \pm\frac{\sqrt{2l_3+1}(i)^{-l_3}}{8\pi\sqrt{\pi}\mu R} \sum_{m=0}^{\infty} \beta_m(0,0) i^{|2m-l_3|}\\
\left[G^{2,4}_{4,6}\left((R\alpha)^2 \left| \begin{matrix} 0, 1/2, 0, 1/2 \\ \frac{z+\rho}{2}, \frac{z-\rho}{2}, \frac{-z+\rho}{2}, \frac{-z-\rho}{2}, \frac{-1-l_3}{2}, \frac{l_3}{2} \end{matrix} \right.\right)\right.
\end{aligned}
$$

$$\text{(4.8) [cont.]}\quad +G^{2,4}_{4,6}\left((R\alpha)^2\left|\begin{matrix}-1,-1/2,0,1/2\\ \frac{z+\rho}{2},\frac{z-\rho}{2},\frac{-z+\rho}{2},\frac{-z-\rho}{2},\frac{-3-l_3}{2},\frac{-2+l_3}{2}\end{matrix}\right.\right)\Bigg].$$

Some other properties of the mutual-interaction tensors are discussed in the Appendix B.

5. Friction relations at $R\alpha > 1$

From (2.8) it follows that the hydrodynamic forces $\mathbf{F}_j, j = 1, ..., N$, exerted by the fluid on the immersed spheres, can be presented in the following form [4]:

$$\mathbf{F}_j = \sum_{k=1}^{N} \xi^{TV}_{jk} \cdot \mathbf{U}, \tag{5.1}$$

where the second-rank friction tensors ξ^{TV}_{jk}, describing the hydrodynamic interactions between the spheres as the multiple scattering process, take into account the effects of the inertia of the fluid and of the geometrical distribution of the spheres. The explicit expressions for the self-friction tensors ξ^{TV}_{jj} and the mutual-friction tensors ξ^{TV}_{jk}, can be obtained through the rearrangement of the contributions to the multiple scattering series (2.8). The rearrangements lead to the following formulae for the self-friction tensors ξ^{TV}_{jj} (including the contributions up to the first order with respect to Re and the second order with respect to σ):

$$\begin{aligned}\xi^{TV}_{jj} = \tilde{\mathbf{T}}_j + \tilde{\mathbf{T}}^1_j + \sum_{l\neq j}\Big[&\tilde{\mathbf{T}}_j \cdot \mathbf{T}_{jl} \cdot \tilde{\mathbf{T}}_l \cdot \mathbf{T}_{lj} \cdot \tilde{\mathbf{T}}_j + \tilde{\mathbf{T}}^1_j \cdot \mathbf{T}_{jl} \cdot \tilde{\mathbf{T}}_l \cdot \mathbf{T}_{lj} \cdot \tilde{\mathbf{T}}_j \\ &+\tilde{\mathbf{T}}_j \cdot \mathbf{T}_{jl} \cdot \tilde{\mathbf{T}}^1_l \cdot \mathbf{T}_{lj} \cdot \tilde{\mathbf{T}}_j + \tilde{\mathbf{T}}_j \cdot \mathbf{T}_{jl} \cdot \tilde{\mathbf{T}}_l \cdot \mathbf{T}_{lj} \cdot \tilde{\mathbf{T}}^1_j\Big] + \ldots .\end{aligned} \tag{5.2}$$

The above contributions depend on the interaction of a single sphere and a pair of spheres. Hence, in contrast with the self-friction tensors, obtained under the assumption $R\alpha < 1$ [4], here the non-additivity of the interactions does not affect the forces considered.

The mutual-friction tensors read:

$$\begin{aligned}\xi^{TV}_{jk} = &-\tilde{\mathbf{T}}_j \cdot \mathbf{T}_{jk} \cdot \tilde{\mathbf{T}}_k - \tilde{\mathbf{T}}_j \cdot \tilde{\mathbf{T}}_{jk} \cdot \mathbf{T}^1_k - \tilde{\mathbf{T}}^1_j \cdot \mathbf{T}_{jk} \cdot \tilde{\mathbf{T}}_k \\ &-\sum_m \tilde{\mathbf{T}}^{1m}_{00}(O_j) \cdot \mathbf{T}^{00}_{1m}(\mathbf{R}_{jk}) \cdot \tilde{\mathbf{T}}_k - \tilde{\mathbf{T}}_j \cdot \sum_m \mathbf{T}^{1m}_{00}(\mathbf{R}_{jk}) \cdot \tilde{\mathbf{T}}^{00}_{1m}(O_k) \\ &+\sum_{l\neq k}\sum_{l\neq j}\Big[\tilde{\mathbf{T}}_j \cdot \mathbf{T}_{jl} \cdot \tilde{\mathbf{T}}_l \cdot \mathbf{T}_{lk} \cdot \tilde{\mathbf{T}}_k + \tilde{\mathbf{T}}^1_j \cdot \mathbf{T}_{jl} \cdot \tilde{\mathbf{T}}_l \cdot \mathbf{T}_{lk} \cdot \tilde{\mathbf{T}}_k + \tilde{\mathbf{T}}_j \\ &\qquad \mathbf{T}_{jl} \cdot \tilde{\mathbf{T}}^1_l \cdot \mathbf{T}_{lk} \cdot \tilde{\mathbf{T}}_k + \tilde{\mathbf{T}}_j \cdot \mathbf{T}_{jl} \cdot \tilde{\mathbf{T}}_l \cdot \mathbf{T}_{lk} \cdot \tilde{\mathbf{T}}^1_k\Big] + \ldots .\end{aligned} \tag{5.3}$$

They depend on the interactions of a pair and of three spheres. Hence, in comparison with the case of $R\alpha < 1$, we have here a smaller number of the spheres, participating in the hydrodynamic interactions. The non-additivity appears starting from the terms of the order of $0(\sigma^2)$.

From (5.2) and (5.3) it follows that in the range $R\alpha > 1$, the non-additivity effects are weaker than in the range $R\alpha < 1$. In the above relations we have introduced the following short-hand notation:

(i) for the self-interaction tensors,

$$\tilde{\mathbf{T}}^{00}_{00}(\mathbf{O}_j) = \tilde{\mathbf{T}}_j + \tilde{\mathbf{T}}^1_j + \cdots, \tag{5.4}$$

$$\tilde{\mathbf{T}}_j = 6\pi\mu a \mathbf{1}, \quad \tilde{\mathbf{T}}^1_j = 6\pi\mu a \left[\frac{3}{16}\mathrm{Re}(\mathbf{3} - \hat{\mathbf{U}}\hat{\mathbf{U}})\right], \tag{5.5}$$

$$\tilde{\mathbf{T}}^{1m}_{00}(\mathbf{O}_j) = 2\sqrt{3}\pi\mu a \mathrm{Re}\,\delta_{m0}\mathbf{1}; \qquad \tilde{\mathbf{I}}^{00}_{1m}(\mathbf{O}_j) = -\tilde{\mathbf{I}}^{1m}_{00}(\mathbf{O}_j) \tag{5.6}$$

(ii) for the mutual-interaction tensors,

$$\mathbf{T}_{jk} = \mathbf{T}^{00}_{00}(\mathbf{R}_{jk}) = \sum_{l_3=0}^{3}\sum_{m_3} \mathbf{T}^{00}_{00,l_3m_3}(R_{jk})Y^{m_3}_{l_3}(\Omega_{jk}), \tag{5.7}$$

$$\mathbf{T}^{00}_{10}(\mathbf{R}_{jk}) = \sum_{l_3=0}^{4}\sum_{m_3} \mathbf{T}^{00}_{10,l_3m_3}(R_{jk})Y^{m_3}_{l_3}(\Omega_{jk}). \tag{5.8}$$

In (5.7) and (5.8) the upper limits of the summations with respect to l_3 follow from the fact that we confine our attention to the hydrodynamic interactions at $\mathrm{Re} < 1$ (comp. App.B).

Here we use the friction tensors (5.2) and (5.3) to calculate the drag forces on the three spheres, rigidly held in the external uniform flow $\mathbf{U}(0, 0, U)$. The centre line of the spheres is parallel to the direction of the external flow. The relative distances between the centres of the spheres are specified in terms of the vectors $\mathbf{R}_{12}(R_{12}, \theta_{12} = 0°, \phi_{12} = 0°)$ and $\mathbf{R}_{13}(R_{13}, \theta_{13} = 0°, \phi_{13} = 0°)$.

The drag forces, obtained according to the expansions (5.2) and (5.3), read:

$$F_j\Big|_z = \sum_{k=1}^{3} \xi^{TV}_{jk}\Big|_{zz} U_z, \qquad j = 1, 2, 3, \tag{5.9}$$

where the respective components of the friction tensors in the approximation considered are given by:

(5.10)
$$\xi_{jj}^{TV}\,|_{zz} = 6\pi\mu a\left\{1+\frac{3}{8}\mathrm{Re}+\left[(6\pi\mu a)^2+6\pi\mu a\frac{9}{8}\mathrm{Re}\right]\sum_{l\neq j} T_{jl}\,|_{zz}T_{lj}|_{zz}+\cdots\right\},$$

(5.11)
$$\xi_{jk}^{TV}\Big|_{zz} = 6\pi\mu a\left\{-6\pi\mu a T_{jk}|_{zz}\left[1+\frac{3}{4}\mathrm{Re}\right]-4\sqrt{3}\pi\mu a\mathrm{Re}T_{10}^{00}\,|_{zz} + \left[(6\pi\mu a)^2+6\pi\mu a\frac{9}{8}\mathrm{Re}\right]\sum_{l\neq k}\sum_{l\neq j} T_{jl}\,|_{zz}T_{lk}|_{zz}+\cdots\right\}.$$

For this particular configuration we have:

(5.12)
$$\mathbf{T}_{jk} = \sum_{l_3=0}^{3}\mathbf{T}_{00,l_30}^{00}Y_{l_3}^{0}, \qquad \mathbf{T}_{10}^{00} = \sum_{l_3=0}^{4}\mathbf{T}_{10,l_30}^{00}Y_{l_3}^{0}.$$

From (4.1) it follows that

(5.13)
$$\mathbf{T}_{00,l_30}^{10}\Big|_{zz} = -\mathbf{T}_{10,l_30}^{00}\Big|_{zz}, \qquad l_3 = 0,1,2,3,4.$$

The relevant mutual interaction tensors are treated in more detail in the Appendix B. We recall that the friction tensors are obtained under the assumptions:

(5.14)
$$a/R < 1/2, \qquad \mathrm{Re} < 1, \qquad R\alpha > 1.$$

It is a direct consequence of (5.10) and (5.11) that the drag forces, exerted on the spheres under the condition of finite but small values of Re, are differentiated stronger than the relevant Stokes drag forces. The inertial contributions to the considered forces are generated both by the self-interactions and the mutual interactions of the spheres. The non-additivity of the interactions affects the mutual-friction tensors ξ_{jk}^{TV}.

According to the results which have been established in ref. [11], in the case of free-fall motion of micron-order particles, the Stokes motion is observed at Re below 0.05 and the Oseen range motion, respectively, at $0.05 < \mathrm{Re} < 0.5$. Hence the convective inertia effects considered here can, in particular cases, affect the hydrodynamic interactions at surprisingly low Re.

Appendix A. The calculations of the coefficients B_1 and B_2, entering the relation (4.2)

In view of the properties of the normalized surface spherical harmonics Y_l^m, described by the formula (4.6.5) from [6], the product $P_{l_1}^{-m_1}P_{l_2}^{m_2}P_{l_3}^{-m_3}\delta_{m_1+m_3,m_2}$ can be expressed in terms of the sum of the P_l^0 functions:

$$P_{l_1}^{-m_1} P_{l_2}^{m_2} P_{l_3}^{-m_3} \delta_{m_1+m_3,m_2} = \sum_{l_4 m_4} \sum_{l_5} B_1 P_{l_5}^0, \tag{A.1}$$

where

$$B_1 = \frac{(-1)^{m_3}(2l_4+1)(2l_5+1)}{4\pi} \sqrt{\frac{(l_1-m_1)!(l_2+m_2)!(l_3-m_3)!}{(l_1+m_1)!(l_2-m_2)!(l_3+m_3)!}}$$

$$\begin{pmatrix} l_1 & l_2 & l_4 \\ -m_1 & m_2 & -m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_3 & l_4 & l_5 \\ -m_3 & m_2-m_1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} l_3 & l_4 & l_5 \\ 0 & 0 & 0 \end{pmatrix} \delta_{m_1+m_3,m_2}, \qquad \begin{array}{l} |l_1-l_2| \leqslant l_4 \leqslant l_1+l_2, \\ |l_3-l_4| \leqslant l_5 \leqslant l_3+l_4. \end{array}$$

Similarly, the product $P_2^{m_6} P_{l_1}^{-m_1} P_{l_2}^{m_2} P_{l_3}^{-m_3} \delta_{m_6+m_2,m_1+m_3}$ can be presented in the following form:

$$P_{l_1}^{-m_1} P_{l_2}^{m_2} P_{l_3}^{-m_3} P_2^{m_6} \delta_{m_1+m_3,m_2+m_6} = \sum_{l_4 m_4} \sum_{l_7 m_7} \sum_{l_8} B_2 P_{l_8}^0, \tag{A.2}$$

where

$$B_2 = (2l_4+1)(2l_7+1)(2l_8+1) \sqrt{\frac{(l_1-m_1)!(l_2+m_2)!(l_3-m_3)!(2+m_6)!}{(l_1+m_1)!(l_2-m_2)!(l_3+m_3)!(2-m_6)!}}$$

$$\begin{pmatrix} l_1 & l_2 & l_4 \\ -m_1 & m_2 & m_4 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_3 & 2 & l_7 \\ -m_3 & m_6 & m_7 \end{pmatrix} \begin{pmatrix} l_3 & 2 & l_7 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} l_4 & l_7 & l_8 \\ -m_4 & -m_7 & 0 \end{pmatrix} \begin{pmatrix} l_4 & l_7 & l_8 \\ 0 & 0 & 0 \end{pmatrix} \delta_{m_1+m_3,m_2+m_6},$$

$$\begin{array}{l} |l_1-l_2| \leqslant l_4 \leqslant l_1+l_2, \\ |l_3-2| \leqslant l_7 \leqslant l_3+2, \\ |l_4-l_7| \leqslant l_8 \leqslant l_4+l_7. \end{array}$$

The above expressions yield the information needed to calculate the mutual-interaction tensors $\mathbf{T}_{l_1m_1,l_3m_3}^{l_2m_2}$ for the arbitrary range of $R\alpha$.

Appendix B. Some properties of the mutual-interaction tensors

In what follows we will calculate a few examples of the tensors $\mathbf{T}_{l_1m_1,l_3m_3}^{l_2m_2}$, which in the range $R\alpha < 1$ describe the Stokes and the 0(Re) hydrodynamic interactions. As it has been discussed in the paper [4], the first group of the tensors is characterized by the following sets of their indices:

$$\text{(B.1)}\qquad \begin{aligned} &\text{(i)} && l_3 = l_1 + l_2, m = 0,\\ &\text{(ii)} && l_3 = l_1 + l_2 + 2, m = 1. \end{aligned}$$

As an example, we present the zz-component of the tensor $\mathbf{T}^{00}_{00,00}$:

$$\text{(B.2)}\qquad \mathbf{T}^{00}_{00,00}|_{zz} = \frac{1}{8\sqrt{\pi}R\mu}\left\{ {}_4F_2\left(x\left|\begin{matrix}1/2\\3/2,3/2\end{matrix}\right.\right) - \frac{4}{3}{}_0F_1\left(x\,|5/2\right) + x^{1/2}\left[-2{}_2F_3\left(x\left|\begin{matrix}1,1\\3/2,2\end{matrix}\right.\right) + {}_1F_2\left(x\left|\begin{matrix}1\\3/2,3\end{matrix}\right.\right)\right]\right\},$$

$$x = (R\alpha)^2,$$

where the appropriate $G^{m,n}_{p,q}$ functions, appearing in (4.4), have been expressed in terms of the generalized hypergeometric functions ${}_pF_{q-1}$ [8, § 5.2]. To establish the asymptotic properties of $\mathbf{T}^{00}_{00,00}|_{zz}$ at $R\alpha > 1$, we can use the relations for the ${}_pF_{q-1}$ functions, given in § 7 of [7]. Thus we obtain:

$$\text{(B.3)}\qquad \mathbf{T}^{00}_{00,00}|_{zz} = \frac{1}{8\sqrt{\pi}\mu R^2\alpha}\left\{\gamma + \ln 2\sqrt{x} - Ei(-2\sqrt{x}) + \frac{1}{4x}\left[1 - 2x - (1+2\sqrt{x})e^{-2\sqrt{x}}\right]\right\},$$

where γ is the Euler constant, $-Ei(-2\sqrt{x})$ is the exponential integral [8, §6.2].

The asymptotic series for the exponential integral for large values of $R\alpha$ reads:

$$\text{(B.4)}\qquad Ei(-2R\alpha) = e^{-2R\alpha}\left[\sum_{k=1}^{n}(-1)^k\frac{(k-1)!}{(2R\alpha)^k} + R_n\right], \qquad R_n < \frac{n!}{|2R\alpha|^{n+1}}.$$

Hence we obtain that for $R\alpha \ll 1$

$$\text{(B.5)}\qquad \mathbf{T}^{00}_{00,00}|_{zz} \sim \frac{1}{R^2\alpha}.$$

On the other hand, the expression (B.3) may be rewritten in the following form:

(B.6) $$\mathbf{T}^{00}_{00,00}|_{zz} = \frac{1}{2\sqrt{\pi}\mu R}\sum_{k=0}^{\infty}\frac{(k+2)(-2R\alpha)^k}{(k+1)(k+3)!}.$$

From (B.6) it follows that for $R\alpha \ll 1$

(B.7) $$\mathbf{T}^{00}_{00,00}|_{zz} \sim \frac{1}{R}.$$

The above example illustrates the diverse properties of the hydrodynamic interactions in the regime $R\alpha < 1$, considered in [4], in comparison with the regime $R\alpha > 1$, analysed in the present paper.

The second group of the tensors has the following characteristic indices:

(B.8) $$\begin{aligned} l_3 &= l_1 + l_2 - 1, \quad m = 0, \\ l_3 &= l_1 + l_2 + 1, \quad m = 0, 1, \\ l_3 &= l_1 + l_2 + 3, \quad m = 1, 2. \end{aligned}$$

Here we present, for example, the zz-components of the tensor $\mathbf{T}^{00}_{10,00}$:

(B.9) $$\mathbf{T}^{00}_{10,00}|_{zz} = -\frac{\sqrt{3}a}{2\sqrt{\pi}\mu R^4\alpha^2}\left[-1 - \frac{2}{3}\sqrt{x} + \gamma + \ln 2\sqrt{x} - Ei(-2\sqrt{x}) + \frac{1}{2x} - \frac{1}{2x}\left(1 + 2\sqrt{x}\right)e^{-2\sqrt{x}}\right].$$

Further examples are given in compact forms, involving the functions ${}_pF_q$:

(B.10) $$T^{00}_{00,20}|_{zz} = \frac{5\sqrt{5}}{96\sqrt{\pi}\mu R}\left[1 - 2\left(\frac{a}{R}\right)^2\right]\cdot\left\{x\frac{d}{dx}\left[x^{5/2}\frac{d^2}{dx^2} \cdot\left[{}_2F_3\left(x\left|\begin{matrix}1,1\\3/2,4,3\end{matrix}\right.\right) - \frac{64}{15}x^{-1/2}{}_1F_2\left(x\left|\begin{matrix}1/2\\5/2,7/2\end{matrix}\right.\right)\right]\right] - \frac{d}{dx}\left[x^{7/2}\frac{d^2}{dx^2}\left[{}_2F_3\left(x\left|\begin{matrix}2,1\\3/2,4,4\end{matrix}\right.\right) - \frac{64}{15}x^{-1/2}{}_1F_2\left(x\left|\begin{matrix}3/2\\7/2,7/2\end{matrix}\right.\right)\right]\right]\right\},$$

$$\mathbf{T}^{00}_{00,10}|_{zz} = -\frac{\sqrt{3}}{8\sqrt{\pi}\mu R}\left\{x\frac{d}{dx}\left[-2{}_2F_3\left(x\left|\begin{matrix}1,1\\3/2,2,2\end{matrix}\right.\right)\right.\right.$$

$$+4x^{-1/2}\left({}_1F_2\left(x\left|\begin{matrix}1/2\\3/2,3/2\end{matrix}\right.\right)-1\right)-\frac{8}{15}x^{1/2}{}_1F_2\left(x\left|\begin{matrix}1\\2,7/2\end{matrix}\right.\right)$$

$$\left.+{}_1F_2\left(x\left|\begin{matrix}1\\3/2,3\end{matrix}\right.\right)\right]+\frac{5}{2}\left[1-2\left(\frac{a}{R}\right)^2\right]\cdot\frac{x^2d^2}{dx^2}\left[{}_2F_3\left(x\left|\begin{matrix}1,1\\3/2,3,2\end{matrix}\right.\right)\right]$$

$$-\frac{8}{3}x^{1/2}\left({}_1F_2\left(x\left|\begin{matrix}1/2\\3/2,5/2\end{matrix}\right.\right)-1\right)-\frac{1}{2}{}_2F_3\left(x\left|\begin{matrix}2,1\\3/2,3,3\end{matrix}\right.\right)$$

$$\left.\left.-\frac{8}{9}x^{-1/2}\left({}_1F_2\left(x\left|\begin{matrix}3/2\\5/2,5/2\end{matrix}\right.\right)-1\right)\right]\right\}. \tag{B.11}$$

Similarly, as for the case of the components $\mathbf{T}^{00}_{00,00}|_{zz}$ and $\mathbf{T}^{00}_{10,00}|_{zz}$, the above hypergeometric functions can be rewritten in the more explicit forms, using the formulae of the Chapter 7 of [7].

Appendix C. On the integral equations (2.1)

The integral equations (2.1) are obtained, starting from the Oseen equations of the motion of the viscous, incompressible fluid

$$\tilde{\rho}\mathbf{U}\cdot\nabla\mathbf{v}-\mu\Delta\mathbf{v}+\nabla p=\sum_{j=1}^{N}\int d\Omega_j\delta[\mathbf{r}-\mathbf{R}_j(\Omega_j)]\mathbf{f}_j(\Omega_j),$$

$$\nabla\cdot\mathbf{v}=0, \tag{C.1}$$

and the no-slip boundary conditions on the surfaces of the spheres:

$$\dot{\mathbf{R}}_j(\Omega_j)=\mathbf{v}(\mathbf{R}_j(\Omega_j)). \tag{C.2}$$

Inside the volumes of the spheres, the following relation holds [12]:

$$\nabla\cdot\mathbf{P}(\mathbf{r}_j)=0, |\mathbf{r}_j|<a. \tag{C.3}$$

In the case considered, the velocity field of the fluid can be expressed in terms of the fundamental tensor $\mathbf{T}(\mathbf{r}-\mathbf{r}')$, acting on the induced forces $\mathbf{f}_j$:

$$\mathbf{v}(\mathbf{r})=\mathbf{U}+\int d\mathbf{r}'\mathbf{T}(\mathbf{r}-\mathbf{r}')\cdot\sum_{j=1}^{N}\int d\Omega'_j\delta[\mathbf{r}'-R'_j(\Omega'_j)]\mathbf{f}'_j(\Omega'_j). \tag{C.4}$$

Applying the boundary conditions (C.2) to the formula (C.4), we arrive at N coupled integral equations (2.1). Starting from these equations, the hydrodynamic interactions can be described as multiple scattering processes of the perturbations of the uniform velocity $\mathbf{U}$. The multiple scattering is the main physical phenomenon, examined in connection with the hydrodynamic drag, exerted by the fluid on the spheres. The scattering events are specified by the series (2.8), describing the dependence of the hydrodynamic forces on the geometrical distribution of the spheres and on the convective inertia of the fluid, expressed in terms of the parameters σ and Re. The subsequent contributions to the series are due to the interaction of a single sphere with the surrounding fluid, the one-fold interactions between two different spheres, the two-fold interactions between two or three different spheres, and so on. The contributions present the series expansion of the induced forces with respect to the parameters σ and Re, where $\sigma < 1$ and $\mathrm{Re} < 1$. Hence the hydrodynamic forces can, in general, be calculated within the required approximation with respect to σ and Re. In this paper, we confine our attention to the first order Oseen effects.

References

1. S. KIM, S.J. KARRILA, *Microhydrodynamics*, Buterworth-Heinemann.
2. F. FEUILLEBOIS, *Some theoretical results for the motion of solid spherical particles in a viscous fluid*, [in:] Multiphase Science and Technology, vol. 4, [Eds.:] G.F. Hewitt, J.M. Delhaye, N. Zuber, Hemisphere 1989.
3. T. KUMAGAI, JSME Int. J. Ser. B, **38**, 563, 1995.
4. I. PIEŃKOWSKA, Arch. Mech.. **48**, 231, 1996.
5. T. YOSHIZAKI, H. YAMAKAWA, J. Chem. Phys., **73**, 578, 1980.
6. A. R. EDMONDS, *Angular momentum in quantum mechanics*, Princeton University Press, 1974.
7. A.P. GRUDNIKOV, Yu.A. BRYČKOW, O.I. MARIČEV, Integrals and Series [in Russian], Nauka, Moskva 1986.
8. Y. L. LUKE, *The spherical functions and their approximations*, vol. I, Academic Press, 1969.
9. P. MAZUR, W. VAN SAARLOOS, Physica A, **115**, 21, 1982.
10. I. PIEŃKOWSKA, Arch. Mech., **36**, 749, 1984.
11. T. KUMAGAI, *Revaluation of Oseen's approximation for prediction of motions of a cluster of spheres in fluid at low Reynolds number*, [in:] XIXth Int. Congress of Theoretical and Applied Mechanics, Kyoto, Japan, August 25-31, 1996.
12. P. MAZUR, A. J. WEISENBORN, Physica A, **123**, 209, 1984.

Received September 28, 1998; revised version December 4, 1998.