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Distortion equation of motion in linear incompatible elastodynamics

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THE PAPER DEALS with an initial-boundary value problem of linear incompatible elastodynamics, based on Kosevich' theory of continuously distributed defects due to prescribed plastic fields, [1, 2, 3]. In analogy to a stress formulation of linear incompatible elastodynamics with continuously distributed defects, [4], a distortion formulation is proposed. In such a formulation the tensorial initial-boundary value problem for an unknown asymmetric tensor field is to be solved. A solution to the problem generates the associated stress and rotation fields.

1. Introduction

THE STRESS FORMULATION of linear elastodynamics is a counterpart of the displacement formulation (cf. IGNACZAK [5, 6]). The stress formulation was applied to a dislocation theory by IGNACZAK and RAO in their fundamental work [4]. In the present paper we give the formulation of a problem of linear elastodynamics with defects in terms of an asymmetric distortion tensor field, and prove an appropriate uniqueness theorem. Also, we show that by a symmetrization, the problem and uniqueness theorem reduce to those of a pure stress formulation of elastodynamics with continuously distributed defects, cf. [4]. In addition, we use a solution to the distortion initial-boundary value problem to obtain a dislocation density formula from [4], see Eq.(2.11) in [4].

If the displacement field of a solid with defects is described by a vector u_i, ($u_i = u_i^T$, where u_i^T is the total displacement introduced in [4]), then the distortion tensor w_{ij} is defined as, cf. LANDAU and LIFSHITZ [2],

$$w_{ij} = u_{j,i}. \tag{1.1}$$

Of course, in general tensor w_{ij} is not symmetric, i.e. $w_{ij} \neq w_{ji}$.

In the defect theory, the integral of w_{ij} along a contour L which encloses the dislocation line D is equal to Burger's vector $\tilde{b}_i$, cf. Fig. 1.

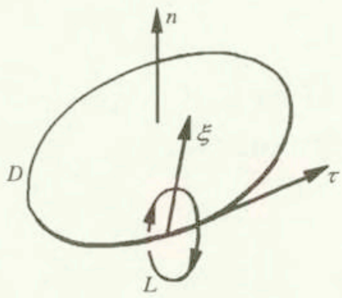

FIG. 1. The contour of integration L enclosing the dislocation line D. The direction of the contour integration and the chosen direction of the tangent vector $\boldsymbol{\tau}$ to the line D are related by the corkscrew rule, after [2]. Vector $\mathbf{n}$ is normal to the dislocation loop surface and vector $\boldsymbol{\xi}$ is a radius vector taken from the line D in the plane perpendicular to the vector $\boldsymbol{\tau}$.

Therefore, the following relation holds true:

$$\oint_L w_{ik} n_i dl = -\tilde{b}_k. \tag{1.2}$$

Using Stokes' theorem, this integral relation can be written in a differential form

$$\varepsilon_{imn} w_{nk,m} = -\tau_i \tilde{b}_k \delta(\boldsymbol{\xi}), \tag{1.3}$$

where ε_{imn} is the permutation symbol (the antisymmetric unit tensor), vector τ_i denotes the unit tangent to the dislocation line D, and $\boldsymbol{\xi}$ is the two-dimensional position vector taken from the dislocation line in the plane perpendicular to the vector τ_i at the point considered; and $\delta = \delta(\boldsymbol{\xi})$ represents the Dirac delta function, cf. also the Appendix.

If we introduce, after [2], the dislocation density tensor ρ_{ik}, defined as

$$\int_{S_L} \rho_{ik} df_i = \tilde{b}_k \tag{1.4}$$

where integration is performed over a surface S_L spanned by the contour L, we write instead of relation (1.3), the following equality

$$\varepsilon_{ijm} w_{mk,j} = -\rho_{ik}. \tag{1.5}$$

Hence, tensor ρ_{ik} satisfies the condition

$$\rho_{ik,i} = 0 \tag{1.6}$$

which for a single dislocation expresses the conservation of the Burgers vector along the dislocation line.

For convenience, and to make comparisons with [4] easier, we use the notation adopted in [4]

$$\alpha_{ik} \equiv -\rho_{ik}. \tag{1.7}$$

Then, Eqs. (1.5) and (1.6) take the forms

$$\varepsilon_{ijm} w_{mk,j} = \alpha_{ik} \tag{1.5$'$}$$

and

$$\alpha_{ik,i} = 0. \tag{1.6$'$}$$

Equations (1.5) or (1.5)$'$ hold true both for dislocations which are at rest and those in motion, if only the elastic displacements are taken into account. However, the motion of a dislocation can be related to a plastic deformation, as well. An exhaustive discussion of the elastodynamic theory of defects in which plastic deformations are prescribed, and the defects are characterized in terms of a symmetric stress tensor field, is presented in [4]. In the following, we present a ramification of the results obtained in [4] in which an asymmetric distortion tensor field plays a central role.

A total distortion tensor w_{ij}^T, denoted in [2] by W_{ij}, is postulated in the form

$$w_{ij}^T = w_{ij}^E + w_{ij}^P, \tag{1.8}$$

where w_{ij}^E and w_{ij}^P represent the elastic and plastic parts of the distortion, respectively,

$$w_{ij}^T = u_{j,i} \tag{1.9}$$

and neither elastic nor plastic part of decomposition (1.8) are obtained by the gradient of a vector field. Clearly, relations (1.1) to (1.6) are valid for $w_{ij} \equiv w_{ij}^E$ and $w_{ij}^P \equiv 0$.

If, in addition, it is postulated that (1.5)$'$ is satisfied by the elastic part w_{ij}^E only, i.e.

$$\varepsilon_{ijm} w_{mk,j}^E = \alpha_{ik} \tag{1.10}$$

then, by (1.8) and (1.9), we obtain

$$\varepsilon_{ijm} w_{mk,j}^P = -\alpha_{ik}. \tag{1.11}$$

The total deformation tensor e_{ij}^T, and elastic and plastic strain tensors e_{ij}^E and e_{ij}^P are defined in terms of w_{ij}^T, w_{ij}^E and w_{ij}^P by {cf. [4], Eqs. (2.1) – (2.2)}

$$2e^T_{ij} = w^T_{ij} + w^T_{ji}, \tag{1.12}$$

$$2e^E_{ij} = w^E_{ij} + w^E_{ji} \, (\equiv 2e_{ij}), \tag{1.13}$$

$$2e^P_{ij} = w^P_{ij} + w^P_{ji}. \tag{1.14}$$

Other fields of the defect theory such as the dislocation moment density can be also defined in terms of the tensor w^P_{mn}, (cf. LANDAU and LIFSHITZ [2], KOSEVICH [3]).

2. Basic field equations and initial boundary value problem

Let a nonhomogeneous anisotropic linear elastic body, occupying a three-dimensional region B, be subject to a dynamic motion. Let the body forces be absent. Let the displacement be described by a vector u_i. Then the distortion tensor w^T_{ij} is defined as, cf. (1.9),

$$w^T_{ij} = u_{j,i}. \tag{2.1}$$

The relation between the strain tensor e^T_{ij} and the distortion tensor w^T_{ij} is given by (1.12), and the relation between the stress tensor s^E_{ij} and the strain tensor e^T_{ij} is given by the generalized Hooke's law, cf. SOKOLNIKOFF [7], NOWACKI [8],

$$s^E_{ij} = C_{ijmn} e^E_{mn}, \tag{2.2}$$

where C_{ijmn} is the elasticity tensor. The inverse relation reads

$$e^E_{ij} = K_{ijmn} s^E_{mn}, \tag{2.3}$$

where K_{ijmn} is the compliance tensor, *i.e.*:

$$C_{ijpq} K_{pqmn} = \delta_{i(m}\delta_{jn)} \qquad \text{on} \quad B$$

and δ_{ij} stands for Kronecker's symbol.

Thus, instead of (2.2) we can write

$$s^E_{ij} = C_{ijmn} w^E_{mn} \tag{2.4}$$

or taking into account (1.8), we get

$$s^E_{ij} = C_{ijmn}(w^T_{mn} - w^P_{mn}) (\equiv s_{ij}). \tag{2.5}$$

In Eqs. (2.2) - (2.5) the symmetry relations are postulated

$$C_{ijmn} = C_{jimn} = C_{ijnm} = C_{mnij}, \quad K_{ijmn} = K_{jimn} = K_{ijnm} = K_{mnij}. \tag{2.6}$$

The equation of motion takes the form

$$s^E_{ik,k} + b_i = \rho \ddot{u}_i, \tag{2.7}$$

where $\rho = \rho(\mathbf{x})$ and $\mathbf{b} = \mathbf{b}(\mathbf{x})$ denote the mass density and body force vector fields.

Finally, the density ρ and compliance K_{ijkl} satisfy the inequalities

$$\rho > 0, \qquad K_{ijkl}\xi_{ij}\xi_{kl} > 0 \tag{2.8}$$

for every tensor field $\xi_{ij}(\mathbf{x}), \mathbf{x} \in B$, not necessarily symmetric.

3. Distortion equation of motion

After differentiation of equation of motion (2.7) and use of definition of distortion (2.1), we get

$$(\rho^{-1} s^E_{ik,k})_{,j} + (\rho^{-1} b_i)_{,j} = \ddot{w}^T_{ij}, \qquad (\mathbf{x}, t) \in B \times [0, \infty) \tag{3.1}$$

or, by (2.5),

$$\left\{\rho^{-1}[C_{ikmn}(w^T_{mn} - w^P_{mn})]_{,k}\right\}_{,j} + (\rho^{-1} b_i)_{,j} = \ddot{w}^T_{ij}, \qquad (\mathbf{x}, t) \in B \times [0, \infty)$$

or

$$[\rho^{-1}(C_{ikmn} w^T_{mn})_{,k}]_{,j} + \hat{b}_{i,j} = \ddot{w}^T_{ij}, \qquad (\mathbf{x}, t) \in B \times [0, \infty). \tag{3.2}$$

This is the distortion equation of elastodynamics we are to deal with. The term

$$\hat{b}_i \equiv [b_i - (C_{ikmn} w^P_{mn})_{,k}]/\rho \tag{3.2a}$$

is to be considered as the given density of body forces distributed in a crystal, (cf. [2]).

The above field equation is subject to the initial conditions

$$w^T_{ij}(\mathbf{x}, 0) = w^o_{ij}(\mathbf{x}), \qquad \dot{w}^T_{ij}(\mathbf{x}, 0) = \dot{w}^o_{ij}(\mathbf{x}) \qquad \mathbf{x} \in B \tag{3.3}$$

and boundary conditions

$$u_i^0(\mathbf{x}) + t\dot{u}_i(\mathbf{x}) + \rho^{-1} t * (C_{ikmn} w_{mn}^T)_{,k}(\mathbf{x},t) = \mathcal{U}_i(\mathbf{x},t) \quad \text{on } \partial B_{\mathcal{U}} \times [0,\infty),$$

(3.4)

$$(C_{ijmn} w_{mn}^T) n_j = \mathcal{F}_i(\mathbf{x},t) \text{ on } \partial B_{\mathcal{F}} \times [0,\infty),$$

where $\partial B_{\mathcal{U}} \cup \partial B_{\mathcal{F}} = \partial B, \partial B_{\mathcal{U}} \cap \partial B_{\mathcal{F}} = \emptyset$.

The fields $w_{ij}^0(\mathbf{x})$ and $\dot{w}_{ij}^0(\mathbf{x})$ are determined by the initial displacement field $u_i(\mathbf{x},0) = u_i^o(\mathbf{x})$ and the initial velocity field $\dot{u}_i(\mathbf{x},0) = \dot{u}_i^o(\mathbf{x})$ through the relations

$$w_{ij}^0(\mathbf{x}) = u_{j,i}^0(\mathbf{x}), \qquad \dot{w}_{ij}^o(\mathbf{x}) = \dot{u}_{j,i}^p(\mathbf{x}) \quad \text{on} \quad B.$$

Moreover, the field $\mathcal{U}_i(\mathbf{x},t)$ is expressed by the boundary displacement $\hat{u}_i(\mathbf{x},t)$, and the initial data $u_i^o(\mathbf{x})$ and $\dot{u}_i^o(\mathbf{x})$, through

$$\mathcal{U}_i(\mathbf{x},t) = \hat{u}_i(\mathbf{x},t) - t\dot{u}_i^o(\mathbf{x}) - u_i^o(\mathbf{x}).$$

Finally, the field $\mathcal{F}_i(\mathbf{x},t)$ represents the surface traction.

THEOREM 1. *(Uniqueness theorem for a distortion initial-boundary value problem of elastodynamics with defects).*
The problem (3.2) – (3.4) *has at most one solution.*

P r o o f. It is sufficient to show that the problem (3.2) – (3.4) corresponding to homogeneous data has a zero solution only.

Introduce the notations

(a) $$s_{ij} = C_{ijmn} w_{mn}^T$$

and

(b) $$W_{ij} = w_{ij}^T.$$

Then, the homogenous counterparts to Eqs. (3.2) – (3.4) read

(3.2)′ $$(\rho^{-1} s_{ik,k})_{,j} = \ddot{W}_{ij}, \qquad (\mathbf{x},t) \in B \times [0,\infty)$$

(3.3)′ $$W_{ij}(\mathbf{x},0) = 0, \qquad \dot{W}_{ij}(\mathbf{x},0) = 0, \qquad \mathbf{x} \in B;$$

(3.4)′ $$\rho^{-1} t * (C_{ikmn} W_{mn})_{,k}(\mathbf{x},t) = 0 \qquad \text{on} \quad \partial B_{\mathcal{U}} \times [0,\infty),$$

$$C_{ijmn} W_{mn} n_j = 0 \qquad \text{on} \quad \partial B_{\mathcal{F}} \times [0,\infty).$$

Moreover, (a), $(3.3)'$ and $(3.4)'$ imply that

(b) $$s_{ij}(\mathbf{x},0)=0, \qquad \dot{s}_{ij}(\mathbf{x},0)=0, \qquad \mathbf{x}\in B$$

and

(c) $$s_{ik,k}(\mathbf{x},t)=0 \qquad \text{on} \qquad \partial B_{\mathcal{U}}\times[0,\infty),$$

(d) $$s_{ij}n_j=0 \qquad \text{on} \qquad \partial B_{\mathcal{F}}\times[0,\infty).$$

Multiply both sides of Eq. $(3.2)'$ by $\dot{s}_{ij}$ and integrate over B, to obtain

$$\int\limits_B [(\rho^{-1}s_{ik,k})_{,j}]\dot{s}_{ij}dB=\int\limits_B \ddot{W}_{ij}\dot{s}_{ij}dB.$$

Hence, by integration by parts, we get

$$\int\limits_B \left\{[(\rho^{-1}s_{ik,k})\dot{s}_{ij}]_{,j}-(\rho^{-1}s_{ik,k})\dot{s}_{ij,j}\right\}dB=\int\limits_B \ddot{W}_{ij}\dot{s}_{ij}dB$$

or, by use of the divergence theorem,

$$\int\limits_{\partial B}(\rho^{-1}s_{ik,k})\dot{s}_{ij}n_j dA-\int\limits_B[(\rho^{-1}s_{ik,k})\dot{s}_{ij,j}]dB=\int\limits_B \ddot{W}_{ij}\dot{s}_{ij}dB.$$

By (c) and (d), we obtain

$$-\int\limits_B[(\rho^{-1}s_{ik,k})\dot{s}_{ij,j}]dB=\int\limits_B \ddot{W}_{ij}\dot{s}_{ij}dB$$

or by (a)

$$\frac{1}{2}\frac{\partial}{\partial t}\int\limits_B\left[(\rho^{-1}s_{ik,k})\dot{s}_{ij,j}+C_{ijmn}\dot{W}_{ij}\dot{W}_{mn}\right]dB=0,$$

and after integration with respect to time and using the initial conditions $(3.3)'$ and (b), we get

(3.5) $$\int\limits_B[(\rho^{-1}s_{ik,k})s_{ij,j}+C_{ijmn}\dot{W}_{ij}\dot{W}_{mn}]dB=0.$$

Hence

(e) $$s_{ij,j}=0$$

and

$$\text{(f)} \qquad \dot{W}_{ij} + \dot{W}_{ji} = 0.$$

Insert (e) into (3.2)′ to get

$$\text{(g)} \qquad 0 = \ddot{W}_{ij} \qquad \forall \quad (\mathbf{x}, t) \in B \times [0, \infty).$$

Integrating the last equation twice and using the initial conditions (3.3)′ we obtain the result

$$\text{(h)} \qquad W_{ij} = 0 \qquad \forall \quad (\mathbf{x}, t) \in B \times [0, \infty).$$

This completes the proof of the uniqueness theorem. □

Note that the uniqueness theorem for a pure stress initial-boundary value problem of elastodynamics in which the elasticity tensor is positive definite and the density is positive was presented in [5], while the uniqueness theorem for a pure stress initial-boundary value problem of incompressible isotropic elastodynamics was given in [9].

4. Remarks

4.1.

Let us, in addition to the fields introduced in Sec. 2, define the elastic rotation vector field

$$\omega_k = \frac{1}{2}\varepsilon_{kab} w^E_{ab} \tag{4.1}$$

and elastic rotation tensor field

$$o_{mk} = \frac{1}{2}(w^E_{mk} - w^E_{km}). \tag{4.2}$$

Also, define the elastic bend-twist tensor as

$$\kappa_{mk} = \omega_{k,m}. \tag{4.3}$$

With these definitions we find that

$$o_{mk} = \frac{1}{2}(o_{mk} - o_{km})$$

or

$$o_{mk} = \frac{1}{2}(\delta_{ma}\delta_{kb} - \delta_{mb}\delta_{ka})o_{ab}$$

or

$$o_{mk} = \frac{1}{2}\varepsilon_{mkq}\varepsilon_{qab}o_{ab}.$$

Hence

$$o_{mk} = \varepsilon_{mkq}\omega_q$$

and, by (4.3)

$$o_{mk,j} = \varepsilon_{mkq}\kappa_{jq}. \tag{4.4}$$

Therefore, Eq. (1.5)′ which, by definitions (1.13) and (4.2) can be written as

$$\varepsilon_{ijm}(e_{mk} + o_{mk})_{,j} = \alpha_{ik}, \tag{4.5}$$

takes the form

$$\varepsilon_{ijm}(e_{mk,j} + \varepsilon_{mkq}\kappa_{jq}) = \alpha_{ik} \tag{4.6}$$

and this is an equation identical with Eq.$(2.11)_1$ in [4].

4.2.

By taking symmetric part of the distortion problem (3.1) - (3.4), we arrive at the pure stress problem of elastodynamics with continuously distributed defects discussed in [4] (cf. THEOREM 3.3 in [4]).

4.3.

If a solution of the pure stress problem discussed in **4.2.** is available, then the skew part of the distortion, *i.e.* a solution of the problem obtained by taking skew part of the problem (3.1) - (3.4), can be easily obtained.

5. Conclusions

1. A pure distortion initial-boundary value problem of linear elastodynamics with continuously distributed defects has been formulated, and a uniqueness theorem for the problem has been proved.

2. By a symmetrization, the problem and uniqueness theorem reduce to those of a pure stress formulation of elastodynamics with continuously distributed defects, cf. [4].

3. By defining the elastic rotation fields in terms of the distortion tensor, a dislocation density formula from [4] has been recovered.

Appendix

To show that the integral condition (1.2) implies the local equation (1.3), Stokes' theorem is used. The proof is due to LANDAU and LIFSHITZ [2]. First, we note that for an arbitrary vector **a** we have

$$\oint_L a_i n_i dl = \int_{S_L} \varepsilon_{imn} a_{n,m} n_i dS, \tag{A.1}$$

where S_L is a surface that spans the contour L. Substituting w_{ik} instead of a_i into (A.1) we get

$$\oint_L w_{ik} n_i dl = \int_{S_L} \varepsilon_{imn} w_{nk,m} n_i dS. \tag{A.2}$$

Next, we note that a constant vector $\tilde{\mathbf{b}}$ may be represented by an integral involving the two-dimensional delta function

$$\tilde{b}_k = \int_{S_L} \tau_i \tilde{b}_k \delta(\boldsymbol{\xi}) n_i dS, \tag{A.3}$$

where $\boldsymbol{\xi}$ is the two-dimensional radius vector taken from the axis of dislocation in the plane perpendicular to the vector $\boldsymbol{\tau}$ at the point considered, cf. Fig. 1. Substituting (A2) and (A3) into (1.2) we get

$$\int_{S_L} \varepsilon_{imn} w_{nk,m} n_i dS = - \int_{S_L} \tau_i \tilde{b}_k \delta(\boldsymbol{\xi}) n_i dS. \tag{A.4}$$

Since the contour L is arbitrary, from (A.4) we obtain (1.3). This completes the proof of implication (1.2) $\Rightarrow$ (1.3).

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