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# Some exact solutions of steady plane MHD non-Newtonian power-law fluid flows*

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EQUATIONS of steady plane MHD flow of a non-Newtonian power-law fluid are transformed to the hodograph plane by means of the Legendre transform function of the stream-function. Results are summarized in the form of a theorem. As applications of the developed theory, four flow problems of physical interest are studied. Exact solutions and the corresponding geometry are obtained in each case.

## 1. Introduction

HODOGRAPH TRANSFORMATION is one of the methods of solving systems of nonlinear partial differential equations. This technique has been widely used in continuum mechanics. AMES [1] present a survey of the hodograph transformation and its applications in fluid mechanics and various other fields. In a series of papers CHANDNA *et al.* [2 – 7] have applied hodograph and Legendre transformations to investigate steady plane viscous flows, non-Newtonian flows and MHD non-Newtonian flows in the presence of constantly inclined, aligned, transverse or orthogonal magnetic field. ADLURI [8, 9] has employed this method to obtain a class of exact solutions plane MHD non-Newtonian power-law fluids and micropolar fluids. In recent years, the interest in problems of non-Newtonian fluid flows has grown considerably due to an extensive use of these fluids in many areas such as chemical processes in industries, food and construction engineering, petroleum production, power engineering and commercial applications. Since most liquid metals, non-Newtonian fluids, and many other second grade fluids to which single fluid model can be applied, accounting for electrical conductivity, makes the flow problem realistic both from the mathematical and the physical point of view.

The present paper deals with application of hodograph transformation technique to obtain a class of exact solutions of the nonlinear partial differential equations governing the steady plane flow of a power-law fluid in the presence of a transverse magnetic field. Equations of the flow are transformed to the hodograph plane interchanging the role of independent variables $x, y$ and the

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components $u, v$ of the velocity field. Introducing a Legendre function of the stream-function, all equations in the hodograph plane are expressed in terms of this transform function. Results are summarized in the form of a theorem and finally, four interesting flow problems are studied to illustrate the developed theory. Exact solutions and geometry of the flow are obtained in each case and it is proved that a spiral flow cannot exist in a non-Newtonian power-law fluid whether or not the fluid is conducting.

## 2. Equations of flow

The steady plane flow of an electrically conducting non-Newtonian fluid which obeys Ostwaald-de Waele power-law model

$$\tau_{ij} = 2K\left[2e_{kl}e_{kl}\right]^{\frac{(n-1)}{2}} e_{ij}$$

where

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

is governed by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

$$\frac{\partial F}{\partial x} = \rho v\omega + K\left\{\omega\frac{\partial I}{\partial y} - I\frac{\partial \omega}{\partial y} + 2\left(\frac{\partial u}{\partial x}\frac{\partial I}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial I}{\partial y}\right)\right\}, \tag{2.2}$$

$$\frac{\partial F}{\partial y} = -\rho u\omega - K\left\{\omega\frac{\partial I}{\partial x} - I\frac{\partial \omega}{\partial x} - 2\left(\frac{\partial v}{\partial x}\frac{\partial I}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial I}{\partial y}\right)\right\}, \tag{2.3}$$

$$u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y} = \frac{1}{\mu_e\sigma}\left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right), \tag{2.4}$$

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \tag{2.5}$$

$$I = \left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right]^{\frac{(n-1)}{2}}, \tag{2.6}$$

where $\tau_{ij}$ denotes the strain rate tensor, $e_{ij}$ – the strain tensor, $(u(x,y)\cdot v(x,y), 0)$ is the velocity vector field, $(0, 0, H(x,y))$ is the magnetic field, $p$ is the pressure, $\rho$ is the fluid density, $\mu_e$ is the magnetic permeability, $\sigma$ is the electrical conductivity, $K$ is the consitutive coefficient, $I$ is the Ostwaald-de Waele parameter and

$$F(x,y) = \frac{1}{2}\rho(u^2+v^2) + p + \frac{1}{2}\mu_e H^2. \tag{2.7}$$

## 3. Equations in the hodograph plane

Assuming $u(x,y), v(x,y)$ to be such that the *Jacobian*

$$J(x,y) = \frac{\partial(u,v)}{\partial(x,y)} \neq 0, \qquad 0 < |J| < \infty \tag{3.1}$$

in the region of the flow and considering $x$ and $y$ as functions of $u$ and $v$, we can derive the following relations

$$J(x,y) = \frac{\partial(u,v)}{\partial(x,y)} = \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^{-1} = \overline{J}(u,v),$$

$$\frac{\partial u}{\partial x} = J\frac{\partial y}{\partial v}, \qquad \frac{\partial u}{\partial y} = -J\frac{\partial x}{\partial v}, \qquad \frac{\partial v}{\partial y} = J\frac{\partial x}{\partial u}, \tag{3.2}$$

$$\frac{\partial f}{\partial x} = \frac{\partial(f,y)}{\partial(x,y)} = \overline{J}\frac{\partial(\overline{f},y)}{\partial(u,v)}, \qquad \frac{\partial f}{\partial y} = -\frac{\partial(f,x)}{\partial(u,v)} = \overline{J}\frac{\partial(x,\overline{f})}{\partial(u,v)},$$

where $f = f(x,y) = f(u(x,y), v(x,y)) = \overline{f}(u,v)$ is any continuously differentiable function.

Employing these relations to Eqs. (2.1) – (2.7) we obtain the following equations in the $(u,v)$ – plane:

$$\frac{\partial x}{\partial u} + \frac{\partial y}{\partial v} = 0, \tag{3.3}$$

$$\overline{J}\frac{\partial(\overline{F},y)}{\partial(u,v)} = \rho v\overline{\omega} + K\overline{J}\left\{\overline{\omega}Q_1 - \overline{I}P_1 + 2\overline{J}\left(Q_2\frac{\partial y}{\partial v} - Q_1\frac{\partial x}{\partial v}\right)\right\}, \tag{3.4}$$

$$\overline{J}\frac{\partial(x,\overline{F})}{\partial(u,v)} = -\rho u\omega - K\overline{J}\left\{\overline{\omega}Q_2 - \overline{I}P_2 + 2\overline{J}\left(Q_2\frac{\partial y}{\partial u} - Q_1\frac{\partial x}{\partial u}\right)\right\}, \tag{3.5}$$

$$uN_1 + vN_2 = \frac{1}{\mu_e\sigma}\left\{\frac{\partial(\overline{J}N_1,y)}{\partial(u,v)} + \frac{\partial(x,\overline{J}N_2)}{\partial(u,v)}\right\}, \tag{3.6}$$

$$\overline{\omega} = \overline{J}\left(\frac{\partial x}{\partial v} - \frac{\partial y}{\partial u}\right), \tag{3.7}$$

$$(3.8)\qquad \overline{I} = \left(\overline{J}\right)^{(n-1)} \left\{ 2\left(\frac{\partial x}{\partial u}\right)^2 + 2\left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\right)^2 \right\}^{\frac{(n-1)}{2}}$$

where

$$P_1(u,v) = \frac{\partial(x,\overline{\omega})}{\partial(u,v)}, \qquad P_2(u,v) = \frac{\partial(\overline{\omega},y)}{\partial(u,v)},$$

$$Q_1(u,v) = \frac{\partial(x,\overline{I})}{\partial(u,v)}, \qquad Q_2(u,v) = \frac{\partial(\overline{I},y)}{\partial(u,v)},$$

$$(3.9)\qquad N_1(u,v) = \frac{\partial(\overline{H},y)}{\partial(u,v)}, \qquad N_2(u,v) = \frac{\partial(x,\overline{H})}{\partial(u,v)}.$$

Equations (3.3) – (3.8) is a system of six equations in six unknown functions $x(u,v), y(u,v), \overline{\omega}$, $\overline{I}$, $\overline{H}$ and $\overline{F}$. Once a solution of this is obtained, we can determine $u(x,y)$, $v(x,y)$ and all other flow variables in the physical plane. Eq. (3.6) is the diffusion equation for a finitely conducting fluid for infinitely conducting fluid flows it should be replaced by

$$(3.10)\qquad uN_1 + vN_2 = 0.$$

## 4. Legendre transformation function and $\overline{H}(u,v)$

The continuity Eq. (2.1) implies the existence of a stream-function $\psi(x,y)$ such that

$$(4.1)\qquad d\psi = -vdx + udy, \qquad \frac{\partial \psi}{\partial x} = v, \qquad \frac{\partial \psi}{\partial y} = u$$

and Eq. (3.3) implies the existence of a function $L(u,v)$ called the Legendre transform function of the stream-function $\psi(x,y)$ such that

$$(4.2)\qquad dL = -ydu + xdv, \qquad \frac{\partial L}{\partial u} = -y, \qquad \frac{\partial L}{\partial v} = x.$$

Functions $L(u,v)$ and $\psi(x,y)$ are related by

$$(4.3)\qquad L(u,v) = vx - uy + \psi(x,y).$$

Using Eq. (4.2), we can transform Eqs. (3.4) – (3.9), respectively, to

$$(4.4)\qquad \overline{J}\frac{\partial\left(\dfrac{\partial L}{\partial u}, \overline{F}\right)}{\partial(u,v)} = \rho v\omega + K\overline{J}\left(\overline{\omega}Q_1 - \overline{I}P_1 - 2\overline{J}R_1\right),$$

$$\overline{J}\frac{\partial\left(\dfrac{\partial L}{\partial v},\overline{F}\right)}{\partial(u,v)} = -\rho u\omega - K\overline{J}\left(\overline{\omega}Q_2 - \overline{I}P_2 - 2\overline{J}R_2\right), \tag{4.5}$$

$$uN_1 + vN_2 = \frac{1}{\mu_e\sigma}\left\{\frac{\partial\left(\dfrac{\partial L}{\partial u},\overline{J}N_1\right)}{\partial(u,v)} + \frac{\partial\left(\dfrac{\partial L}{\partial v},\overline{J}N_2\right)}{\partial(u,v)}\right\}, \tag{4.6}$$

$$\begin{aligned}\overline{J} &= \left[\frac{\partial^2 L}{\partial u^2}\cdot\frac{\partial^2 L}{\partial v^2} - \left(\frac{\partial^2 L}{\partial u\partial v}\right)^2\right]^{-1},\\ \overline{\omega} &= \overline{J}\left(\frac{\partial^2 L}{\partial u^2} + \frac{\partial^2 L}{\partial v^2}\right),\end{aligned} \tag{4.7}$$

$$\overline{I} = (\overline{J})^{(n-1)}\left\{\left(\frac{\partial^2 L}{\partial u^2} - \frac{\partial^2 L}{\partial v^2}\right)^2 + 4\left(\frac{\partial^2 L}{\partial u\partial v}\right)^2\right\}^{\frac{(n-1)}{2}}, \tag{4.8}$$

where

$$P_1(u,v) = \frac{\partial\left(\dfrac{\partial L}{\partial v},\overline{\omega}\right)}{\partial(u,v)}, \qquad P_2(u,v) = \frac{\partial\left(\dfrac{\partial L}{\partial u},\overline{\omega}\right)}{\partial(u,v)},$$

$$Q_1(u,v) = \frac{\partial\left(\dfrac{\partial L}{\partial v},\overline{I}\right)}{\partial(u,v)}, \qquad Q_2(u,v) = \frac{\partial\left(\dfrac{\partial L}{\partial u},\overline{I}\right)}{\partial(u,v)},$$

$$\begin{aligned}N_1(u,v) &= \frac{\partial\left(\dfrac{\partial L}{\partial u},\overline{H}\right)}{\partial(u,v)}, \qquad N_2(u,v) = \frac{\partial\left(\dfrac{\partial L}{\partial v},\overline{H}\right)}{\partial(u,v)},\\ R_1(u,v) &= Q_1\frac{\partial^2 L}{\partial v^2} + Q_2\frac{\partial^2 L}{\partial u\partial v}, \qquad R_2 = Q_1\frac{\partial^2 L}{\partial u\partial v} + Q_2\frac{\partial^2 L}{\partial u^2}.\end{aligned} \tag{4.9}$$

To eliminate $\overline{F}(u,v)$ from Eqs. (4.4) and (4.5), we use the integrability condition

$$\left(\overline{J}\frac{\partial^2 L}{\partial u\partial v}\frac{\partial}{\partial v}-\overline{J}\frac{\partial^2 L}{\partial v^2}\frac{\partial}{\partial u}\right)\left(\overline{J}\frac{\partial\left(\dfrac{\partial L}{\partial u},\overline{F}\right)}{\partial(u,v)}\right)$$

$$=\left(\overline{J}\frac{\partial^2 L}{\partial u^2}\frac{\partial}{\partial v}-\overline{J}\frac{\partial^2 L}{\partial u\partial v}\frac{\partial}{\partial u}\right)\left(\overline{J}\frac{\partial\left(\dfrac{\partial L}{\partial v},\overline{F}\right)}{\partial(u,v)}\right)$$

and obtain

$$\rho(vP_1+uP_2)+K(\overline{\omega}W_1-\overline{I}W_2-2W_3)=0, \tag{4.10}$$

where

$$W_1=\frac{\partial\left(\dfrac{\partial L}{\partial v},\overline{J}Q_1\right)}{\partial(u,v)}+\frac{\partial\left(\dfrac{\partial L}{\partial u},\overline{J}Q_2\right)}{\partial(u,v)},$$

$$W_2=\frac{\partial\left(\dfrac{\partial L}{\partial v},\overline{J}P_1\right)}{\partial(u,v)}+\frac{\partial\left(\dfrac{\partial L}{\partial u},\overline{J}P_2\right)}{\partial(u,v)}, \tag{4.11}$$

$$W_3=\frac{\partial\left(\dfrac{\partial L}{\partial v},\overline{J}^2R_1\right)}{\partial(u,v)}+\frac{\partial\left(\dfrac{\partial L}{\partial u},\overline{J}^2R_2\right)}{\partial(u,v)}.$$

We can summarize the above results in the form of a theorem:

THEOREM: *If $L(u,v)$ is the Legendre transform function of a stream-function of a steady plane transverse finitely conducting non-Newtonian power-law fluid flow and $\overline{H}(u,v)$ is the transformed magnetic field, then functions $L(u,v)$ and $\overline{H}(u,v)$ have to satisfy Eqs. (4.6) and (4.10) where $\overline{\omega}$, $\overline{I}$, $\overline{J}$, $P_i$, $Q_i$, $R_i$ and $W_i$ are given by (4.7) – (4.9) and (4.11).*

To solve $L(u,v)$ and $\overline{H}(u,v)$ from Eqs. (4.6) and (4.10), it is convenient to express Eqs. (4.6) – (4.11) in polar coordinates $(q,\theta)$ in the hodograph plane by defining

$$u=q\cos\theta,\quad v=q\sin\theta,\quad q^2=u^2+v^2,\quad \theta=\tan^{-1}\left(\frac{v}{u}\right). \tag{4.12}$$

Using (4.12), Eqs. (4.6) – (4.11) can be transformed to

$$(4.13)\qquad q\left(N_1^*\cos\theta + N_2^*\sin\theta\right) = \frac{1}{\mu_e\sigma q}\left\{\frac{\partial\left(\cos\theta\dfrac{\partial L^*}{\partial q} - \dfrac{\sin\theta}{q}\dfrac{\partial L^*}{\partial\theta}, J^*N_1^*\right)}{\partial(q,\theta)}\right.$$

$$\left. + \frac{\partial\left(\sin\theta\dfrac{\partial L^*}{\partial q} + \dfrac{\cos\theta}{q}\dfrac{\partial L^*}{\partial\theta}, J^*N_2^*\right)}{\partial(q,\theta)}\right\},$$

$$(4.14)\qquad \rho(P_1^*\sin\theta + P_2^*\cos\theta) + K(\omega^*W_1^* - I^*W_2^* - 2W_3^*) = 0,$$

$$(4.15)\qquad \omega^*(q,\theta) = J^*\left(\frac{\partial^2 L^*}{\partial q^2} + \frac{1}{q}\frac{\partial L^*}{\partial q} + \frac{1}{q^2}\frac{\partial^2 L^*}{\partial\theta^2}\right),$$

$$(4.16)\qquad I^*(q,\theta) = (J^*)^{(n-1)}\left\{\left(\frac{\partial^2 L^*}{\partial q^2} - \frac{1}{q}\frac{\partial L^*}{\partial q} - \frac{1}{q^2}\frac{\partial^2 L^*}{\partial\theta^2}\right)^2\right.$$

$$\left. + \frac{4}{q^4}\left(\frac{\partial L^*}{\partial\theta} - q\frac{\partial^2 L^*}{\partial q\partial\theta}\right)^2\right\}^{\frac{(n-1)}{2}},$$

$$(4.17)\qquad J^*(q,\theta) = q^4\left\{q^2\frac{\partial^2 L^*}{\partial q^2}\left(q\frac{\partial L^*}{\partial q} + \frac{\partial^2 L^*}{\partial\theta^2}\right) - \left(\frac{\partial L^*}{\partial\theta} - q\frac{\partial^2 L^*}{\partial q\partial\theta}\right)^2\right\}^{-1},$$

$$(4.18)\qquad \begin{aligned} P_1^*(q,\theta) &= \frac{1}{q}\frac{\partial\left(\sin\theta\dfrac{\partial L^*}{\partial q} + \dfrac{\cos\theta}{q}\dfrac{\partial L^*}{\partial\theta}, \omega^*\right)}{\partial(q,\theta)},\\ P_2^*(q\theta) &= \frac{1}{q}\frac{\partial\left(\cos\theta\dfrac{\partial L^*}{\partial q} - \dfrac{\sin\theta}{q}\dfrac{\partial L^*}{\partial\theta}, \omega^*\right)}{\partial(q,\theta)},\\ Q_1^*(q,\theta) &= \frac{1}{q}\frac{\partial\left(\sin\theta\dfrac{\partial L^*}{\partial q} + \dfrac{\cos\theta}{q}\dfrac{\partial L^*}{\partial\theta}, I^*\right)}{\partial(q,\theta)},\\ Q_2^*(q,\theta) &= \frac{1}{q}\frac{\partial\left(\cos\theta\dfrac{\partial L^*}{\partial q} - \dfrac{\sin\theta}{q}\dfrac{\partial L^*}{\partial\theta}, I^*\right)}{\partial(q,\theta)}, \end{aligned}$$

$$R_1^*(q,\theta) = Q_2^*\left(\sin\theta\cos\theta\frac{\partial^2 L^*}{\partial q^2} - \frac{\sin\theta\cos\theta}{q}\frac{\partial L^*}{\partial q} - \frac{\cos 2\theta}{q^2}\frac{\partial L^*}{\partial \theta}\right.$$
$$\left.+\frac{\cos 2\theta}{q}\frac{\partial^2 L^*}{\partial q\partial\theta} - \frac{\sin\theta\cos\theta}{q^2}\frac{\partial^2 L^*}{\partial\theta^2}\right) + Q_2^*\left(\sin\theta\cos\theta\frac{\partial^2 L^*}{\partial q^2} - \frac{\sin\theta\cos\theta}{q}\frac{\partial L^*}{\partial q}\right.$$
$$\left.-\frac{\cos 2\theta}{q^2}\frac{\partial L^*}{\partial \theta} + \frac{\cos 2\theta}{q}\frac{\partial^2 L^*}{\partial q\partial\theta} - \frac{\sin\theta\cos\theta}{q^2}\frac{\partial^2 L^*}{\partial\theta^2}\right),$$

(4.19)
$$R_2^*(q,\theta) = Q_1^*\left(\sin\theta\cos\theta\frac{\partial^2 L^*}{\partial q^2} - \frac{\sin\theta\cos\theta}{q}\frac{\partial L^*}{\partial q} - \frac{\cos 2\theta}{q^2}\frac{\partial L^*}{\partial \theta}\right.$$
$$\left.+\frac{\cos 2\theta}{q}\frac{\partial^2 L^*}{\partial q\partial\theta} - \frac{\sin\theta\cos\theta}{q^2}\frac{\partial^2 L^*}{\partial\theta^2}\right)$$
$$+Q_2^*\left(\cos^2\theta\frac{\partial^2 L^*}{\partial q^2} + \frac{\sin^2\theta}{q}\frac{\partial L^*}{\partial q} + \frac{\sin 2\theta}{q^2}\frac{\partial L^*}{\partial \theta} - \frac{\sin 2\theta}{q^2}\frac{\partial^2 L^*}{\partial q\partial\theta} + \frac{\sin^2\theta}{q^2}\frac{\partial^2 L^*}{\partial\theta^2}\right),$$

$$W_1^*(q,\theta) = \frac{1}{q}\left\{\frac{\partial\left(\sin\theta\dfrac{\partial L^*}{\partial q} + \dfrac{\cos\theta}{q}\dfrac{\partial L^*}{\partial\theta}, J^*Q_1^*\right)}{\partial(q,\theta)}\right.$$
$$\left.+\frac{\partial\left(\cos\theta\dfrac{\partial L^*}{\partial q} - \dfrac{\sin\theta}{q}\dfrac{\partial L^*}{\partial\theta}, J^*Q_2^*\right)}{\partial(q,\theta)}\right\},$$

(4.20)
$$W_2^*(q,\theta) = \frac{1}{q}\left\{\frac{\partial\left(\sin\theta\dfrac{\partial L^*}{\partial q} + \dfrac{\cos\theta}{q}\dfrac{\partial L^*}{\partial\theta}, J^*P_1^*\right)}{\partial(q,\theta)}\right.$$
$$\left.+\frac{\partial\left(\cos\theta\dfrac{\partial L^*}{\partial q} - \dfrac{\sin\theta}{q}\dfrac{\partial L^*}{\partial\theta}, J^*P_2^*\right)}{\partial(q,\theta)}\right\},$$

$$W_3^*(q,\theta) = \frac{1}{q}\left\{\frac{\partial\left(\sin\theta\dfrac{\partial L^*}{\partial q} + \dfrac{\cos\theta}{q}\dfrac{\partial L^*}{\partial\theta}, J^{*2}R_1^*\right)}{\partial(q,\theta)}\right.$$
$$\left.+\frac{\partial\left(\cos\theta\dfrac{\partial L^*}{\partial q} - \dfrac{\sin\theta}{q}\dfrac{\partial L^*}{\partial\theta}, J^{*2}R_2^*\right)}{\partial(q,\theta)}\right\},$$

$$N_1^*(q,\theta) = \frac{1}{q}\frac{\partial\left(\cos\theta\dfrac{\partial L^*}{\partial q} - \dfrac{\sin\theta}{q}\dfrac{\partial L^*}{\partial\theta}, H^*\right)}{\partial(q,\theta)},$$

(4.21)

$$N_2^*(q,\theta) = \frac{1}{q}\frac{\partial\left(\sin\theta\dfrac{\partial L^*}{\partial q} + \dfrac{\cos\theta}{q}\dfrac{\partial L^*}{\partial\theta}, H^*\right)}{\partial(q,\theta)}$$

where $L^*(q,\theta) = L(u,v)$, $\omega^*(q,\theta) = \omega(u,v)$, $I^*(q,\theta) = I(u,v)$, and $J^*(q,\theta) = J(u,v)$.

Once $L^*(q,\theta)$ and $H^*(q,\theta)$ are determined, we use relations

$$x = \sin\theta\frac{\partial L^*}{\partial q} + \frac{\cos\theta}{q}\frac{\partial L^*}{\partial\theta}, \qquad y = \cos\theta\frac{\partial L^*}{\partial q} - \frac{\sin\theta}{q}\frac{\partial L^*}{\partial\theta} \tag{4.22}$$

to find the velocity components and the remaining flow variables in the physical plane.

## 5. Applications

In this section, we investigate some flow problems of physical interest as applications of the theorem.

### 5.1 Hyperbolic flow

Let

$$L(u,v) = A_1u^2 + B_1v^2 + C_1u + D_1v + E_1 \tag{5.1}$$

be the Legendre transform function where $A_1 \neq B_1 \neq 0$, $C_1$, $D_1$ and $E_1$ are arbitrary constants.

Substituting (5.1) in (4.6) – (4.9) we have

$$\overline{J} = \frac{1}{4A_1B_1}, \qquad \overline{\omega} = \frac{A_1+B_1}{2A_1B_1}, \qquad P_1 = 0, \quad P_2 = 0,$$

$$\overline{I} = \left(\frac{A_1-B_1}{2A_1B_1}\right)^{(n-1)}, \qquad Q_1 = 0, \quad Q_2 = 0, \qquad R_1 = 0, \quad R_2 = 0, \tag{5.2}$$

$$N_1 = 2A_1\frac{\partial\overline{H}}{\partial v}, \qquad N_2 = -2B_1\frac{\partial\overline{H}}{\partial u}.$$

*Finitely conducting fluid flow*: Substituting (5.2) in Eqs. (4.6) and (4.10), we find that (4.10) is satisfied identically and (4.6) simplifies to

$$A_1 u\frac{\partial\overline{H}}{\partial v} - B_1 v\frac{\partial\overline{H}}{\partial u} = \frac{\mu_e\sigma}{2A_1B_1}\left(B_1^2\frac{\partial^2\overline{H}}{\partial u^2} + A_1^2\frac{\partial^2\overline{H}}{\partial v^2}\right). \tag{5.3}$$

Solving this equation we get

$$\overline{H}(u,v) = A_1u^2 + B_1v^2, \qquad A_1 = -B_1, \tag{5.4}$$

and

$$L(u,v) = A_1(u^2-v^2) + C_1u + D_1v + E_1. \tag{5.5}$$

Substitution of (5.5) in (4.1) and (4.3) yields

$$u(x,y) = -\frac{(y+C_1)}{2A_1}, \qquad v(x,y) = -\frac{(x-D_1)}{2A_1}. \tag{5.6}$$

Using (5.6) in (5.4), (2.2), (2.3), (2.6) and (4.1), we get the flow variables $H(x,y)$, $I(x,y)$ and $p(x,y)$ in the physical plane in the following form:

$$H(x,y) = \frac{(y+C_1)^2-(x-D_1)^2}{4A_1^2}, \tag{5.7}$$

$$I(x,y) = \frac{(-1)^{(n-1)}}{A_1^{(n-1)}}, \tag{5.8}$$

$$p(x,y) = -\frac{\rho}{2}\left\{\frac{(x-D_1)^2+(y+C_1)^2}{4A_1^2}\right\} - \frac{\mu_e}{2}\left\{\frac{(y+C_1)^2-(x-D_1)^2}{4A_1^2}\right\}^2 + \pi_1 \tag{5.9}$$

where $\pi_1$ is an arbitrary constant.

*Infinitely conducting fluid flow*: In this case, Eq. (4.10) is again satisfied identically and Eq. (3.10) simplifies to

$$A_1u\frac{\partial\overline{H}}{\partial v} - B_1v\frac{\partial\overline{H}}{\partial u} = 0. \tag{5.10}$$

A general solution of Eq. (5.10) is $\overline{H}(u,v) = f(u^2+v^2)$, but without loss of generality, we can take

$$\overline{H}(u,v) = A_1u^2 + B_1v^2. \tag{5.11}$$

Proceeding as in the case of finitely conducting fluid flow, we obtain

$$u(x,y) = -\frac{(y+C_1)}{2A_1}, \qquad v(x,y) = \frac{(x-D_1)}{2B_1}, \tag{5.12}$$

$$H(x,y) = \frac{(x-D_1)^2}{4B_1} + \frac{(y+C_1)^2}{4A_1}, \tag{5.13}$$

$$I(x,y) = \frac{1}{2^{(n-1)}}\left(\frac{1}{A_1} + \frac{1}{B_1}\right)^{(n-1)}, \tag{5.14}$$

$$p(x,y) = -\frac{\rho}{2}\left\{\frac{(x-D_1)^2}{4B_1^2} + \frac{(y+C_1)^2}{4A_1^2}\right\} - \frac{\mu_e}{2}\left\{\frac{(x-D_1)^2}{4B_1} + \frac{(y+C_1)^2}{4A_1}\right\}^2 + \pi_2 \tag{5.15}$$

where $\pi_2$ is arbitrary constant.

If $L(u,v) = A_1u^2 + B_1v^2 + C_1u + D_1v + E_1$ is the Legendre transform function of a stream-function of a steady plane transverse flow of a finitely conducting non-Newtonian power-law fluid, then the flow variables are given by (5.6) – (5.9) and the stream-lines are hyperbolas

$$\frac{(x-D_1)^2}{4A_1} - \frac{(y+C_1)^2}{4A_1} = \text{const.} \tag{5.16}$$

If the fluid is infinitely conducting, the flow in the physical plane is given by (5.12) – (5.15) with stream-lines

$$\frac{(x-D_1)^2}{4B_1} + \frac{(y+C_1)^2}{4A_1} = \text{const.} \tag{5.17}$$

#### 5.2. Parabolic flow

Letting

$$L(u,v) = (A_2v + B_2u)u + C_2u + D_2, \tag{5.18}$$

where $A_2 \neq 0$, $B_2$, $C_2$ and $D_2$ are arbitrary constants, and using it in (4.7) – (4.9) we can obtain

$$\overline{J} = -\frac{1}{A_2^2}, \quad \overline{\omega} = -\frac{2B_2}{A_2^2}, \quad \overline{I} = (-2)^{(n-1)} \left( \frac{A_2^2 + B_2^2}{A_2^4} \right)^{\frac{(n-1)}{2}},$$

(5.19) $$P_1 = 0, \quad P_2 = 0, \qquad Q_1 = 0, \quad Q_2 = 0, \quad R_1 = 0, \quad R_2 = 0,$$

$$N_1 = 2B_2 \frac{\partial \overline{H}}{\partial v} - A_2 \frac{\partial \overline{H}}{\partial u}, \quad N_2 = A_2 \frac{\partial \overline{H}}{\partial v}.$$

*Finitely conducting fluid flows*: Results (5.19) satisfy Eq. (4.10) identically and simplify (4.6) to

(5.20) $$(2B_2 u + A_2 v) \frac{\partial \overline{H}}{\partial v} - A_2 u \frac{\partial \overline{H}}{\partial u} = -\frac{1}{\mu_e \sigma} \left\{ \frac{\partial^2 \overline{H}}{\partial u^2} - \frac{4B_2}{A_2} \frac{\partial^2 \overline{H}}{\partial u \partial v} + \left( 1 + \frac{4B_2^2}{A_2^2} \right) \frac{\partial^2 \overline{H}}{\partial v^2} \right\}.$$

Solving the above equation we get

(5.21) $$\overline{H}(u, v) = c_1 \int e^{\frac{(A_2 \sigma \mu_e u^2)}{2}} du + c_2$$

where $c_1$ and $c_2$ are arbitrary constants.

Proceeding as in the previous application, we find

(5.22) $$u(x, y) = \frac{x}{A_2}, \quad v(x, y) = -\left( \frac{2B_2}{A_2^2} x + \frac{y}{A_2} + \frac{C_2}{A_2} \right),$$

(5.23) $$H(x, y) = \frac{c_1}{A_2} \int e^{\frac{x^2}{2A_2 \sigma \mu_e}} dx + c_2,$$

(5.24) $$p(x, y) = -\frac{\rho}{2} \left\{ \frac{(x^2 + y^2) + (y + C_2)^2}{A_2^2} \right\} - \frac{\mu_e}{2} \left\{ \frac{c_1}{A_2} \int e^{\frac{x^2}{2A_2 \sigma \mu_e}} dx + c_2 \right\}^2 + \pi_3,$$

where $\pi_3$ is an arbitrary constant.

*Infinitely conducting fluid*: In this case, Eq. (4.10) is again satisfied identically and Eq. (5.19) takes the form

$$(5.25)\qquad (2B_2u + A_2v)\frac{\partial \overline{H}}{\partial v} - A_2u = 0.$$

A general solution of this equation is

$$(5.26)\qquad \overline{H}(u,v) = \Phi(B_2u^2 + A_2uv)$$

which, on substituting the velocity components given in (5.22), becomes

$$(5.27)\qquad H(x,y) = \Phi\left(-\frac{B_2}{A_2^2}x^2 - \frac{xy}{A_2} - \frac{C_2}{A_2}x\right),$$

where $\Phi$ is an arbitrary function of argument.

The pressure function is given by

$$(5.28)\qquad p(x,y) = -\frac{\rho}{2}\left\{\frac{(x^2+y^2)+(y+C_2)^2}{A_2^2}\right\} - \frac{\mu_e}{2}\left[\Phi\left(-\frac{B_2}{A_2^2}x^2 - \frac{xy}{A_2} - \frac{C_2}{A_2}\right)\right]^2 + \pi_4$$

where $\pi_4$ is an arbitrary constant.

If $L(u,v) = (A_2v + B_2u)u + C_2u + D_2$ is the Legendre transform function of a steady plane transverse flow of a non-Newtonian power-law fluid of finite conductivity, the flow variables are given by (5.22) – (5.24) and the streamlines are parabolic curves

$$(5.29)\qquad B_2x^2 + A_2xy + A_2C_2x = \text{const.}$$

In case of infinitely conducting fluid flow, Eqs. (5.22), (5.27) and (5.28) express the flow variables in the physical plane with streamlines given by (5.29).

### 5.3. Radial flow

Let

$$(5.30)\qquad L^*(q,\theta) = A_3\theta + B_3$$

where $A_3$ and $B_3$ are arbitrary constants in the Legendre transform function of stream-function.

Using (5.30) in (4.15) – (4.21), we obtain

$$(5.31)\quad J^* = -\frac{q^4}{A_3^2}, \quad \omega^* = 0, \quad I^* = \frac{2^{(n-1)} q^{2(n-1)}}{A_3^{(n-1)}}, \quad P_2^* = 0,$$

$$Q_1^* = \frac{A_3 I^{*'} \sin\theta}{q^2}, \qquad Q_2^* = \frac{A_3 I^{*'} \cos\theta}{q^2},$$

$$R_1^* = -\frac{A_3^2 I^{*'} \cos\theta}{q^4}, \quad R_2^* = \frac{A_3^2 I^{*'} \sin\theta}{q^4},$$

$$W_1^* = \frac{I^{*'}}{q} - \frac{1}{q^2}(q^2 I^{*'})', \quad W_2^* = 0, \quad W_3^* = 0,$$

$$N_1^* = \frac{A_3 \cos\theta}{q^2}\frac{\partial H^*}{\partial q} + \frac{A_3 \sin\theta}{q^3}\frac{\partial H^*}{\partial \theta}, \; N_2^* = \frac{A_3 \sin\theta}{q^2}\frac{\partial H^*}{\partial q} - \frac{A_3 \cos\theta}{q^3}\frac{\partial H^*}{\partial \theta}.$$

*Finitely conducting fluid flows*: Using results (5.31) in (4.13) and (4.14), we find that (4.14) is satisfied identically and (4.13) becomes

$$(5.32)\qquad \frac{\partial^2 H^*}{\partial q^2} + (1 + A_3\mu_e\sigma)\frac{1}{q}\frac{\partial H^*}{\partial q} + \frac{1}{q^2}\frac{\partial^2 H^*}{\partial \theta^2} = 0.$$

Assuming the solution of Eq. (5.32) in the form $H^*(q,\theta) = h_1(q) + h_2(\theta)$ we can obtain

$$(5.33)\qquad H^*(q,\theta) = c_3\left(\frac{1}{2}\theta^2 + \frac{1}{A_3\mu_e\sigma}\ln q\right) + c_4 q^{-A_3\mu_e\sigma} + c_5\theta + c_6,$$

where $c_3$, $c_4$, $c_5$, and $c_6$ are arbitrary constants.

Proceeding as in the previous applications, we obtain the flow variables in the physical plane in the form

$$(5.34)\qquad u(x,y) = \frac{A_3 x}{(x^2+y^2)}, \qquad v(x,y) = \frac{A_3 y}{(x^2+y^2)},$$

$$(5.35)\qquad H(x,y) = \frac{c_3}{2}\left\{\left(\tan^{-1}\frac{y}{x}\right)^2 + \frac{1}{A_3\mu_{e\sigma}}\ln(x^2+y^2)\right\}$$

$$+ c_4(x^2+y^2)^{-\frac{A_3\mu_e\sigma}{2}} + c_5\tan^{-1}\frac{y}{x} + c_6,$$

$$(5.36)\qquad I(x,y) = \frac{2^{(n-1)}}{A_3^{(n-1)}}(x^2+y^2)^{(n-1)},$$

$$p(x,y) = -\frac{2^n(n-1)K}{(n-2)A_3^{(n-2)}}(x^2+y^2)^{(n-2)} - \frac{\rho A_3^2}{2(x^2+y^2)} - \frac{\mu_e}{2}H^2 + \pi_4, \tag{5.37}$$

where $\pi_4$ is an arbitrary constant.

*Infinitely conducting fluid flows*: In this case, the diffusion equation reduces to

$$\frac{\partial H^*}{\partial q} = 0 \tag{5.38}$$

which yields

$$H^*(q,\theta) = \Psi(\theta), \tag{5.39}$$

where $\Psi$ is an arbitrary function of $\theta$.

The expressions of velocity components and $I(x,y)$ remain the same whereas $H(x,y)$ and $p(x,y)$ are given by

$$H(x,y) = \Psi\left(\tan^{-1}\frac{y}{x}\right), \tag{5.40}$$

$$p(x,y) = -\frac{2^n K(n-1)}{A_3^{(n-2)}(n-2)}\left(x^2+y^2\right)^{(n-2)} - \frac{\rho A_3^2}{2(x^2+y^2)} - \frac{\mu_e}{2}\left\{\Psi\left(\tan^{-1}\frac{y}{x}\right)\right\}^2 + \pi_5, \tag{5.41}$$

where $\pi_5$ is an arbitrary constant.

If $L^*(q,\theta) = A_3\theta + B_3$, $A_3 \neq 0$ is the Legendre transform function of a stream-function of a steady plane transverse flow of a finitely conducting non-Newtonian fluid of power-law model, then the Eqs. (5.34) – (5.37) express the flow variables in the physical plane. If the fluid conductivity is infinite, the flow variables are given by (5.34), (5.36), (5.40) and (5.41).

#### 5.4. Spiral flow

$$L^*(q,\theta) = A_4 \ln q + B_4\theta, \qquad A_4 \neq 0, \qquad B_4 \neq 0 \tag{5.42}$$

and proceeding as in the previous applications, we can obtain

$$J^* = -\frac{q^4}{(A_4^2+B_4^2)}, \quad \omega^* = 0, \quad I^*\frac{(-2)^{(n-1)}q^{(2n-2)}}{(A_4^2+B_4^2)^{\frac{(n-1)}{2}}}, \quad P_1^* = 0, \quad P_2^* = 0, \tag{5.43}$$

$$Q_1^* = -\frac{I^{*'}}{q^2}\left(A_4\cos\theta - B_4\sin\theta\right), \quad Q_2^* = \frac{I^{*'}}{q^2}\left(A_4\sin\theta - B_4\cos\theta\right),$$

$$R_1^* = -\frac{I^{*'}}{q^4}\left(A_4^2 + B_4^2\right)\cos\theta, \quad R_2^* = \frac{I^{*'}}{q^4}\left(A_4^2 + B_4^2\right)\sin\theta, \tag{5.43}$$

[cont.]

$$N_1^* = \frac{1}{q^2}\left(A_4\sin\theta + B_4\cos\theta\right)\frac{\partial H^*}{\partial q} - \frac{1}{q^3}\left(A_4\cos\theta - B_4\sin\theta\right)\frac{\partial H^*}{\partial \theta},$$

$$N_2^* = -\frac{1}{q^2}\left(A_4\cos\theta - B_4\sin\theta\right)\frac{\partial H^*}{\partial q} - \frac{1}{q^3}\left(A_4\sin\theta + B_4\cos\theta\right)\frac{\partial H^*}{\partial \theta}.$$

These equations together with (5.42) yield

$$W_1^* = 0, \quad W_2^* = 0, \quad W_3^* = \frac{A_4}{(A_4^2 + B_4^2)}\left\{\frac{(I^{*'}q^4)'}{q^2} - I^{*'}q\right\}. \tag{5.44}$$

Employing (5.43) and (5.44), Eq. (4.414) simplifies to

$$\frac{4A_4K(-2)^{(n-1)}n(n-1)}{(A_4^2 + B_4^2)^{\frac{(n+1)}{2}}}q^{(2n-2)} = 0. \tag{5.45}$$

From this equation it follows that a steady plane flow of a non-Newtonian power-law fluid cannot be spiral whether the fluid conductivity is finite or infinite. However, a spiral can exist in a steady plane viscous fluid ($n = 1$) whether it is conducting or non-conducting. In the case of finitely conducting fluid, Eq. (5.45) is identically satisfied and Eq. (4.13) becomes

$$\frac{\partial^2 H^*}{\partial q^2} + (1 + B_4\mu_e\sigma)\frac{1}{q}\frac{\partial H^*}{\partial q} - \frac{A_4\mu_e\sigma}{q^2}\frac{\partial H^*}{\partial \theta} + \frac{1}{q^2}\frac{\partial^2 H^*}{\partial \theta^2} = 0. \tag{5.46}$$

Assuming the solution of this equation in the form $H^*(q,\theta) = g_1(q) + g_2(\theta)$, where $g_1$ and $g_2$ are arbitrary functions, we obtain

$$H^*(q,\theta) = \frac{\lambda_1}{\mu_e\sigma}\left(\frac{\ln q}{B_4} + \frac{\theta}{A_4}\right) + d_1 q^{-B_4\sigma\mu_e} + d_2 e^{A_4\sigma\mu_e\theta} + d_3, \tag{5.47}$$

where $\lambda_1$, $d_1$, $d_2$ and $d_3$ are arbitrary constants.

Proceeding as in the previous applications, we can determine $u(x,y)$, $v(x,y)$ and $p(x,y)$ in the following form:

$$u(x,y) = \frac{(B_4x - A_4y)}{(x^2 + y^2)}, \qquad v(x,y) = \frac{(A_4x + B_4y)}{(x^2 + y^2)}, \tag{5.48}$$

$$p(x,y) = -\frac{\rho}{2}\frac{(A_4^2+B_4^2)}{(x^2+y^2)} - \frac{\mu_e}{2}\left\{\frac{\lambda_1}{\mu_e\sigma}\left(\frac{\ln q}{B_4}+\frac{\theta}{A_4}\right) + d_1 q^{-B_4\mu_e\sigma} + d_2 e^{A_4\mu_e\sigma\theta} + d_3\right\}^2 + \pi_6, \tag{5.49}$$

where $\pi_6$ is an arbitrary constant.

If the viscous fluid is infinitely conducting, the the diffusion Eq. (4.13) reduces to

$$\frac{B_4}{q}\frac{\partial H^*}{\partial q} - \frac{A_4}{q^2}\frac{\partial H^*}{\partial \theta} = 0. \tag{5.50}$$

A solution of this equation is

$$H^*(q,\theta) = \lambda_1\left(\frac{\ln q}{B_4}+\frac{\theta}{A_4}\right) + d_4, \tag{5.51}$$

where $d_4$ is an arbitrary constant.

The pressure function is given by

$$p(x,y) = -\frac{\rho}{2}\frac{(A_4^2+B_4^2)}{(x^2+y^2)} - \frac{\mu_e}{2}\left\{\lambda_1\left(\frac{\ln q}{B_4}+\frac{\theta}{A_4}\right) + d_4\right\}^2 + \pi_7, \tag{5.52}$$

where $\pi_7$ is an arbitrary constant.

## 6. Discussion

In each of the applications studied, the velocity components $\omega(x,y)$ and $I(x,y)$ are independent of the effect of electrical conductivity. Therefore, the solutions of the flow equations of steady plane flow of a non-Newtonian power-law non-conducting fluid can be derived as special cases of the problems investigated by discarding the term - $\mu_e H^2/2$ from the pressure functions. A steady plane flow of a non-Newtonian power-law fluid cannot be a spiral flow whether or not it is conducting. The solution of the problem of steady plane flow of an ordinary viscous fluid can be obtained by setting $n = 1$.

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