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Dimensional analysis and asymptotic expansions of the equilibrium equations in nonlinear elasticity Part II: The two-dimensional von Kármán model

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IN THE SECOND PART of this article, which is a continuation of [8] which dealt with a plate subjected to large loads, we consider a plate subjected to moderate applied forces level within the framework of nonlinear elasticity. We then apply the new constructive asymptotic approach developed in the first part which needs no *a priori* assumption. For these moderate forces, we prove that the two-dimensional model we obtain by asymptotic expansions is the von Kármán one. Finally the two-dimensional stress field in the plate is deduced from the three-dimensional constitutive equations without any *a priori* assumption.

1. Introduction

THIS PAPER IS A CONTINUATION of [8] to which we will refer for the definitions and the notations not explained here.

In the first part we have proved that for a plate subjected to large applied forces such as $\mathcal{F}_3 = \varepsilon^4$ and $\mathcal{G}_t = \mathcal{G}_3 = \varepsilon$, the asymptotic expansion of the three-dimensional equilibrium equations leads to the nonlinear membrane model. In this part we assume the plate to be subjected to the same moderate applied forces as in the linear case [6]: $\mathcal{F}_3 = \mathcal{G}_3 = \varepsilon^4$ and $\mathcal{G}_t = \varepsilon^3$. We recall that in the linear case such forces lead to the two-dimensional linear Kirchhoff-Love model.

The aim of this second part is now to prove, as in the first part, that the reference scales of the displacement and the corresponding two-dimensional models we obtain by asymptotic expansions are determined by the magnitude of forces. Indeed, we prove that the force magnitude considered here ($\mathcal{F}_3 = \mathcal{G}_3 = \varepsilon^4$ and $\mathcal{G}_t = \varepsilon^3$) leads to new reference scales ($u_{3r} = h_0$ and $V_r = \varepsilon h_0$) for the normal and tangential displacement. These new reference scales we obtain as a consequence of the forces applied, are formally equivalent to the scaling assumptions generally made in the literature [1–4, 9]. They naturally lead to the two-dimensional nonlinear "von Kármán model", even if the designation "von Kármán" is often reserved for particular boundary conditions. Finally, the two-dimensional stress field in the plate is deduced from the three-dimensional constitutive equations without any *a priori* assumption.

2. Determination of the reference scales

Let us consider now the same applied forces level as in the linear case [6], such as $\mathcal{G}_t = \varepsilon^3$, $\mathcal{F}_t = 0$ and $\mathcal{G}_3 = \mathcal{F}_3 = \varepsilon^4$. Then for these forces, the variational formulation of the membrane equations obtained in the first part of this article (result 2 of [8]) becomes:

$$\psi^0 \in \mathcal{Q}(\omega_0) = \Big\{\psi : \omega_0 \mapsto \mathbb{R}^3, \quad \text{“smooth”}, \quad \psi = I_2 \quad \text{on } \gamma_0\Big\},$$

$$\int\limits_{\omega_0} \mathrm{Tr}\left[N_t^0 \overline{\frac{\partial \psi^0}{\partial x}} \, \frac{\partial v^0}{\partial x}\right] d\omega_0 = 0 \,, \qquad \forall v^0 \in V(\omega_0).$$

The solutions of this membrane problem without a right-hand term are the smooth enough inextensional, mapping functions defined on ω_0 ([5]). So we have $\psi^0 \in \mathcal{S}_0$ where $\mathcal{S}_0 = \Big\{\psi^0 : \omega_0 \to \mathbb{R}^3 \;;\; \overline{\frac{\partial \psi^0}{\partial x}} \frac{\partial \psi^0}{\partial x} = I_2\Big\}$ denotes the space of the inextensional mapping functions defined on ω_0. On the other hand, as the plate is assumed to be clamped on its lateral surface, the space of the inextensional mapping fuctions $\mathcal{S}_0$ reduces to the identity I_2 of $\mathbb{R}^2$. Hence we have $\psi^0 = I_2$ which implies that $V^0 = u_3^0 = 0$ (see relation (4.11) of [8]).

Since we have proved that $V^0 = u_3^0 = 0$, we get

$$V = \frac{V^*}{V_r} = \frac{V^*}{L_0} = \varepsilon V^1 + \varepsilon^2 V^2 + \cdots \,,$$

$$u_3 = \frac{u_3^*}{u_{3r}} = \frac{u_3^*}{L_0} = \varepsilon u_3^1 + \varepsilon^2 u_3^2 + \cdots \,,$$

which is equivalent to

$$\widetilde{V} = \frac{V^*}{\varepsilon V_r} = \frac{V^*}{h_0} = V^1 + \varepsilon V^2 + \cdots = \widetilde{V^0} + \varepsilon \widetilde{V^1} + \varepsilon^2 \widetilde{V^2} + \cdots \,,$$

$$\widetilde{u_3} = \frac{u_3^*}{\varepsilon u_{3r}} = \frac{u_3^*}{h_0} = u_3^1 + \varepsilon u_3^2 + \cdots = \widetilde{u_3^0} + \varepsilon \widetilde{u_3^1} + \varepsilon^2 \widetilde{u_3^2} + \cdots \,.$$

Hence for these forces, the reference scales of the tangential displacement $V_r = L_0$ and of the normal displacement $u_{3r} = L_0$ are not properly chosen. $\widetilde{V}$ and $\widetilde{u_3}$ will be of the order of one unit provided the reference scales of the displacement satisfy the condition $V_r = u_{3r} = h_0$. Therefore the new reference scales of the tangential displacement V^* and of the normal displacement u_3^* we have to consider are $V_r = u_{3r} = h_0$.

Consequently, the dimensionless equilibrium equations are written again with $V_r = u_{3r} = h_0$ as reference scales. The dimensionless components of the displacement will still be denoted by V and u_3. Thus for the magnitude of forces such as $\mathcal{F}_3 = \mathcal{G}_3 = \varepsilon^4$, $\mathcal{F}_t = 0$ and $\mathcal{G}_t = \varepsilon^3$, the dimensionless equilibrium Eqs. (2.4) and (2.5) of [8] become in $\Omega_0 = \omega_0 \times]-1,1[$:

$$\varepsilon^2\left[(1+\beta)\mathrm{grad}(\mathrm{div}V) + \Delta V\right] + \varepsilon\left[(1+\beta)\mathrm{grad}\frac{\partial u_3}{\partial x_3} + \mathrm{div}\,(E_{ut} + \Gamma_t)\right] + \frac{\partial}{\partial x_3}(Q + \Gamma_s) + \frac{\partial^2 V}{\partial x_3^2} = 0\,,$$

(2.1)

$$\varepsilon^2 \Delta u_3 + \varepsilon\left[(1+\beta)\frac{\partial}{\partial x_3}\mathrm{div}\;V + \mathrm{div}\,(E_{us} + \Gamma_s)\right] + \frac{\partial}{\partial x_3}(E_{un} + \Gamma_n) + (2+\beta)\frac{\partial^2 u_3}{\partial x_3^2} = -\varepsilon^4 f_3\,.$$

The boundary conditions on the upper and the lower faces become:

$$\varepsilon\;\mathrm{grad}\,u_3 + Q + \Gamma_s + \frac{\partial V}{\partial x_3} = \pm\varepsilon^3 g_t^{\pm} \quad \text{for } x_3 = \pm 1\,,$$

(2.2)

$$\varepsilon\;\beta\,\mathrm{div}\;V + E_{un} + \Gamma_n + (2+\beta)\frac{\partial u_3}{\partial x_3} = \pm\varepsilon^4 g_3^{\pm} \quad \text{for } x_3 = \pm 1\,,$$

where $\beta = \lambda/\mu$ and

$$\Gamma_t = \varepsilon^2\left[\frac{\beta}{2}\mathrm{Tr}\left(\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right) I_2 + \frac{\beta}{2}\|\mathrm{grad}\;u_3\|^2 I_2 + \overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x} + \mathrm{grad}\;u_3\;\overline{\mathrm{grad}\;u_3}\right] + \frac{\beta}{2}\left[\left\|\frac{\partial V}{\partial x_3}\right\|^2 + \left(\frac{\partial u_3}{\partial x_3}\right)^2\right] I_2\,,$$

$$\Gamma_s = \varepsilon\left[\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x_3} + \frac{\partial u_3}{\partial x_3}\mathrm{grad}\;u_3\right],$$

$$\Gamma_n = \varepsilon^2\left[\frac{\beta}{2}\mathrm{Tr}\left(\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right) + \frac{\beta}{2}\|\mathrm{grad}\;u_3\|^2\right] + \left(1 + \frac{\beta}{2}\right)\left[\left\|\frac{\partial V}{\partial x_3}\right\|^2 + \left(\frac{\partial u_3}{\partial x_3}\right)^2\right],$$

$$E_{ut} = \varepsilon^3\left[\frac{\beta}{2}\mathrm{Tr}\left(\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right) I_2 + \overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x} + \frac{\beta}{2}\|\mathrm{grad}\;u_3\|^2\;I_2 + \;\mathrm{grad}\;u_3\overline{\mathrm{grad}\;u_3}\right]\overline{\frac{\partial V}{\partial x}}$$

[cont.]

$$+\varepsilon^2\left[\beta\,\mathrm{div}V\;I_2+\frac{\partial V}{\partial x}+\overline{\frac{\partial V}{\partial x}}\right]\overline{\frac{\partial V}{\partial x}}+\varepsilon\left\{\left(1+\frac{\partial u_3}{\partial x_3}\right)\mathrm{grad}\;u_3\;\overline{\frac{\partial V}{\partial x_3}}\right.$$

$$\left.+\beta\frac{\partial u_3}{\partial x_3}\overline{\frac{\partial V}{\partial x}}+\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x_3}\overline{\frac{\partial V}{\partial x_3}}+\frac{\beta}{2}\left[\left\|\frac{\partial V}{\partial x_3}\right\|^2+\left(\frac{\partial u_3}{\partial x_3}\right)^2\right]\overline{\frac{\partial V}{\partial x}}\right\}+\frac{\partial V}{\partial x_3}\overline{\frac{\partial V}{\partial x_3}},$$

$$E_{us}=\varepsilon^3\left[\frac{\beta}{2}\mathrm{Tr}\left(\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right)I_2+\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}+\left(1+\frac{\beta}{2}\right)\|\mathrm{grad}\;u_3\|^2\;I_2\right]\mathrm{grad}\;u_3$$

$$\varepsilon^2\left[\beta\,\mathrm{div}V\;I_2+\frac{\partial V}{\partial x}+\overline{\frac{\partial V}{\partial x}}\right]\mathrm{grad}\;u_3+\varepsilon\left\{\left[\frac{\beta}{2}\left\|\frac{\partial V}{\partial x_3}\right\|^2+\left(1+\frac{\beta}{2}\right)\right.\right.$$

$$\left.\left.\times\left(\frac{\partial u_3}{\partial x_3}\right)^2+(1+\beta)\frac{\partial u_3}{\partial x_3}\right]\mathrm{grad}\;u_3+\frac{\partial u_3}{\partial x_3}\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x_3}\right\}+\frac{\partial u_3}{\partial x_3}\frac{\partial V}{\partial x_3},$$

$$Q=\varepsilon^2\left\{\left(1+\frac{\partial u_3}{\partial x_3}\right)\frac{\partial V}{\partial x}\;\mathrm{grad}\;u_3+\left[\frac{\beta}{2}\mathrm{Tr}\left(\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right)I_2+\frac{\partial V}{\partial x}\overline{\frac{\partial V}{\partial x}}\right.\right.$$

$$\left.\left.+\frac{\beta}{2}\|\mathrm{grad}\;u_3\|^2\;I_2\right]\frac{\partial V}{\partial x_3}\right\}+\varepsilon\left[\beta\mathrm{div}V\;I_2+\frac{\partial V}{\partial x}\right]\frac{\partial V}{\partial x_3}$$

$$+\left(1+\frac{\beta}{2}\right)\left[\left\|\frac{\partial V}{\partial x_3}\right\|^2+\left(\frac{\partial u_3}{\partial x_3}\right)^2+2\frac{\partial u_3}{\partial x_3}\right]\frac{\partial V}{\partial x_3},$$

$$E_{un}=\varepsilon^2\left\{\left[\frac{\beta}{2}\mathrm{Tr}\left(\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right)+\frac{\beta}{2}\|\mathrm{grad}\;u_3\|^2\right]\frac{\partial u_3}{\partial x_3}+\left(1+\frac{\partial u_3}{\partial x_3}\right)\|\mathrm{grad}\;u_3\|^2\right.$$

$$\left.+\overline{\frac{\partial V}{\partial x_3}}\frac{\partial V}{\partial x}\mathrm{grad}\;u_3\right\}+\varepsilon\left\{\beta\,\mathrm{div}V\;\frac{\partial u_3}{\partial x_3}+\overline{\frac{\partial V}{\partial x_3}}\;\mathrm{grad}\;u_3\right\}$$

$$+\left(1+\frac{\beta}{2}\right)\left[\left\|\frac{\partial V}{\partial x_3}\right\|^2+\left(\frac{\partial u_3}{\partial x_3}\right)^2+2\frac{\partial u_3}{\partial x_3}\right]\frac{\partial u_3}{\partial x_3}.$$

We assume again that there exists a formal expansion with respect to ε of the new dimensionless solution (V, u_3) of the dimensionless problem (2.1)–(2.2):

$$V = V^0 + \varepsilon V^1 + \varepsilon^2 V^2 + \cdots ,$$
$$u_3 = u_3^0 + \varepsilon u_3^1 + \varepsilon^2 u_3^2 + \cdots . \tag{2.3}$$

We then obtain the following result:

RESULT 1

For applied forces magnitude such as $\mathcal{F}_3 = \mathcal{G}_3 = \varepsilon^4$ and $\mathcal{G}_t = \varepsilon^3$, we have $V^0 = 0$.

P r o o f. The proof of this result is divided into two steps.

i) u_3^0 and V^0 depend on (x_1, x_2) only.

Replacing V and u_3 by their expansions (2.3) in the dimensionless equilibrium Eqs. (2.1) and (2.2), problem $\mathcal{P}_0$ can be written in $\Omega_0 = \omega_0 \times] - h_0, h_0[$ in the form

$$\frac{\partial}{\partial x_3}(Q^0 + \Gamma_s^0) + \frac{\partial^2 V^0}{\partial x_3^2} = 0 , \tag{2.4}$$

$$\frac{\partial}{\partial x_3}(\Gamma_n^0 + E_{un}^0) + (2+\beta)\frac{\partial^2 u_3^0}{\partial x_3^2} = 0 , \tag{2.5}$$

with the boundary conditions on the upper and the lower faces

$$Q^0 + \Gamma_s^0 + \frac{\partial V^0}{\partial x_3} = 0 \quad \text{for } x_3 = \pm 1 , \tag{2.6}$$

$$\Gamma_n^0 + E_{un}^0 + (2+\beta)\frac{\partial u_3^0}{\partial x_3} = 0 \quad \text{for } x_3 = \pm 1 , \tag{2.7}$$

and where

$$Q^0 = \left(1 + \frac{\beta}{2}\right)\left[\left\|\frac{\partial V^0}{\partial x_3}\right\|^2 + \left(\frac{\partial u_3^0}{\partial x_3}\right)^2 + 2\frac{\partial u_3^0}{\partial x_3}\right]\frac{\partial V^0}{\partial x_3} ,$$

$$E_{un}^0 = \left(1 + \frac{\beta}{2}\right)\left[\left\|\frac{\partial V^0}{\partial x_3}\right\|^2 + \left(\frac{\partial u_3^0}{\partial x_3}\right)^2 + 2\frac{\partial u_3^0}{\partial x_3}\right]\frac{\partial u_3^0}{\partial x_3} ,$$

$$\Gamma_n^0 = \left(1+\frac{\beta}{2}\right)\left[\left\|\frac{\partial V^0}{\partial x_3}\right\|^2 + \left(\frac{\partial u_3^0}{\partial x_3}\right)^2\right],$$

$$\Gamma_s^0 = 0\,.$$

It is then possible to introduce the mapping function

$$\psi^1 = V^0 + (x_3 + u_3^0)e_3 = \begin{pmatrix} V^0 \\ x_3 + u_3^0 \end{pmatrix}$$

to simplify the expressions of Q^0 and E_{un}^0. So we have:

$$Q^0 = \left(1+\frac{\beta}{2}\right)\left(\left\|\frac{\partial \psi^1}{\partial x_3}\right\|^2 - 1\right)\frac{\partial V^0}{\partial x_3}\,,$$

$$E_{un}^0 = \left(1+\frac{\beta}{2}\right)\left(\left\|\frac{\partial \psi^1}{\partial x_3}\right\|^2 - 1\right)\frac{\partial u_3^0}{\partial x_3}\,.$$

REMARK 1

It is natural to introduce ψ^1 as a function of (V^0, u_3^0). Indeed, going back to the mapping function ψ^*, we get:

$$\psi^* = I + U^* = x_\alpha^*\, e_\alpha + V^* + (x_3^* + u_3^*)\, e_3.$$

Taking into account the expansion of (V, u_3) with respect to ε, the dimensionless form of the previous equation with $V_r = u_{3r} = h_0$ can be written as

$$\psi = \frac{\psi^*}{L} = \underbrace{x_\alpha\, e_\alpha}_{\psi^0 = I_2} + \varepsilon\,\Big[\underbrace{(x_3 + u_3^0)e_3 + V^0}_{\psi^1}\Big] + \cdots\,. \tag{2.8}$$

Thus we obtain $\psi^0 = I_2$ which complies with the new reference scales of the displacements. ■

According to the boundary conditions (2.7), Eqs. (2.5) leads to

$$\Gamma_n^0 + E_{un}^0 + (2+\beta)\frac{\partial u_3^0}{\partial x_3} = 0 \qquad \text{in } \Omega_0\,,$$

which can also be written as

$$\left(1 + \frac{\partial u_3^0}{\partial x_3}\right)\left(\left\|\frac{\partial \psi^1}{\partial x_3}\right\|^2 - 1\right) = 0 \qquad \text{in } \Omega_0\,.$$

Consequently we obtain:

(2.9)
$$\frac{\partial u_3^0}{\partial x_3} = -1, \qquad \text{or}$$
$$\left\| \frac{\partial \psi^1}{\partial x_3} \right\| = 1.$$

We must use again the mass conservation law to conclude the analysis. Indeed, taking into account the new reference scales of the displacements, the expansion of the condition (3.3) of [8] gives

$$\frac{\partial \psi^1}{\partial x_3} . \left(\frac{\partial \psi^0}{\partial x_1} \wedge \frac{\partial \psi^0}{\partial x_2} \right) > 0 \, .$$

As $\psi^0 = I_2$, we get $\dfrac{\partial \psi^1}{\partial x_3} . e_3 > 0$ which becomes

(2.10)
$$\left(1 + \frac{\partial u_3^0}{\partial x_3} \right) > 0 \qquad \text{in } \Omega_0 \, ,$$

according to the expression of ψ^1. Hence Eq. (2.9) gives

(2.11)
$$\left\| \frac{\partial \psi^1}{\partial x_3} \right\| = 1 \, .$$

Finally, according to the boundary conditions (2.6) and to Eq. (2.11), Eq. (2.4) becomes

$$\frac{\partial V^0}{\partial x_3} = 0 \qquad \text{in } \Omega_0$$

and yields

(2.12)
$$V^0 = V^0(x_1, x_2) \qquad \text{in } \Omega_0 \, .$$

Therefore equation (2.11) becomes

$$\left(\frac{\partial u_3^0}{\partial x_3} \right)^2 + 2 \, \frac{\partial u_3^0}{\partial x_3} = 0$$

and according to the condition (2.10) we have $\dfrac{\partial u_3^0}{\partial x_3} = 0$, or equivalently

(2.13)
$$u_3^0 = u_3^0(x_1, x_2) \qquad \text{in } \Omega_0 \, .$$

ii) Equation satisfied by V^0.

Taking into account Eqs. (2.12) and (2.13), we have $E^0_{us} = 0$. Thus the second equation and the second boundary condition of problem $\mathcal{P}_1$ can be written as:

$$\frac{\partial}{\partial x_3}(\Gamma^1_n + E^1_{un}) + (2+\beta)\frac{\partial^2 u^1_3}{\partial x_3^2} = 0 \qquad \text{in } \Omega_0, \tag{2.14}$$

$$\Gamma^1_n + E^1_{un} + (2+\beta)\frac{\partial u^1_3}{\partial x_3} + \beta \operatorname{div} V^0 = 0 \qquad \text{for } x_3 = \pm 1 . \tag{2.15}$$

These two equations lead to a relation which couples u^1_3 and V^0 :

$$\frac{\partial u^1_3}{\partial x_3} = -\frac{\beta}{2+\beta} \operatorname{div} V^0 . \tag{2.16}$$

The cancellation of the coefficient of ε^2 leads to the problem $\mathcal{P}_2$ whose first equation and first boundary condition can be written in the form

$$(1+\beta)\operatorname{grad}(\operatorname{div} V^0) + \Delta V^0 + (1+\beta)\operatorname{grad}\frac{\partial u^1_3}{\partial x_3} + \frac{\partial}{\partial x_3}(Q^2 + \Gamma^2_s) + \frac{\partial^2 V^2}{\partial x_3^2} = 0 , \tag{2.17}$$

$$\operatorname{grad} u^1_3 + Q^2 + \Gamma^2_s + \frac{\partial V^2}{\partial x_3} = 0 \qquad \text{for } x_3 = \pm 1 . \tag{2.18}$$

Using the boundary condition (2.18), the integration from -1 to 1 of Eq. (2.17) gives:

$$\int_{-1}^{1} \left[(1+\beta)\operatorname{grad}(\operatorname{div} V^0) + \Delta V^0 + \beta \operatorname{grad}\left(\frac{\partial u^1_3}{\partial x_3}\right)\right] dx_3 = 0 .$$

Then replacing $\partial u^1_3/\partial x_3$ by expression (2.16), the previous equation becomes

$$\operatorname{div}(n^0_t) = 0$$

with

$$n^0_t = \frac{4\beta}{2+\beta}\operatorname{Tr} e^0_t(V^0)\, I_2 + 4 e^0_t(V^0) ,$$

$$e^0_t(V^0) = \frac{1}{2}\left(\frac{\partial V^0}{\partial x} + \overline{\frac{\partial V^0}{\partial x}}\right) = \frac{1}{2}\left(\operatorname{grad} V^0 + \overline{\operatorname{grad} V^0}\right) .$$

As the plate is assumed to be clamped on its lateral surface, we obtain a linear boundary problem:

$$\begin{aligned} \operatorname{div}(n_t^0) &= 0 \qquad \text{in } \omega_0 , \\ V^0|_{\partial\omega_0} &= 0 , \end{aligned}$$

which has (if we assume that V^0 is smooth enough) a unique solution $V^0 = 0$ in $\left[H_0^1(\omega_0)\right]^2$.

Therefore for these forces level, the reference scale of the tangential displacement $V_r = h_0$ is still not properly chosen. We have to consider $V_r = \varepsilon h_0$. Thus the dimensionless equilibrium equations must be written again with $u_{3r} = h_0$ and $V_r = \varepsilon h_0$ as the reference scales. The dimensionless components of the displacement will still be denoted by V and u_3.

3. The two-dimensional von Kármán model

With the new reference scales of the displacement ($u_{3r} = h_0$ and $V_r = \varepsilon h_0$) which are obtained as a consequence of the order of the forces applied, the dimensionless equilibrium equations assume in $\Omega_0 = \omega_0 \times]-1, 1[$ the form

$$(3.1) \qquad \begin{aligned} &\varepsilon^2[(1+\beta)\operatorname{grad}\operatorname{div}V + \Delta V] + (1+\beta)\operatorname{grad}\frac{\partial u_3}{\partial x_3} + \frac{\partial^2 V}{\partial x_3^2} \\ &\qquad + \operatorname{div}(E_{ut} + \Gamma_t) + \frac{1}{\varepsilon}\frac{\partial}{\partial x_3}(Q + \Gamma_s) = 0 , \\ &\varepsilon^2[(1+\beta)\frac{\partial}{\partial x_3}\operatorname{div}V + \Delta u_3] + \varepsilon \operatorname{div}(E_{us} + \Gamma_s) + (2+\beta)\frac{\partial^2 u_3}{\partial x_3^2} \\ &\qquad + \frac{\partial}{\partial x_3}(E_{un} + \Gamma_n) = -\varepsilon^4 f_3 . \end{aligned}$$

The boundary conditions on the upper and the lower faces $\Gamma_{0\pm}$ become:

$$(3.2) \qquad \begin{aligned} \operatorname{grad} u_3 + \frac{\partial V}{\partial x_3} + \frac{1}{\varepsilon}(Q + \Gamma_s) &= \pm\varepsilon^2 g_t^{\pm} \qquad \text{for } x_3 = \pm 1 , \\ (2+\beta)\frac{\partial u_3}{\partial x_3} + \beta\varepsilon^2 \operatorname{div}V + E_{un} + \Gamma_n &= \pm\varepsilon^4 g_3^{\pm} \qquad \text{for } x_3 = \pm 1 , \end{aligned}$$

where

$$\Gamma_t = \varepsilon^4\left\{\frac{\beta}{2}\mathrm{Tr}\left(\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right)I_2 + \overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right\} + \varepsilon^2\left\{\frac{\beta}{2}\left(\|\mathrm{grad}\ u_3\|^2 + \left\|\frac{\partial V}{\partial x_2}\right\|^2\right)I_2 + \mathrm{grad}\ u_3\ \overline{\mathrm{grad}\ u_3}\right\} + \frac{\beta}{2}\left(\frac{\partial u_3}{\partial x_3}\right)^2 I_2\ ,$$

$$\Gamma_s = \varepsilon^3\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x_3} + \varepsilon\frac{\partial u_3}{\partial x_3}\mathrm{grad}\ u_3\ ,$$

$$\Gamma_n = \varepsilon^4\ \frac{\beta}{2}\mathrm{Tr}\left(\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right) + \varepsilon^2\left\{\frac{\beta}{2}\|\mathrm{grad}\ u_3\|^2 + \left(1+\frac{\beta}{2}\right)\left\|\frac{\partial V}{\partial x_3}\right\|^2\right\} + \left(1+\frac{\beta}{2}\right)\left(\frac{\partial u_3}{\partial x_3}\right)^2,$$

$$E_{ut} = \varepsilon^6\left\{\frac{\beta}{2}\mathrm{Tr}\left(\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right)\overline{\frac{\partial V}{\partial x}} + \overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\overline{\frac{\partial V}{\partial x}}\right\} + \varepsilon^4\left\{\left[\beta\ \mathrm{div}V\ I_2 + \frac{\beta}{2}\left(\left\|\frac{\partial V}{\partial x_3}\right\|^2 + \|\mathrm{grad}\ u_3\|^2\ \right)I_2 + \left(\frac{\partial V}{\partial x} + \overline{\frac{\partial V}{\partial x}}\right) + \mathrm{grad}\ u_3\ \overline{\mathrm{grad}\ u_3}\right]\overline{\frac{\partial V}{\partial x}} + \overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x_3}\overline{\frac{\partial V}{\partial x_3}}\right\}$$
$$+\ \varepsilon^2\left\{\left[\frac{\partial V}{\partial x_3} + \mathrm{grad}\ u_3 + \frac{\partial u_3}{\partial x_3}\mathrm{grad}\ u_3\right]\overline{\frac{\partial V}{\partial x_3}} + \left[\beta\frac{\partial u_3}{\partial x_3} + \frac{\beta}{2}\left(\frac{\partial u_3}{\partial x_3}\right)^2\right]\overline{\frac{\partial V}{\partial x}}\right\},$$

$$E_{us} = \varepsilon^5\left\{\frac{\beta}{2}\mathrm{Tr}\left(\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right)I_2 + \overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x}\right\}\mathrm{grad}\ u_3 + \varepsilon^3\left\{\left[\beta\mathrm{div}\ V\ I_2 + \frac{\beta}{2}\left\|\frac{\partial V}{\partial x_3}\right\|^2 I_2 + \left(1+\frac{\beta}{2}\right)\|\mathrm{grad}\ u_3\|^2\ I_2 + \left(\frac{\partial V}{\partial x} + \overline{\frac{\partial V}{\partial x}}\right)\right]\mathrm{grad}\ u_3 + \frac{\partial u_3}{\partial x_3}\overline{\frac{\partial V}{\partial x}}\frac{\partial V}{\partial x_3}\right\}$$
$$+\ \varepsilon\left\{\left[\left(1+\frac{\beta}{2}\right)\left(\frac{\partial u_3}{\partial x_3}\right)^2 + (1+\beta)\frac{\partial u_3}{\partial x_3}\right]\mathrm{grad}\ u_3 + \frac{\partial u_3}{\partial x_3}\frac{\partial V}{\partial x_3}\right\},$$

$$Q = \varepsilon^5 \left\{ \frac{\beta}{2} \mathrm{Tr} \left(\overline{\frac{\partial V}{\partial x}} \frac{\partial V}{\partial x} \right) \frac{\partial V}{\partial x_3} + \frac{\partial V}{\partial x} \overline{\frac{\partial V}{\partial x}} \frac{\partial V}{\partial x_3} \right\} + \varepsilon^3 \left\{ \left[\beta \, \mathrm{div}\, V \; I_2 \right. \right.$$

$$\left. + \frac{\beta}{2} \| \mathrm{grad}\; u_3 \|^2 dis + \left(1 + \frac{\beta}{2} \right) \left\| \frac{\partial V}{\partial x_3} \right\|^2 \right] \frac{\partial V}{\partial x_3} + \frac{\partial V}{\partial x} \left[\frac{\partial V}{\partial x_3} + \mathrm{grad}\; u_3 \right.$$

$$\left. \left. + \frac{\partial u_3}{\partial x_3} \mathrm{grad}\; u_3 \right] \right\} + \varepsilon \left[\left(1 + \frac{\beta}{2} \right) \left(\frac{\partial u_3}{\partial x_3} \right)^2 + (2 + \beta) \frac{\partial u_3}{\partial x_3} \right] \frac{\partial V}{\partial x_3} ,$$

$$E_{un} = \varepsilon^4 \left\{ \frac{\beta}{2} \mathrm{Tr} \left(\overline{\frac{\partial V}{\partial x}} \frac{\partial V}{\partial x} \right) \frac{\partial u_3}{\partial x_3} + \overline{\frac{\partial V}{\partial x_3}} \frac{\partial V}{\partial x} \mathrm{grad}\; u_3 \right\}$$

$$+ \varepsilon^2 \left\{ \left[\beta \, \mathrm{div}\, V + \left(1 + \frac{\beta}{2} \right) \| \mathrm{grad}\; u_3 \|^2 + \left(1 + \frac{\beta}{2} \right) \left\| \frac{\partial V}{\partial x_3} \right\|^2 \right] \frac{\partial u_3}{\partial x_3} \right.$$

$$\left. + \overline{\frac{\partial V}{\partial x_3}} \mathrm{grad}\, u_3 + \| \mathrm{grad}\; u_3 \|^2 \right\} + \left[(2 + \beta) \frac{\partial u_3}{\partial x_3} + \left(1 + \frac{\beta}{2} \right) \left(\frac{\partial u_3}{\partial x_3} \right)^2 \right] \frac{\partial u_3}{\partial x_3} .$$

The asymptotic expansion method enables us to write again the new dimensionless solution (V, u_3) of the problem (3.1) – (3.2) as a formal expansion with respect to ε :

$$\begin{aligned} V &= V^0 + \varepsilon V^1 + \varepsilon^2 V^2 + \cdots , \\ u_3 &= u_3^0 + \varepsilon u_3^1 + \varepsilon^2 u_3^2 + \cdots . \end{aligned} \tag{3.3}$$

Then replacing V and u_3 by their expansions in equations (3.1) – (3.2) and equating to zero the coefficients of the powers of ε, we obtain again a chain of coupled problems.

RESULT 2

For the applied given forces of the magnitude such as $\mathcal{F}_3 = \mathcal{G}_3 = \varepsilon^4$ and $\mathcal{G}_t = \varepsilon^3$, the leading term (V^0, u_3^0) of the expansion of (V, u_3) is a Kirchhoff-Love displacement which satisfies the following conditions

i) $u_3^0 = \zeta_3^0(x_1, x_2)$, $\qquad V^0 = \zeta_t^0(x_1, x_2) - x_3 \; \mathrm{grad}\; \zeta_3^0$;

ii) $\zeta^0 = (\zeta_t^0, \zeta_3^0)$ is a solution of the following problem:

$$\operatorname{div} n_t^0 = -p_t \quad \text{in } \omega_0 \ ,$$

$$\operatorname{div}(\operatorname{div} m_t^0) + \operatorname{div}(n_t^0 \operatorname{grad} u_3^0) = -p_3 - \operatorname{div} M_t \quad \text{in} \quad \omega_0 \ ,$$

$$\zeta_3^0 = \frac{\partial \zeta_3^0}{\partial \nu} = 0 \quad \text{and} \quad \zeta_t^0 = 0 \quad \text{on } \gamma_0 \ ,$$

where ν denotes the external unit normal to γ_0, and where

$$n_t^0 = \frac{4\beta}{2+\beta} \operatorname{Tr} E_t^0(\zeta^0) I_2 + 4 E_t^0(\zeta^0) \ ,$$

$$E_t^0(\zeta^0) = \frac{1}{2}\Big(\operatorname{grad}\zeta_t^0 + \overline{\operatorname{grad}\zeta_t^0} + \operatorname{grad}\zeta_3^0 \ \overline{\operatorname{grad}\zeta_3^0}\Big) \ ,$$

$$m_t^0 = -\Big\{\frac{4\beta}{3(2+\beta)} \Delta u_3^0 I_2 + \frac{4}{3} \operatorname{grad}(\operatorname{grad} u_3^0)\Big\} \ ,$$

$$p_3 = \int_{-1}^{1} f_3 dx_3 + g_3^+ + g_3^- \ , \qquad p_t = g_t^+ + g_t^- \ , \qquad M_t = g_t^+ - g_t^- \ .$$

P r o o f. The proof of this result is split into several steps from i) to iii).

i) u_3^0 et V^0 is a Kirchhoff-Love displacement.

The cancellation of the factor of ε^0 leads to the problem $\mathcal{P}_0$:

(3.4) $$(1+\beta) \operatorname{grad} \frac{\partial u_3^0}{\partial x_3} + \frac{\partial^2 V^0}{\partial x_3^2} + \frac{\partial}{\partial x_3}(Q^1 + \Gamma_s^1) + \operatorname{div}(\Gamma_t^0) = 0 \quad \text{in } \Omega_0 \ ,$$

(3.5) $$(2+\beta) \frac{\partial^2 u_3^0}{\partial x_3^2} + \frac{\partial}{\partial x_3}(E_{un}^0 + \Gamma_n^0) = 0 \quad \text{in } \Omega_0 \ ,$$

(3.6) $$\frac{\partial V^0}{\partial x_3} + \operatorname{grad} u_3^0 + Q^1 + \Gamma_s^1 = 0 \quad \text{for } x_3 = \pm 1 \ ,$$

(3.7) $$(2+\beta) \frac{\partial u_3^0}{\partial x_3} + E_{un}^0 + \Gamma_n^0 = 0 \quad \text{for } x_3 = \pm 1 \ ,$$

with

$$Q^1 = \left[\left(1+\frac{\beta}{2}\right)\left(\frac{\partial u_3^0}{\partial x_3}\right)^2 + (2+\beta)\frac{\partial u_3^0}{\partial x_3}\right]\overline{\frac{\partial V^0}{\partial x_3}}\,, \quad \Gamma_s^1 = \frac{\partial u_3^0}{\partial x_3}\,\mathrm{grad}\, u_3^0,$$

$$E_{un}^0 = \left(1+\frac{\beta}{2}\right)\left(\frac{\partial u_3^0}{\partial x_3}\right)^3 + (2+\beta)\left(\frac{\partial u_3^0}{\partial x_3}\right)^2\,, \quad \Gamma_t^0 = \frac{\beta}{2}\left(\frac{\partial u_3^0}{\partial x_3}\right) I_2,$$

$$\Gamma_n^0 = (1+\frac{\beta}{2})\left(\frac{\partial u_3^0}{\partial x_3}\right)^2 .$$

Equations (3.5) and (3.7) lead to:

$$(2+\beta)\frac{\partial u_3^0}{\partial x_3} + E_{un}^0 + \Gamma_n^0 = 0 \qquad \text{in } \Omega_0 \, .$$

Equivalently we get

$$(2+\beta)\frac{\partial u_3^0}{\partial x_3}\left[1+\frac{3}{2}\frac{\partial u_3^0}{\partial x_3}+\frac{1}{2}\left(\frac{\partial u_3^0}{\partial x_3}\right)^2\right] = 0 \qquad \text{in } \Omega_0$$

which gives

$$\frac{\partial u_3^0}{\partial x_3} = 0 \quad \text{or} \qquad \frac{\partial u_3^0}{\partial x_3} = -1 \quad \text{or} \qquad \frac{\partial u_3^0}{\partial x_3} = -2 \qquad \text{in } \Omega_0 \, .$$

Then using the mass conservation law condition (2.10):

$$\left(1+\frac{\partial u_3^0}{\partial x_3}\right) > 0 \qquad \text{in } \Omega_0 \, ,$$

which is still valid with the new reference scales, we have $\dfrac{\partial u_3^0}{\partial x_3} = 0$ which implies

$$u_3^0 = \zeta_3^0(x_1, x_2) \qquad \text{in } \Omega_0 \, . \tag{3.8}$$

On the other hand, since the condition (2.10) can be extended to the upper and the lower faces $\Gamma_\pm$, equation (3.7) leads to

$$\frac{\partial u_3^0}{\partial x_3} = 0 \qquad \text{on } \Gamma_\pm \, . \tag{3.9}$$

REMARK 2

In [2] and [3] appears a similar indeterminacy in the evaluation of $\partial u_3^0/\partial x_3$, even if a variational formulation of the problem is used. To remove this problem, P.G. Ciarlet and P. Destuynder assume that $\dfrac{\partial u_3^0}{\partial x_3} \in C^0(\overline{\Omega}_0)$. With the new approach presented in this paper, the condition (2.10), which is obtained as a consequence of the mass conservation law, enables us to drop this additional assumption concerning the regulariy of $\partial u_3^0/\partial x_3$. ∎

Then according to (3.8) and (3.9), equations (3.4) and (3.6) become

$$\frac{\partial^2 V^0}{\partial x_3^2} = 0 \qquad \text{in } \Omega_0 ,$$

$$\frac{\partial V^0}{\partial x_3} + \text{grad } u_3^0 = 0 \qquad \text{for } x_3 = \pm 1 ,$$

and lead to

$$V^0 = \zeta_t^0(x_1, x_2) - x_3 \text{grad } \zeta_3^0 , \tag{3.10}$$

where $\zeta^0 = (\zeta_t^0, \zeta_3^0)$ represents the displacement of the middle surface ω_0. Hence step i) of the RESULT 2 is proved.

Since the problems are two-two coupled in the linear case, problem $\mathcal{P}_1$ leads to a similar result with u_3^1 and V^1. Even if this result is not necessary to prove the steps ii) and iii), it will be used for the stress calculus. So it is explained here.

Taking into account (3.8), (3.9) and (3.10), problem $\mathcal{P}_1$ can be written as

$$(1+\beta)\text{grad}\frac{\partial u_3^1}{\partial x_3} + \frac{\partial^2 V^1}{\partial x_3^2} + \frac{\partial}{\partial x_3}(Q^2 + \Gamma_s^2) = 0 \qquad \text{in } \Omega_0 , \tag{3.11}$$

$$(2+\beta)\frac{\partial^2 u_3^1}{\partial x_3^2} = 0 \qquad \text{in } \Omega_0 , \tag{3.12}$$

$$\frac{\partial V^1}{\partial x_3} + \text{grad } u_3^1 + Q^2 + \Gamma_s^2 = 0 \qquad \text{for } x_3 = \pm 1 , \tag{3.13}$$

$$(2+\beta)\frac{\partial u_3^1}{\partial x_3} = 0 \qquad \text{for } x_3 = \pm 1 , \tag{3.14}$$

with

$$Q^2 + \Gamma_s^2 = (2+\beta)\frac{\partial u_3^1}{\partial x_3}\frac{\partial V^0}{\partial x_3} + \frac{\partial u_3^1}{\partial x_3}\text{grad } u_3^0 .$$

Then equations (3.12) and (3.14) immediately lead to

$$u_3^1 = \zeta_3^1(x_1, x_2) . \tag{3.15}$$

Hence equations (3.11) and (3.13) give

$$\frac{\partial V^1}{\partial x_3} + \text{grad } u_3^1 = 0 \qquad \text{in } \Omega_0 \ ,$$

or equivalently

$$V^1 = \zeta_t^1(x_1, x_2) - x_3 \text{ grad } \zeta_3^1 \ . \tag{3.16}$$

Therefore the second term (V^1, u_3^1) of the expansion of (V, u_3) is a Kirchhoff-Love displacement as well.

ii) Extension-compression equation

According to the results previously obtained, the cancellation of the coefficient of ε^2 leads to the problem $\mathcal{P}_2$ in Ω_0 :

$$(1+\beta)\text{grad div}V^0 + \Delta V^0 + \beta\text{grad } \frac{\partial u_3^2}{\partial x_3} + \frac{\partial}{\partial x_3}(\text{grad } u_3^2 + \frac{\partial V^2}{\partial x_3}) + \text{div } \Gamma_t^2 + \frac{\partial}{\partial x_3}(Q^3 + \Gamma_s^3) = 0 \ , \tag{3.17}$$

$$(1+\beta)\frac{\partial}{\partial x_3}\text{div}V^0 + \Delta u_3^0 + (2+\beta)\frac{\partial^2 u_3^2}{\partial x_3^2} + \frac{\partial}{\partial x_3}\Gamma_n^2 = 0 \ , \tag{3.18}$$

$$\text{grad } u_3^2 + \frac{\partial V^2}{\partial x_3} + Q^3 + \Gamma_s^3 = \pm g_t^\pm \qquad \text{for } x_3 = \pm 1 \ , \tag{3.19}$$

$$(2+\beta)\frac{\partial u_3^2}{\partial x_3} + \beta\,\text{div}V^0 + \Gamma_n^2 = 0 \qquad \text{for } x_3 = \pm 1 \ , \tag{3.20}$$

with

$$Q^3 = \left[\beta\,\text{div } V^0 + \frac{\beta}{2}\|\text{grad } u_3^0\|^2 + \left(1+\frac{\beta}{2}\right)\left\|\frac{\partial V^0}{\partial x_3}\right\|^2\right]\frac{\partial V^0}{\partial x_3} + (2+\beta)\frac{\partial u_3^2}{\partial x_3}\frac{\partial V^0}{\partial x_3}$$

$$\Gamma_t^2 = \frac{\beta}{2}\Big(\|\text{grad } u_3^0\|^2 + \left\|\frac{\partial V^0}{\partial x_3}\right\|^2 \Big) I_2 + \text{grad } u_3^0\ \overline{\text{grad } u_3^0} \ ,$$

$$\Gamma_n^2 = \frac{\beta}{2}\|\text{grad } u_3^0\|^2 + \left(1+\frac{\beta}{2}\right)\left\|\frac{\partial V^0}{\partial x_3}\right\|^2 \ , \qquad \Gamma_s^3 = \frac{\partial u_3^2}{\partial x_3}\text{grad } u_3^0 + \overline{\frac{\partial V^0}{\partial x}}\frac{\partial V^0}{\partial x_3}$$

Then replacing the expression (3.10) of V^0 in (3.17) and integrating the equation so obtained from -1 to 1, we have

$$(3.21)\qquad 2(1+\beta)\ \text{grad}\ (\text{div}\ \zeta_t^0) + 2\Delta\ \zeta_t^0 + \int_{-1}^{1} \beta\ \text{grad}\ \frac{\partial u_3^2}{\partial x_3} dx_3 + \int_{-1}^{1} \text{div}\ (\Gamma_t^2) dx_3 = -(g_t^+ + g_t^-)$$

thanks to the boundary condition (3.19).

On the other hand, taking into account (3.10), Eq. (3.18) can be written as

$$\frac{\partial}{\partial x_3}[(2+\beta)\frac{\partial u_3^2}{\partial x_3} + \beta\ \text{div} V^0 + \Gamma_n^2] = 0\ ,$$

and becomes, in view of (3.20),

$$\frac{\partial u_3^2}{\partial x_3} = -\frac{\beta}{2+\beta} \text{div} V^0 - \frac{1}{2+\beta} \Gamma_n^2\ ,$$

or equivalently

$$(3.22)\qquad \frac{\partial u_3^2}{\partial x_3} = -\frac{\beta}{2+\beta}\ \text{div}\ V^0 - \frac{\beta}{2(2+\beta)} \left\|\text{grad}\ u_3^0\right\|^2 - \frac{1}{2}\left\|\frac{\partial V^0}{\partial x_3}\right\|^2.$$

Then using (3.22) and the relation

$$\text{div}(\alpha I_2) = \text{grad}\ \alpha \quad \text{for all scalar fields}\ \alpha\ ,$$

equation (3.21) finally becomes

$$\frac{4+6\beta}{2+\beta}\ \text{grad div}\ \zeta_t^0 + 2\Delta\zeta_t^0 + \text{div}\ \Big(\frac{2\beta}{2+\beta}\|\text{grad}\ u_3^0\|^2 I_2 + 2\text{grad}\ u_3^0\ \overline{\text{grad}\ u_3^0}\ \Big) = -(g_t^+ + g_t^-)\ ,$$

or equivalently

$$\text{div}\ n_t^0 = -p_t\ ,$$

with

$$n_t^0 = \frac{4\beta}{2+\beta} \text{Tr}\ E_t^0(\zeta^0) I_2 + 4E_t^0(\zeta^0)\ ,$$

$$E_t^0(\zeta^0) = \frac{1}{2}(\text{grad}\ \zeta_t^0 + \overline{\text{grad}\ \zeta_t^0} + \text{grad}\ u_3^0\ \overline{\text{grad}\ u_3^0})\ ,$$

$$p_t = g_t^+ + g_t^-\ .$$

Step ii) of the result 2 is then proved.

iii) Equation of bending

Now replacing the expression (3.10) of V^0 in (3.17), we obtain

$$-x_3(2+\beta)\text{grad } \Delta\, u_3^0 + \beta \text{ grad } \frac{\partial u_3^2}{\partial x_3} + \frac{\partial}{\partial x_3}\left(\text{grad } u_3^2 + \frac{\partial V^2}{\partial x_3}\right)$$

$$+[(1+\beta)\text{grad }(\text{div }\zeta_t^0)+\Delta\,\zeta_t^0]+\text{div } \Gamma_t^2+\frac{\partial}{\partial x_3}(Q^3+\Gamma_s^3)=0\,.$$

Then after multiplying the last equation by x_3 and taking its divergence, an integration between -1 and 1 leads to

$$(3.23)\qquad -\frac{2}{3}(2+\beta)\Delta^2 u_3^0 + \beta\int_{-1}^{1} x_3\frac{\partial}{\partial x_3}\Delta u_3^2 dx_3 + \int_{-1}^{1} x_3\frac{\partial}{\partial x_3}(\Delta u_3^2$$

$$+\frac{\partial}{\partial x_3}\text{div}V^2)dx_3 + \int_{-1}^{1} x_3\frac{\partial}{\partial x_3}\text{div }(Q^3+\Gamma_s^3)=0\,.$$

Now using (3.22) it is possible to express the second term of (3.23) in terms of u_3^0. Indeed we have

$$(3.24)\qquad \int_{-1}^{1} x_3\frac{\partial}{\partial x_3}\Delta u_3^2 dx_3 = \frac{2\beta}{3(2+\beta)}\Delta^2 u_3^0\,.$$

On the other hand, integration by parts of the third and fourth terms of (3.23) leads to

$$\int_{-1}^{1} x_3\frac{\partial}{\partial x_3}\Big[\Delta u_3^2 + \frac{\partial}{\partial x_3}\text{div}V^2 + \text{div }(Q^3+\Gamma_s^3)\Big]dx_3 = \Big[x_3 \text{ div }(\text{grad } u_3^2$$

$$+\frac{\partial V^2}{\partial x_3}+Q^3+\Gamma_s^3)\Big]_{-1}^{1} - \int_{-1}^{1}\Big[\Delta\, u_3^2 + \frac{\partial}{\partial x_3}\text{div}V^2 + \text{div}(Q^3+\Gamma_s^3)\Big]dx_3\,,$$

and with the boundary condition (3.19) we obtain

$$\Big[x_3 \text{ div}(\text{grad } u_3^2 + \frac{\partial V^2}{\partial x_3} + Q^3 + \Gamma_s^3)\Big]_{-1}^{1} = \text{div}(g_t^+ - g_t^-)\,.$$

Finally equation (3.23) becomes

$$\text{(3.25)}\qquad \frac{8}{3}\frac{\beta+1}{\beta+2}\Delta^2 u_3^0 + \int_{-1}^{1}\Big\{\Delta\, u_3^2 + \frac{\partial}{\partial x_3}\mathrm{div}V^2 + \mathrm{div}(Q^3 + \Gamma_s^3)\Big\}dx_3 = \mathrm{div}M_t$$

where

$$M_t = g_t^+ - g_t^- \ .$$

Now in order to eliminate the unknowns u_3^2 and V^2 we have to analyse the problem $\mathcal{P}_4$. The second equation and the second boundary condition of the problem $\mathcal{P}_4$ are given by

$$\text{(3.26)}\qquad (1+\beta)\frac{\partial}{\partial x_3}\mathrm{div}V^2 + \Delta u_3^2 + (2+\beta)\frac{\partial^2 u_3^4}{\partial x_3^2} + \mathrm{div}(E_{us}^3 + \Gamma_s^3) + \frac{\partial}{\partial x_3}(E_{un}^4 + \Gamma_n^4) = -f_3 \ \text{ on } \ \Omega_0 \ ,$$

$$\text{(3.27)}\qquad (2+\beta)\frac{\partial u_3^4}{\partial x_3} + \beta\,\mathrm{div}V^2 + E_{un}^4 + \Gamma_n^4 = \pm g_3^\pm \quad \text{for} \quad x_3 = \pm 1 \ ,$$

with

$$\text{(3.28)}\qquad E_{us}^3 = \Big[\beta\,\mathrm{div}\ V^0 + \Big(1+\frac{\beta}{2}\Big)\|\mathrm{grad}\ u_3^0\|^2 + \frac{\beta}{2}\Big\|\frac{\partial V^0}{\partial x_3}\Big\|^2\Big]\mathrm{grad}\ u_3^0 + \Big(\frac{\partial V^0}{\partial x} + \overline{\frac{\partial V^0}{\partial x}}\Big)\ \mathrm{grad}\ u_3^0 + (1+\beta)\frac{\partial u_3^2}{\partial x_3}\mathrm{grad}\ u_3^0 + \frac{\partial u_3^2}{\partial x_3}\frac{\partial V^0}{\partial x_3} \ ,$$

$$\Gamma_s^3 = \overline{\frac{\partial V^0}{\partial x}}\frac{\partial V^0}{\partial x_3} + \frac{\partial u_3^2}{\partial x_3}\mathrm{grad}\ u_3^0 \ .$$

Then integrating (3.26) between -1 and 1, and using the boundary condition (3.27), we get

$$\int_{-1}^{1}\Big[\frac{\partial}{\partial x_3}\,\mathrm{div}V^2 + \Delta u_3^2\Big]dx_3 + \int_{-1}^{1}\mathrm{div}(E_{us}^3 + \Gamma_s^3)dx_3 = -p_3$$

where

$$p_3 = \int_{-1}^{1} f_3 dx_3 + g_3^+ + g_3^- .$$

Hence we have:

$$\int_{-1}^{1} \Big(\Delta u_3^2 + \frac{\partial}{\partial x_3}\mathrm{div}V^2 + \mathrm{div}(Q^3 + \Gamma_s^3\Big) dx_3 = -p_3 + \int_{-1}^{1} \mathrm{div}(Q^3 - E_{us}^3) dx_3 \ .$$

Replacing then Q^3 and E_{us}^3 by their respective expressions and using the coupling relation (3.22) we obtain

$$Q^3 - E_{us}^3 = -\left[\frac{2\beta}{2+\beta}\,\mathrm{div}\,V^0 + \frac{2+2\beta}{2+\beta}\|\mathrm{grad}\ u_3^0\|^2\right] \mathrm{grad}\ u_3^0$$

$$-\left(\overline{\frac{\partial V^0}{\partial x}} + \frac{\partial V^0}{\partial x}\right) \mathrm{grad}\ u_3^0 \ .$$

Finally the bending equation (3.25) becomes

$$\frac{8}{3}\frac{\beta+1}{\beta+2}\Delta^2 u_3^0 - \mathrm{div}\left[\left(\frac{4\beta}{2+\beta}\mathrm{div}\ \zeta_t^0 + \frac{4+4\beta}{2+\beta}\|\mathrm{grad}\ u_3^0\|^2\right) \mathrm{grad}\ u_3^0\right]$$

$$-2\mathrm{div}\left[\left(\overline{\frac{\partial V^0}{\partial x}} + \frac{\partial V^0}{\partial x}\right) \mathrm{grad}\ u_3^0\right] = p_3 + \mathrm{div}M_t \ ,$$

or equivalently

$$\frac{8}{3}\frac{\beta+1}{\beta+2}\Delta^2 u_3^0 - \mathrm{div}(n_t^0\ \mathrm{grad}\ u_3^0) = p_3 + \mathrm{div}M_t \ ,$$

with

$$n_t^0 = \frac{4\beta}{2+\beta}\mathrm{Tr}E_t^0(\zeta^0)I_2 + 4E_t(\zeta^0) \ ,$$

$$E_t^0(\zeta^0) = \frac{1}{2}(\mathrm{grad}\ \zeta_t^0 + \overline{\mathrm{grad}\ \zeta_t^0} + \mathrm{grad}\ u_3^0\ \overline{\mathrm{grad}\ u_3^0}) \ ,$$

$$p_3 = \int_{-1}^{1} f_3 dx_3 + g_3^+ + g_3^- \ , \qquad M_t = g_t^+ - g_t^- \ .$$

It is also possible to write the last equation in a more classical form. To this end, let us define the dimensionless tensor of bending moments:

$$m_t^0 = -\left\{\frac{4\beta}{3(2+\beta)}\Delta u_3^0 I_2 + \frac{4}{3}\mathrm{grad}(\mathrm{grad}\ u_3^0)\right\} \ .$$

Therefore we have

$$\operatorname{div}(\operatorname{div} m_t^0) + \operatorname{div}(n_t^0 \operatorname{grad} u_3^0) = -p_3 - \operatorname{div} M_t \ ,$$

which concludes the proof of the RESULT 2.

4. Stress analysis

Now the components of the stress tensor can be deduced from the three-dimensional constitutive equations and from the previous results. We recall that the constitutive equation of Saint-Venant-Kirchhoff materials is given by

$$\Sigma^* = \lambda \operatorname{Tr} E^* I + 2\mu E^* \ ,$$

$$E^* = \frac{1}{2}(\overline{F^*} F^* - I) = \frac{1}{2}\left(\overline{\frac{\partial U^*}{\partial x^*}} + \frac{\partial U^*}{\partial x^*}\right) + \frac{1}{2}\overline{\frac{\partial U^*}{\partial x^*}}\frac{\partial U^*}{\partial x^*} = e^* + \gamma^* \ ,$$

where e^* and γ^* denote, respectively, the linear and the nonlinear parts of the Green-Lagrange strain tensor E^*. So it is possible to write Σ^* in the following form:

$$\Sigma^* = \sigma^* + \Gamma^* \ ,$$

where

$$\sigma^* = \lambda \operatorname{Tr} e^* I + 2\mu\, e^* \ ,$$

$$\Gamma^* = \lambda \operatorname{Tr} \gamma^* I + 2\mu \gamma^* \ .$$

Then decomposing U^* as $U^* = V^* + u_3^* e_3$ and writing the components of Σ^* in a dimensionless form with $u_{3r} = h_0$ and $V_r = \varepsilon u_{3r}$, we obtain

$$\frac{\Sigma_t^*}{\mu} = \beta \frac{\partial u_3}{\partial x_3} I_2 + \varepsilon^2 \left\{ 2e_t(V) + \frac{\beta}{2}\left(2 \operatorname{div} V + \|\operatorname{grad} u_3\|^2 + \|\frac{\partial V}{\partial x_3}\|^2\right) I_2 + \operatorname{grad} u_3\, \overline{\operatorname{grad} u_3} \right\} + O(\varepsilon^3) \ ,$$

$$\frac{\Sigma_s^*}{\mu} = \varepsilon \left\{ \operatorname{grad} u_3 + \frac{\partial V}{\partial x_3} + \frac{\partial u_3}{\partial x_3} \operatorname{grad} u_3 \right\} + \varepsilon^3 \overline{\frac{\partial V}{\partial x}} \frac{\partial V}{\partial x_3} + O(\varepsilon^4) \ ,$$

$$\frac{\Sigma_n^*}{\mu} = (2+\beta)\frac{\partial u_3}{\partial x_3} + \left(1 + \frac{\beta}{2}\right)\left(\frac{\partial u_3}{\partial x_3}\right)^2 + \varepsilon^2 \left\{ \beta \operatorname{div} V + \frac{\beta}{2}\|\operatorname{grad} u_3\|^2 + \left(1+\frac{\beta}{2}\right)\left\|\frac{\partial V}{\partial x_3}\right\|^2 \right\} + \varepsilon^4 \left\{ \frac{\beta}{2} \operatorname{Tr}\left(\overline{\frac{\partial V}{\partial x}} \frac{\partial V}{\partial x} \right) \right\} + O(\varepsilon^5) \ .$$

Now let us replace the dimensionless components of the displacement (V, u_3) by their expansions (3.3) and let us use some of the previous results to simplify the formulas. On the one hand, the first and the second terms of the expansion of (V, u_3) satisfy the kinematic Kirchhoff-Love hypothesis :

$$\frac{\partial u_3^0}{\partial x_3} = 0 , \qquad \frac{\partial V^0}{\partial x_3} + \text{grad } u_3^0 = 0 \quad \text{and} \quad \frac{\partial u_3^1}{\partial x_3} = 0 , \qquad \frac{\partial V^1}{\partial x_3} + \text{grad } u_3^1 = 0 .$$

On the other hand, we easily remark that the problem $\mathcal{P}_3$ leads to a coupling equation similar to (3.22):

$$\frac{\partial u_3^3}{\partial x_3} = -\frac{\beta}{2+\beta} \text{ div } V^1 - \frac{\beta}{2(2+\beta)}\Big[\|\text{grad } u_3^1\|^2 + 2\overline{\text{grad } u_3^0} \text{ grad } u_3^2\Big] - \frac{1}{2}\left[\left\|\frac{\partial V^1}{\partial x_3}\right\|^2 + 2\frac{\overline{\partial V^0}}{\partial x_3}\frac{\partial V^2}{\partial x_3}\right]. \tag{4.1}$$

Then using the previous relations, the formulas for the components of the stress tensor assume the form :

$$\frac{\Sigma_t^*}{\varepsilon^2\mu} = \frac{1}{2}n_t^0 + \frac{3}{2}x_3 m_t^0 + O(\varepsilon) ,$$

$$\frac{\Sigma_s^*}{\varepsilon^2\mu} = \varepsilon\left\{\text{grad } u_3^2 + \frac{\partial V^2}{\partial x_3} + \frac{\partial u_3^2}{\partial x_3}\text{grad } u_3^0 + \frac{\overline{\partial V^0}}{\partial x}\frac{\partial V^0}{\partial x_3}\right\} + O(\varepsilon^2) ,$$

$$\begin{aligned}\frac{\Sigma_n^*}{\varepsilon^2\mu} = \varepsilon^2\Bigg\{&(2+\beta)\frac{\partial u_3^4}{\partial x_3} + \Big(1+\frac{\beta}{2}\Big)\Big(\frac{\partial u_3^2}{\partial x_3}\Big)^2 + \beta \text{ div}V^2 + \frac{\beta}{2}\Big[\|\text{grad } u_3^1\|^2 \\ &+ 2\overline{\text{grad } u_3^0} \text{ grad } u_3^2\Big] + \Big(1+\frac{\beta}{2}\Big)\left[\left\|\frac{\partial V^1}{\partial x_3}\right\|^2 + 2\frac{\overline{\partial V^0}}{\partial x_3}\frac{\partial V^2}{\partial x_3}\right] \\ &+ \frac{\beta}{2}\text{Tr}\Big(\frac{\overline{\partial V^0}}{\partial x}\frac{\partial V^0}{\partial x}\Big)\Bigg\} + O(\varepsilon^3) .\end{aligned}$$

The reference scale $\Sigma_r = \varepsilon^2\mu$ of the stresses naturally appears so that at least one component of $\Sigma = \dfrac{\Sigma^*}{\Sigma_r}$ is of order 1.

The existence of an expansion of the solution (V, u_3) into a power series of ε then implies the existence of an expansion of Σ with respect to ε :

$$\Sigma = \Sigma^0 + \varepsilon\Sigma^1 + \varepsilon^2\Sigma^2 + \cdots , \tag{4.2}$$

where

$$
\begin{aligned}
&\Sigma_t^0 = \frac{1}{2}n_t^0 + \frac{3}{2}x_3 m_t^0 , \\
&\Sigma_s^1 = \text{grad } u_3^2 + \frac{\partial V^2}{\partial x_3} + \frac{\partial u_3^2}{\partial x_3}\text{grad } u_3^0 + \overline{\frac{\partial V^0}{\partial x}}\frac{\partial V^0}{\partial x_3} , \\
(4.3) \quad &\Sigma_n^2 = (2+\beta)\frac{\partial u_3^4}{\partial x_3} + \left(1+\frac{\beta}{2}\right)\left(\frac{\partial u_3^2}{\partial x_3}\right)^2 + \beta\,\text{div } V^2 + \frac{\beta}{2}\Big[\|\text{grad } u_3^1\|^2 \\
&\qquad + 2\overline{\text{grad } u_3^0}\text{ grad } u_3^2\Big] + (1+\frac{\beta}{2})\Big[\|\frac{\partial V^1}{\partial x_3}\|^2 + 2\overline{\frac{\partial V^0}{\partial x_3}}\frac{\partial V^2}{\partial x_3}\Big] \\
&\qquad + \frac{\beta}{2}\text{Tr}\left(\overline{\frac{\partial V^0}{\partial x}}\frac{\partial V^0}{\partial x}\right) , \\
&\Sigma_s^0 = 0, \qquad \Sigma_n^i = 0 \quad i = 0, 1 .
\end{aligned}
$$

The explicit expressions of Σ_s^1 and Σ_n^2 are obtained by integration across the thickness. This process is similar to the one developed by P. G. CIARLET and P. DESTUYNDER in [2].

4.1. Evaluation of Σ_s^1

Taking the gradient of (3.22) and inserting the obtained expression in (3.17), we get, according to (4.3),

$$
\begin{aligned}
\frac{\partial}{\partial x_3}\Sigma_s^1 &= -\text{div } \Sigma_t^0 \quad \text{on } \Omega_0 , \\
\Sigma_s^1 &= \pm g_t^{\pm} \qquad \text{on } \Gamma_{0\pm} .
\end{aligned}
$$

The integration of Σ_s^1 leads then to the classical expression:

$$
\Sigma_s^1 = \frac{3}{4}\left(1 - x_3^2\right)\text{div}(m_t^0) + \frac{g_t^+ - g_t^-}{2} + \frac{x_3}{2}\left(g_t^+ + g_t^-\right) .
$$

4.2. Evaluation of Σ_n^2

The equation (3.26) can be written as

$$(4.4)\qquad \frac{\partial}{\partial x_3}\Big[(2+\beta)\frac{\partial^2 u_3^4}{\partial x_3^2}+\beta\,\mathrm{div}V^2\Big]+\mathrm{div}\Big[\frac{\partial V^2}{\partial x_3}+\mathrm{grad}\ u_3^2\Big] + \mathrm{div}(E_{us}^3+\Gamma_s^3)+\frac{\partial}{\partial x_3}(E_{un}^4+\Gamma_n^4)=-f_3\,,$$

with

$$E_{us}^3=\left[\beta\ \mathrm{div}V^0+\left(1+\frac{\beta}{2}\right)\|\mathrm{grad}\ u_3^0\|^2+\frac{\beta}{2}\left\|\frac{\partial V^0}{\partial x_3}\right\|^2+\beta\frac{\partial u_3^2}{\partial x_3}\right]\mathrm{grad}\ u_3^0 + \left(\frac{\partial V^0}{\partial x}+\overline{\frac{\partial V^0}{\partial x}}\right)\mathrm{grad}\ u_3^0\,,$$

$$E_{un}^4=\left[\overline{\frac{\partial V^0}{\partial x_3}}\frac{\partial V^0}{\partial x}+\overline{\frac{\partial V^2}{\partial x_3}}+\overline{\mathrm{grad}\ u_3^2}\right]\mathrm{grad}\ u_3^0+\frac{\partial u_3^2}{\partial x_3}\|\mathrm{grad}\ u_3^0\|^2\,,$$

$$\Gamma_s^3=\overline{\frac{\partial V^0}{\partial x}}\frac{\partial V^0}{\partial x_3}+\frac{\partial u_3^2}{\partial x_3}\mathrm{grad}\ u_3^0\,,$$

$$\Gamma_n^4=\left(1+\frac{\beta}{2}\right)\left(\frac{\partial u_3^2}{\partial x_3}\right)^2+\frac{\beta}{2}\left[\left\|\mathrm{grad}\ u_3^1\right\|^2+2\overline{\mathrm{grad}\ u_3^0}\ \mathrm{grad}\ u_3^2\right] +\left(1+\frac{\beta}{2}\right)\left[\left\|\frac{\partial V^1}{\partial x_3}\right\|^2+2\overline{\frac{\partial V^0}{\partial x_3}}\frac{\partial V^2}{\partial x_3}\right]+\frac{\beta}{2}\mathrm{Tr}\left(\overline{\frac{\partial V^0}{\partial x}}\frac{\partial V^0}{\partial x}\right)\,.$$

The intermediate steps of the previous calculation are not difficult, so they will not be discussed in details. In these steps, use is made of relations (3.8), (3.10), (3.15), (3.16) and (3.22).

On the other hand, a simple analysis leads to

$$\frac{\partial V^2}{\partial x_3}+\mathrm{grad}\ u_3^2+\Gamma_s^3=\Sigma_s^1\,,$$

$$E_{us}^3=\Sigma_t^0\ \mathrm{grad}\ u_3^0\,,$$

$$(2+\beta)\frac{\partial u_3^4}{\partial x_3}+\beta\,\mathrm{div}\ V^2+\Gamma_n^4=\Sigma_n^2\,,$$

$$E_{un}^4=\overline{\Sigma_s^1}\,\mathrm{grad}\ u_3^0\,.$$

Hence Eq. (4.4) and the boundary conditions on the upper and the lower faces give us

$$\frac{\partial}{\partial x_3}\,\Sigma_n^2 = -\text{div}(\Sigma_t^0\text{grad } u_3^0) \ -\text{div}(\Sigma_s^1) - \frac{\partial}{\partial x_3}(\overline{\Sigma}_s^1\text{grad } u_3^0) - f_3 \ \text{ on } \ \Omega_0 \ ,$$

$$\Sigma_n^2 = \pm g_3^\pm \ \text{ on } \ \Gamma_{0\pm}.$$

Finally, the integration of Σ_n^2 leads to the classical result:

$$\Sigma_n^2 = -\frac{x_3}{4}(1-x_3^2)\text{div}(\text{div } m_t^0) + \frac{3}{4}(1-x_3^2)\text{Tr}\Big[m_t^0 \text{ grad grad } u_3^0\Big]$$
$$+\frac{1+x_3}{2}\int_{-1}^{1} f_3 dx_3 - \int_{-1}^{x_3} f_3(z)dz + \frac{1}{2}(g_3^+ - g_3^-) + \frac{x_3}{2}(g_3^+ + g_3^-)$$
$$+\frac{1}{4}(1-x_3^2)\text{div }(g_t^+ + g_t^-) - \text{grad } \zeta_3^0 . \left\{\frac{1}{2}(g_t^+ - g_t^-) + \frac{x_3}{2}(g_t^+ + g_t^-)\right\} .$$

5. Passage to the initial variables

Now let us go back to the initial domain Ω_0^* and to the physical variables V^*, u_3^*, f^* and g^*. To do this, let us define

$$V^{*0} = V_r V^0 = \varepsilon h_0 V^0 \ ,$$

$$u_3^{*0} = u_{3r} u_3^0 = h_0 u_3^0 \ .$$

Thus we have the following result :

RESULT 3

For applied forces of order f^* and g^* such that $\mathcal{F}_3 = \mathcal{G}_3 = \varepsilon^4$ and $\mathcal{G}_t = \varepsilon^3$, (V^{*0}, u_3^{*0}) is a Kirchhoff-Love displacement which satisfies

$$u_3^{*0} = \zeta_3^{*0}(x_1^*, x_2^*) \ , \qquad V^{*0} = \zeta_t^{*0}(x_1^*, x_2^*) - x_3^* \text{ grad}^* \zeta_3^{*0} \ ;$$

where $\zeta^{*0} = (\zeta_t^{*0}, \zeta_3^{*0})$ is the solution of the following nonlinear problem :

$$h_0^3 \text{ div}^*(\text{div}^* m_t^{0*}) + h_0 \text{ div}^*\left(n_t^{0*}\text{grad}^*\zeta_3^{*0}\right) = -p_3^* - \text{div}M_t^* \ ,$$

$$h_0 \text{ div}^*(n_t^{*0}) = -p_t^* \ ,$$

$$\zeta_3^{*0} = \frac{\partial \zeta_3^{*0}}{\partial \nu} = 0 \qquad \text{and} \qquad \zeta_t^{*0} = 0 \quad \text{on } \gamma_0^* ,$$

with

$$n_t^{*0} = \frac{4\lambda\mu}{\lambda + 2\mu} \mathrm{Tr} E_t^{*0}(\zeta^{*0}) I_2 + 4\mu E_t^{*0}(\zeta^{*0}) ,$$

$$m_t^{*0} = -\Big\{ \frac{4\lambda\mu}{3(\lambda+2\mu)} \Delta^* \zeta_3^{*0} I_2 + \frac{4\mu}{3} \mathrm{grad}^* (\mathrm{grad}^* \zeta_3^{*0}) \Big\} ,$$

$$E_t^{*0}(\zeta^{*0}) = \frac{1}{2}\Big(\mathrm{grad}^* \zeta_t^{*0} + \overline{\mathrm{grad}^* \zeta_t^{*0}} + \mathrm{grad}\ \zeta_3^{*0}\ \overline{\mathrm{grad}\ \zeta_3^{*0}} \Big) ,$$

$$p_3^* = \int_{-h_0}^{h_0} f_3^* dx_3^* + g_3^{*+} + g_3^{*-} , \qquad p_t^* = g_t^{*+} + g_t^{*-} , \qquad M_t^* = h_0(g_t^{*+} - g_t^{*-}) .$$

In the same way, let us define

$$\Sigma_t^{*0} = \varepsilon^2 \mu \Sigma_t^0 , \qquad \Sigma_s^{*0} = \varepsilon^3 \mu \Sigma_s^1 , \qquad \Sigma_n^{*0} = \varepsilon^4 \mu \Sigma_n^2 .$$

Then the expression of the physical stresses are as follows :

RESULT 4

$$\Sigma_t^{*0} = n_t^{*0} + \frac{3x_3^*}{2} m_t^{*0} ,$$

$$\Sigma_s^{*0} = -\frac{3}{4}(h_0^2 - (x_3^*)^2) \mathrm{div}^*(m_t^{*0}) + \frac{1}{2}(g_t^{*+} - g_t^{*-}) + \frac{x_3^*}{2h_0}(g_t^{*+} + g_t^{*-}) ,$$

$$\begin{aligned} \Sigma_n^{*0} = & -\frac{x_3^*}{4}(h_0^2 - (x_3^*)^2) \mathrm{div}^*\ (\mathrm{div}^*\ m_t^{*0}) + \frac{3}{4}(h_0^2 - (x_3^*)^2) \mathrm{Tr}\Big[m_t^{*0} \mathrm{grad}^* (\mathrm{grad}^*\ u_3^{*0}) \Big] \\ & + \frac{1}{2}(1 + \frac{x_3^*}{h_0}) \int_{-h_0}^{h_0} f_3^* dx_3^* - \int_{-h_0}^{x_3^*} f_3^*(z^*) dz^* + \frac{1}{2}(g_3^{*+} - g_3^{*-}) \\ & + \frac{x_3^*}{2h_0}(g_3^{*+} + g_3^{*-}) + \frac{1}{4}\left(1 - \left(\frac{x_3^*}{h_0}\right)^2\right) \mathrm{div}^* \Big[h_0(g_t^{*+} + g_t^{*-}) \Big] \\ & - \frac{1}{2} \mathrm{grad}\ \zeta_3^{*0} . \Big\{ g_t^{*+} - g_t^{*-} + \frac{x_3^*}{h_0}(g_t^{*+} + g_t^{*-}) \Big\} . \end{aligned}$$

The return to the initial variables does not create any difficulty. The procedures are similar to those exposed in [6] and [7].

6. Conclusion

The results obtained in the first part [8] and in the present article prove that the reference scales of the displacement and the corresponding two-dimensional model we obtain are determined by the magnitude of forces, mainly by the surface forces. Thus the scalings of the displacements generally used in the nonlinear case cannot be considered anymore as a simple change of functions. Indeed, as it has been proved in the first part, there exists another change of functions (corresponding to $V_r = u_{3r} = L_0$) which leads to the nonlinear membrane model for large forces and not to the von Kármán's one.

On the other hand, we have proved in this second part that for moderate applied forces, the standard asymptotic expansion method leads to the nonlinear two-dimensional von Kármán model. Its domain of validity is then specified thanks to the dimensionless numbers naturally introduced. Indeed the von Kármán model is valid for applied forces magnitude so that $\mathcal{F}_3 = h_0 f_{3r}/\mu$ and $\mathcal{G}_3 = g_{3r}/\mu$ are of order $\left(\dfrac{h_0}{L}\right)^4$, $\mathcal{G}_t = g_{3r}/\mu$ of order $\left(\dfrac{h_0}{L}\right)^3$ (where $\mathcal{F}_3$, $\mathcal{G}_3$ and $\mathcal{G}_t$ are known data of the problem). These forces lead to deflections of the order of thickness.

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