

Dimensional analysis and asymptotic expansions of the equilibrium equations in nonlinear elasticity Part I : The membrane model

O. MILLET, A. HAMDOUNI and A. CIMETIERE (POITIERS)

IN THIS PAPER, we develop a new asymptotic constructive approach in nonlinear plate theory. The dimensional analysis of the three-dimensional equilibrium equations naturally leads to dimensionless numbers which reflect the geometry of the structure and the magnitude of forces. These numbers also define the domain of validity of the two-dimensional models which will later be obtained by asymptotic expansions. For nonlinear plates, we prove that the two-dimensional models we obtain by asymptotic expansions are determined by the magnitude of the forces applied. In this first part, we consider a plate subjected to large loads. In this case, we prove that the nonlinear plate model we obtain by asymptotic expansions is a membrane model. In the second part of this article, we will consider a plate subjected to smaller applied forces.

1. Introduction

THE FIRST RIGOROUS JUSTIFICATION of the nonlinear plate model has been obtained at the end of the 70's by P.G. CIARLET and P. DESTUYNDER [1, 2]. In these works, the asymptotic expansion method is applied to a mixed variational formulation (in terms of stresses and displacements) of the three-dimensional elasticity problem. Afterwards, a variational approach was formulated only in terms of displacements [3, 16]. These works have also been extended to some linear [4 - 7], [8, 12], [17 - 20] and nonlinear [11, 13] shell models.

Nevertheless, to our knowledge, two problems still exist concerning the justification of the nonlinear plate model by asymptotic expansions:

1 - The choice of *a priori* scalings on the components of the displacements. These scalings determine the order of magnitude of the ratio between the normal and the tangential displacement.

2 - The difficult physical interpretation of the scalings on the applied loads. Indeed with these scalings, the applied body forces density depends on the thickness. However in usual elasticity problems, the plate is often subjected only to the gravity body force (ρg) which is independent of the thickness.

We propose to solve the two previous problems by extending to the nonlinear case, the new asymptotic approach we have developed in the linear case [14]. This new constructive approach, directly derived from the asymptotic expansion method, successfully used in fluid mechanics, can be considered as a continuation

of A.L. GOLDENVEIZER'S works [10]. It is based on the dimensional analysis of the three-dimensional equilibrium equations and needs no *a priori* assumption on the ratio between the normal and the tangential displacements. The dimensional analysis of the three-dimensional equilibrium equations naturally leads to dimensionless numbers which reflect the geometry of the structure and the forces level. These numbers also define the domain of validity of the two-dimensional models which will later be obtained by asymptotic expansions.

In this paper, we prove that the reference scales of the normal and the tangential displacement and the corresponding two-dimensional model we obtain are determined by the forces level. Indeed, in order to allow for large displacements, the reference scales of the normal displacement u_{3r} and of the tangential displacement V_r are first assumed to be equal to L_0 , diameter of the middle surface of the plate whose thickness is $2h_0$. For large applied forces, the asymptotic expansion leads to a membrane model which differs from the von Kármán model. Then going back to the variational formulation, we prove that this membrane model and the one obtained by D. FOX *et al.* [5] from a three-dimensional variational formulation are identical.

In the second part of this article we will consider a plate subjected to moderate forces. Then we prove that these moderate forces lead to new reference scales ($u_{3r} = h_0$ and $V_r = \varepsilon h_0$) for the normal and the tangential displacement. The new reference scales we obtain as a consequence of the forces magnitude are formally equivalent to the scaling assumptions generally made in the literature and naturally lead to the two-dimensional nonlinear von Kármán model.

2. The three-dimensional problem

In what follows, Greek indices take their values in $\{1, 2\}$ and Latin indices take their values in $\{1, 2, 3\}$. We assume that an origin O and an orthonormal basis (e_1, e_2, e_3) have been chosen in the three-dimensional Euclidian space which will later be identified with \mathbb{R}^3 . We denote by a superscribed asterisk (*) all the dimensional variables. On the other hand, within the framework of large displacements, the reference and the current configuration cannot be confused. Thus the reference configuration variables will be marked by $(\cdot)_0$.

Let ω_0^* be an open bounded connected set of \mathbb{R}^2 in the plane spanned by the vectors (e_α) with a "smooth enough" boundary γ_0^* . Let L_0 be the diameter of γ_0^* . Let us consider now a thin plate of thickness $2h_0$, whose middle surface is ω_0^* . The plate itself occupies the open set $\Omega_0^* = \omega_0^* \times]-h_0, h_0[$ of \mathbb{R}^3 in its reference configuration.

We denote by $X^* = (x^*, x_3^*)$ the generic point of Ω_0^* where $x^* \in \omega_0^*$. Let $\Gamma_{0\pm}^* = \bar{\omega}_0^* \times \{\pm h_0\}$ be the upper and the lower faces of the plate. In what follows,

we consider only thin plates ($h_0 \ll L_0$) subjected to dead loads (independent of the configuration). Finally, in this paper we will use the following notations:

$\partial/\partial X^*$ and Div^* denote the gradient and the divergence in the three-dimensional space,

$\partial/\partial x^*$ and div^* denote the two-dimensional gradient and the two-dimensional divergence.

If U and V are two vectors of \mathbb{R}^3 , we denote by $U\bar{V}$ the tensorial product of U and V (the overbar denotes the transposition operator).

We assume that the plate, subjected to the applied body forces $f^* = f_t^* + f_3^*e_3 : \bar{\Omega}_0^* \rightarrow \mathbb{R}^3$ and surface forces $g^{*\pm} = g_t^{*\pm} + g_3^{*\pm}e_3 : \Gamma_{0\pm}^* \rightarrow \mathbb{R}^3$, occupies the set Ω^* in its deformed configuration.

The unknown of the problem is then the displacement $U^* : \bar{\Omega}_0^* \rightarrow \mathbb{R}^3$ such that if $X^* \in \bar{\Omega}_0^*$ denotes the initial position of a material point, its position in the deformed configuration is $X^* + U^*(X^*)$. Moreover, as the plate is assumed to be clamped on its lateral surface $\Gamma_0^* = \gamma_0^* \times [-h_0, h_0]$, we have $U^* = 0$ on Γ_0^* .

Within the framework of nonlinear elasticity, the displacement $U^* : \bar{\Omega}_0^* \rightarrow \mathbb{R}^3$ and the second Piola-Kirchhoff tensor Σ^* solve the equilibrium equations

$$(2.1) \quad \begin{aligned} \text{Div}^*(\Sigma^* \bar{F}^*) &= -f^* \quad \text{in } \Omega_0^*, \\ U^* &= 0 \quad \text{on } \Gamma_0^*, \\ (F^* \Sigma^*) \cdot n^\pm &= g^{*\pm} \quad \text{on } \Gamma_{0\pm}^*, \end{aligned}$$

where

$$F^* = \frac{\partial \psi^*(X^*)}{\partial X^*} = I + \frac{\partial U^*}{\partial X^*}$$

denotes the linear map tangent to the mapping function $X^* \rightarrow \psi^*(X^*) = X^* + U^*(X^*)$, and n^\pm is the external unit normal to the upper and the lower faces $\Gamma_{0\pm}^*$.

These equilibrium equations can be completed with the mass conservation law $\rho^* \det F^* = \rho_0^*$ where ρ_0^* and ρ^* denote, respectively, the voluminal mass of the material in the reference and the deformed configuration. In what follows, we assume ρ^* to be bounded, which can be written as:

$$(2.2) \quad \det F^* = \det \left(\frac{\partial \psi^*}{\partial X^*} \right) \geq a > 0 \quad \text{in } \Omega_0^*,$$

where $a > 0$ is a constant. This condition will be used later.

Limiting our study to Saint-Venant-Kirchhoff materials, the constitutive relation takes the following form:

$$(2.3) \quad \begin{aligned} \Sigma^* &= \lambda \text{Tr} E^* I + 2\mu E^*, \\ E^* &= \frac{1}{2}(\bar{F}^* F^* - I) = \frac{1}{2} \left(\frac{\partial U^*}{\partial X^*} + \frac{\partial U^*}{\partial X^*} \right) + \frac{1}{2} \frac{\partial U^*}{\partial X^*} \frac{\partial U^*}{\partial X^*} = e^* + \gamma^*, \end{aligned}$$

where e^* and γ^* denote respectively the linear and nonlinear part of the Green-Lagrange strain tensor E^* and I the identity of \mathbb{R}^3 .

REMARK 1

Constitutive relation of Saint-Venant-Kirchhoff is obtained by linearizing more general Lagrangian constitutive relations with respect to E^* . Therefore our model is limited to small deformations even if large displacements are allowed. ■

Let us decompose the equilibrium equations so as to separate the linear from the nonlinear part. Writing E^* as $E^* = e^* + \gamma^*$, we get

$$\begin{aligned} \text{Div}^*(\Sigma^* \overline{F}^*) &= \text{Div}^* \Sigma^* + \text{Div}^* \left(\Sigma^* \frac{\partial \overline{U}^*}{\partial X^*} \right) \\ &= \text{Div}^* (\lambda \text{Tr} e^* I + 2\mu e^*) + \text{Div}^* \Gamma^* + \text{Div}^* E_u^* \end{aligned}$$

which leads to

$$\text{Div}^*(\Sigma^* \overline{F}^*) = (\lambda + \mu) \text{Grad}^*(\text{Div}^* U^*) + \mu \Delta_3^* U^* + \text{Div}^* \Gamma^* + \text{Div}^* E_u^*$$

where:

$$\begin{aligned} \Gamma^* &= \lambda \text{Tr} \gamma^* I + 2\mu \gamma^*, \\ E_u^* &= \lambda \text{Tr} E^* \frac{\partial \overline{U}^*}{\partial X^*} + 2\mu E^* \frac{\partial \overline{U}^*}{\partial X^*}. \end{aligned}$$

Now decomposing U^* into a tangential and a normal component :

$$U^* = V^* + u_3^* e_3,$$

we get:

$$\frac{\partial U^*}{\partial X^*} = \begin{pmatrix} \frac{\partial V^*}{\partial x^*} & \frac{\partial V^*}{\partial x_3^*} \\ \overline{\text{grad}} u_3^* & \frac{\partial u_3^*}{\partial x_3^*} \end{pmatrix} \quad \text{and} \quad \frac{\partial \overline{U}^*}{\partial X^*} = \begin{pmatrix} \frac{\partial V^*}{\partial x^*} & \text{grad } u_3^* \\ \overline{\text{grad}} u_3^* & \frac{\partial u_3^*}{\partial x_3^*} \end{pmatrix}.$$

Thus we have

$$\gamma^* = \frac{1}{2} \frac{\partial \overline{U}^*}{\partial x^*} \frac{\partial U^*}{\partial x^*}$$

$$= \frac{1}{2} \begin{pmatrix} \frac{\partial \overline{V}^*}{\partial x^*} \frac{\partial V^*}{\partial x^*} + \text{grad } u_3^* \overline{\text{grad}} u_3^* & \frac{\partial \overline{V}^*}{\partial x^*} \frac{\partial V^*}{\partial x_3^*} + \frac{\partial u_3^*}{\partial x_3^*} \text{grad } u_3^* \\ \frac{\partial \overline{V}^*}{\partial x_3^*} \frac{\partial V^*}{\partial x^*} + \frac{\partial u_3^*}{\partial x_3^*} \overline{\text{grad}} u_3^* & \frac{\partial \overline{V}^*}{\partial x_3^*} \frac{\partial V^*}{\partial x_3^*} + \left(\frac{\partial u_3^*}{\partial x_3^*} \right)^2 \end{pmatrix}$$

and

$$2\text{Tr } \gamma^* = \text{Tr} \left(\frac{\partial V^*}{\partial x_3^*} \frac{\partial V^*}{\partial x^*} \right) + \|\text{grad } u_3^*\|^2 + \left\| \frac{\partial V^*}{\partial x_3^*} \right\|^2 + \left(\frac{\partial u_3^*}{\partial x_3^*} \right)^2.$$

Therefore we can express Γ^* and E_u^* as a function of V^* , u_3^* and of their derivatives. We will use the following matrix notations:

$$\Gamma^* = \begin{pmatrix} \Gamma_t^* & \Gamma_s^* \\ \Gamma_s^* & \Gamma_n^* \end{pmatrix} \quad \text{and} \quad E_u^* = \begin{pmatrix} E_{ut}^* & E_{us}^* \\ Q^* & E_{un}^* \end{pmatrix}.$$

The explicit expressions of E_u^* and Γ^* are not written here to simplify our derivations. We will directly introduce their nondimensional expressions.

So the equilibrium equations can be written in $\Omega_0^* = \omega_0^* \times] - h_0, h_0[$ as:

$$(2.4) \quad \begin{aligned} (\lambda + \mu)\text{grad}^*(\text{div}^* V^*) + \mu\Delta^* V^* + (\lambda + \mu)\text{grad}^* \frac{\partial u_3^*}{\partial x_3^*} + \mu \frac{\partial^2 V^*}{\partial x_3^{2*}} \\ + \text{div}^*[E_{ut}^* + \Gamma_t^*] + \frac{\partial}{\partial x_3^*}(\Gamma_s^* + Q^*) = -f_t^*, \\ (\lambda + \mu) \frac{\partial}{\partial x_3^*} \text{div}^* V^* + \mu\Delta^* u_3^* + (\lambda + 2\mu) \frac{\partial^2 u_3^*}{\partial x_3^{2*}} + \text{div}^*[E_{us}^* + \Gamma_s^*] \\ + \frac{\partial}{\partial x_3^*}(E_{un}^* + \Gamma_n^*) = -f_3^*, \end{aligned}$$

with the boundary conditions on the upper and the lower faces:

$$(2.5) \quad \begin{aligned} \mu(\text{grad}^* u_3^* + \frac{\partial V^*}{\partial x_3^*}) + Q^* + \Gamma_s^* &= \pm g_t^{*\pm} \quad \text{on } \bar{\omega}_0^* \times \{\pm h_0\}, \\ (\lambda + 2\mu) \frac{\partial u_3^*}{\partial x_3^*} + \lambda \text{div}^* V^* + E_{un}^* + \Gamma_n^* &= \pm g_3^{*\pm} \quad \text{on } \bar{\omega}_0^* \times \{\pm h_0\}. \end{aligned}$$

3. Dimensional analysis of the equilibrium equations

Like in the linear case [14], let us define the following dimensionless physical data and dimensionless unknowns of the problem:

$$\begin{aligned} V &= \frac{V^*}{V_r}, & u_3 &= \frac{u_3^*}{u_{3r}^*}, & x &= \frac{x^*}{L_0}, & x_3 &= \frac{x_3^*}{h_0}, \\ f_3 &= \frac{f_3^*}{f_{3r}^*}, & g_3^\pm &= \frac{g_{3r}^{*\pm}}{g_{3r}^*}, & f_t &= \frac{f_t^*}{f_{tr}^*}, & g_t^\pm &= \frac{g_{tr}^{*\pm}}{g_{tr}^*}, \end{aligned}$$

where the variables with subscript r are the reference ones. The new variables which appear (without the asterisk) are dimensionless.

To avoid any assumption concerning the order of magnitude of the displacement components, the reference scales u_{3r} and V_r are firstly assumed to be equal to L_0 . Thus we *a priori* allow large displacements. If necessary, it will always be possible to define new reference scales for the tangential and the normal displacement.

Introducing the previously defined dimensionless variables into Eqs. (2.4) and (2.5), we obtain a new nondimensional problem posed on $\Omega_0 = \omega_0 \times]-1, 1[$:

$$\begin{aligned} \varepsilon^2 \left[(1 + \beta) \text{grad}(\text{div} V) + \Delta V + \text{div} (E_{ut} + \Gamma_t) \right] \\ + \varepsilon \left[(1 + \beta) \text{grad} \frac{\partial u_3}{\partial x_3} + \frac{\partial}{\partial x_3} (Q + \Gamma_s) \right] + \frac{\partial^2 V}{\partial x_3^2} = -\varepsilon \mathcal{F}_t f_t, \end{aligned} \quad (3.1)$$

$$\begin{aligned} \varepsilon^2 \left[\Delta u_3 + \text{div} (E_{us} + \Gamma_s) \right] + \varepsilon \left[(1 + \beta) \frac{\partial}{\partial x_3} \text{div} V + \frac{\partial}{\partial x_3} (E_{un} + \Gamma_n) \right] \\ + (2 + \beta) \frac{\partial^2 u_3}{\partial x_3^2} = -\varepsilon \mathcal{F}_3 f_3. \end{aligned}$$

The boundary conditions on the upper and the lower faces become:

$$\begin{aligned} \varepsilon \left[\text{grad} u_3 + Q + \Gamma_s \right] + \frac{\partial V}{\partial x_3} = \pm \varepsilon \mathcal{G}_t g_t^\pm \quad \text{for } x_3 = \pm 1, \\ \varepsilon \left[\beta \text{div} V + E_{un} + \Gamma_n \right] + (2 + \beta) \frac{\partial u_3}{\partial x_3} = \pm \varepsilon \mathcal{G}_3 g_3^\pm \quad \text{for } x_3 = \pm 1, \end{aligned} \quad (3.2)$$

with $\beta = \frac{\lambda}{\mu}$ and

$$\begin{aligned} \Gamma_t = \frac{\beta}{2} \text{Tr} \left(\frac{\partial \overline{V}}{\partial x} \frac{\partial V}{\partial x} \right) I_2 + \frac{\beta}{2} \|\text{grad} u_3\|^2 I_2 + \frac{\partial \overline{V}}{\partial x} \frac{\partial V}{\partial x} + \text{grad} u_3 \overline{\text{grad} u_3} \\ + \frac{\beta}{2\varepsilon^2} \left[\left\| \frac{\partial V}{\partial x_3} \right\|^2 + \left(\frac{\partial u_3}{\partial x_3} \right)^2 \right] I_2, \end{aligned}$$

$$\Gamma_s = \frac{1}{\varepsilon} \left\{ \frac{\partial \overline{V}}{\partial x} \frac{\partial V}{\partial x_3} + \frac{\partial u_3}{\partial x_3} \text{grad} u_3 \right\},$$

$$\begin{aligned}
 \Gamma_n &= \frac{\beta}{2} \text{Tr} \left(\frac{\partial \overline{V}}{\partial x} \frac{\partial V}{\partial x} \right) + \frac{\beta}{2} \|\text{grad } u_3\|^2 + \frac{1}{\varepsilon^2} \left\{ \left(1 + \frac{\beta}{2}\right) \left[\left\| \frac{\partial V}{\partial x_3} \right\|^2 + \left(\frac{\partial u_3}{\partial x_3} \right)^2 \right] \right\}, \\
 E_{ut} &= \left[\frac{\beta}{2} \text{Tr} \left(\frac{\partial \overline{V}}{\partial x} \frac{\partial V}{\partial x} \right) I_2 + \frac{\partial \overline{V}}{\partial x} \frac{\partial V}{\partial x} + \frac{\partial V}{\partial x} + \frac{\partial \overline{V}}{\partial x} + \beta \text{div} V I_2 + \frac{\beta}{2} \|\text{grad } u_3\|^2 I_2 \right. \\
 &\quad \left. + \text{grad } u_3 \overline{\text{grad } u_3} \right] \frac{\partial \overline{V}}{\partial x} + \frac{1}{\varepsilon} \left\{ \text{grad } u_3 \frac{\partial \overline{V}}{\partial x_3} + \beta \frac{\partial u_3}{\partial x_3} \frac{\partial \overline{V}}{\partial x} \right\} \\
 &\quad + \frac{1}{\varepsilon^2} \left\{ \frac{\beta}{2} \left[\left\| \frac{\partial V}{\partial x_3} \right\|^2 + \left(\frac{\partial u_3}{\partial x_3} \right)^2 \right] \frac{\partial \overline{V}}{\partial x} + \left[\left(I_2 + \frac{\partial \overline{V}}{\partial x} \right) \frac{\partial V}{\partial x_3} + \frac{\partial u_3}{\partial x_3} \text{grad } u_3 \right] \frac{\partial \overline{V}}{\partial x_3} \right\}, \\
 E_{us} &= \left[\frac{\beta}{2} \text{Tr} \left(\frac{\partial \overline{V}}{\partial x} \frac{\partial V}{\partial x} \right) I_2 + \frac{\partial \overline{V}}{\partial x} \frac{\partial V}{\partial x} + \frac{\partial V}{\partial x} + \frac{\partial \overline{V}}{\partial x} + \beta \text{div } V I_2 \right. \\
 &\quad \left. + \left(1 + \frac{\beta}{2}\right) \|\text{grad } u_3\|^2 I_2 \right] \text{grad } u_3 + \frac{1}{\varepsilon} \left\{ \left(1 + \beta\right) \frac{\partial u_3}{\partial x_3} \text{grad } u_3 \right\} \\
 &\quad + \frac{1}{\varepsilon^2} \left\{ \left[\frac{\beta}{2} \left\| \frac{\partial V}{\partial x_3} \right\|^2 + \left(1 + \frac{\beta}{2}\right) \left(\frac{\partial u_3}{\partial x_3} \right)^2 \right] \text{grad } u_3 + \frac{\partial u_3}{\partial x_3} \left[I_2 + \frac{\partial \overline{V}}{\partial x} \right] \frac{\partial V}{\partial x_3} \right\}, \\
 Q &= \frac{\partial V}{\partial x} \text{grad } u_3 + \frac{1}{\varepsilon} \left\{ \left[\frac{\beta}{2} \text{Tr} \left(\frac{\partial \overline{V}}{\partial x} \frac{\partial V}{\partial x} \right) I_2 + \frac{\partial V}{\partial x} \frac{\partial \overline{V}}{\partial x} + \beta \text{div} V I_2 \right. \right. \\
 &\quad \left. \left. + \frac{\beta}{2} \|\text{grad } u_3\|^2 I_2 + \frac{\partial V}{\partial x} \right] \frac{\partial V}{\partial x_3} + \frac{\partial V}{\partial x} \frac{\partial u_3}{\partial x_3} \text{grad } u_3 \right\} + \frac{1}{\varepsilon^2} (2 + \beta) \frac{\partial u_3}{\partial x_3} \frac{\partial V}{\partial x_3} \\
 &\quad + \frac{1}{\varepsilon^3} \left(1 + \frac{\beta}{2}\right) \left[\left\| \frac{\partial V}{\partial x_3} \right\|^2 + \left(\frac{\partial u_3}{\partial x_3} \right)^2 \right] \frac{\partial V}{\partial x_3}, \\
 E_{un} &= \|\text{grad } u_3\|^2 + \frac{1}{\varepsilon} \left\{ \left[\frac{\beta}{2} \text{Tr} \left(\frac{\partial \overline{V}}{\partial x} \frac{\partial V}{\partial x} \right) + \beta \text{div} V \right. \right. \\
 &\quad \left. \left. + \left(1 + \frac{\beta}{2}\right) \|\text{grad } u_3\|^2 \right] \frac{\partial u_3}{\partial x_3} + \frac{\partial \overline{V}}{\partial x_3} \left[I_2 + \frac{\partial V}{\partial x} \right] \text{grad } u_3 \right\} \\
 &\quad + \frac{1}{\varepsilon^2} (2 + \beta) \left(\frac{\partial u_3}{\partial x_3} \right)^2 + \frac{1}{\varepsilon^3} \left[\left(1 + \frac{\beta}{2}\right) \left[\left\| \frac{\partial V}{\partial x_3} \right\|^2 + \left(\frac{\partial u_3}{\partial x_3} \right)^2 \right] \frac{\partial u_3}{\partial x_3} \right],
 \end{aligned}$$

where I_2 denotes the identity of \mathbb{R}^2 .

Hence, the dimensional analysis of the equilibrium equations leads naturally to the same dimensionless numbers as in the linear case [14]:

$$\varepsilon = \frac{h_0}{L_0}, \quad \mathcal{F}_3 = \frac{h_0 f_{3r}}{\mu}, \quad \mathcal{F}_t = \frac{h_0 f_{tr}}{\mu}, \quad \mathcal{G}_3 = \frac{g_{3r}}{\mu}, \quad \mathcal{G}_t = \frac{g_{tr}}{\mu}.$$

On the other hand, writing the condition (2.2) in a nondimensional form, we obtain

$$\det \left(L_0 \frac{\partial \psi}{\partial X} \frac{\partial X}{\partial X^*} \right) = L_0^3 \det \left(\frac{\partial \psi}{\partial X} \right) \det \left(\frac{\partial X}{\partial X^*} \right) \geq a \quad \text{in } \Omega_0$$

with $\psi = \frac{\psi^*}{L_0}$. Since we have

$$\det \left(\frac{\partial X}{\partial X^*} \right) = \frac{1}{h_0 L_0^2},$$

the condition (2.2) becomes:

$$(3.3) \quad \det F = \det \left(\frac{\partial \psi}{\partial X} \right) \geq \varepsilon a > 0 \quad \forall \varepsilon > 0.$$

3.1. Interpretation of the dimensionless numbers

i) The shape ratio $\varepsilon = \frac{h_0}{L_0}$ of the initial thickness of the plate to the diameter of the middle surface ω_0^* is a known parameter of the problem. For thin plates, ε is a small parameter.

ii) The ratios of forces $\mathcal{F}_t = \frac{h_0 f_{tr}}{\mu}$, $\mathcal{F}_3 = \frac{h_0 f_{3r}}{\mu}$ and $\mathcal{G}_t = \frac{g_{tr}}{\mu}$, $\mathcal{G}_3 = \frac{g_{3r}}{\mu}$ represent respectively the ratios of the resultant body forces (acting across the thickness of the plate) or of the surface forces to μ considered as a reference stress. These numbers depend only on known physical quantities and are known data of the problem.

3.2. Reduction to a single-scale problem

In order to obtain a single-scale problem, ε is chosen as the reference parameter. Therefore the other dimensionless numbers must be linked to ε .

In usual elasticity problems, the body force f^* is often due to the gravity. As an example, let us consider a thin steel plate, of 1m diameter and 10^{-2} m thickness, whose Young's modulus, Poisson's ratio and voluminal mass are, respectively, $E = 2.1 \cdot 10^{11}$ Pa, $\nu = 0.285$ and $\rho = 7800$ kg/m³. If we assume that

the plate is subjected only to the gravity force, we find $\mathcal{F}_3 = \frac{\rho gh}{\mu} = 10^{-8} = \varepsilon^4$. In this example the tangential component of the weight is equal to zero, so that we have $\mathcal{F}_t = 0$. Accordingly, in this paper we consider the body forces level to be of the order of magnitude of the weight which satisfies $\mathcal{F}_3 = \varepsilon^4$ and $\mathcal{F}_t = 0$.

Nevertheless we will distinguish two different magnitudes of surface forces which lead to two different two-dimensional models. In this first part of the article, we consider large surface forces such as $\mathcal{G}_3 = \mathcal{G}_t = \varepsilon$ to obtain large displacements. In the second part of this article, we will consider a plate subjected to the same moderate surface forces as in the linear case [14]: $\mathcal{G}_3 = \varepsilon^4$ and $\mathcal{G}_t = \varepsilon^3$.

However it would be possible to take into account other body forces (like centrifugal acceleration) whose tangential component is not equal to zero. In this case, \mathcal{F}_t must be linked to ε .

4. The nonlinear two-dimensional membrane model

Let us consider in this section a thin plate subjected to body forces such as $\mathcal{F}_3 = \varepsilon^4$, $\mathcal{F}_t = 0$ and to the important surface forces of the magnitude $\mathcal{G}_3 = \mathcal{G}_t = \varepsilon$. Problem (3.1)–(3.2) being reduced to a single-scale problem with ε as a small parameter, the standard asymptotic expansion method leads to write the nondimensional solution (V, u_3) as a formal expansion with respect to ε :

$$(4.1) \quad \begin{aligned} V &= V^0 + \varepsilon V^1 + \varepsilon^2 V^2 + \dots, \\ u_3 &= u_3^0 + \varepsilon u_3^1 + \varepsilon^2 u_3^2 + \dots. \end{aligned}$$

Therefore, we obtain the following result:

RESULT 1

For a plate subjected to applied forces such as $\mathcal{F}_3 = \varepsilon^4$ and $\mathcal{G}_t = \mathcal{G}_3 = \varepsilon$, the leading term (V^0, u_3^0) of the asymptotic expansion of (V, u_3) depends only on $x = (x_1, x_2)$ and solves the following membrane problem:

$$\begin{aligned} \operatorname{div} \left(N_t^0 \left[I_2 + \frac{\partial V^0}{\partial x} \right] \right) &= -p_t, \\ \operatorname{div} (N_t^0 \operatorname{grad} u_3^0) &= -p_3, \\ V^0 &= 0 \quad \text{and} \quad u_3^0 = 0 \quad \text{on} \quad \gamma_0 = \partial\omega_0, \end{aligned}$$

where:

$$\begin{aligned} N_t^0 &= \frac{4\beta}{2 + \beta} \operatorname{Tr} E_t^0 I_2 + 4E_t^0, \\ E_t^0 &= \frac{1}{2} \left(\frac{\partial V^0}{\partial x} + \overline{\frac{\partial V^0}{\partial x}} + \frac{\partial V^0}{\partial x} \frac{\partial V^0}{\partial x} + \operatorname{grad} u_3^0 \overline{\operatorname{grad} u_3^0} \right), \end{aligned}$$

$$p_3 = g_3^+ + g_3^- \quad p_t = g_t^+ + g_t^-.$$

P r o o f. The proof of this result is divided into several steps from i) to iv).

i) V^0 and u_3^0 depends only on (x_1, x_2)

Replacing V and u_3 by their expansions (4.1) in the nondimensional equilibrium equations (3.1) and (3.2) and equating to zero the factors of successive powers of ε , we obtain the coupled problems $\mathcal{P}_{-2}, \mathcal{P}_{-1}, \mathcal{P}_0, \dots$

The problem \mathcal{P}_{-2} can be written as

$$(4.2) \quad \frac{\partial}{\partial x_3}(Q^{-3} + \Gamma_s^{-3}) = 0 \quad \text{in } \Omega_0,$$

$$(4.3) \quad \frac{\partial}{\partial x_3}(E_{un}^{-3} + \Gamma_n^{-3}) = 0 \quad \text{in } \Omega_0,$$

$$(4.4) \quad Q^{-3} + \Gamma_s^{-3} = 0 \quad \text{for } x_3 = \pm 1,$$

$$(4.5) \quad E_{un}^{-3} + \Gamma_n^{-3} = 0 \quad \text{for } x_3 = \pm 1,$$

with:

$$Q^{-3} = \left(1 + \frac{\beta}{2}\right) \left[\left\| \frac{\partial V^0}{\partial x_3} \right\|^2 + \left(\frac{\partial u_3^0}{\partial x_3} \right)^2 \right] \frac{\partial V^0}{\partial x_3},$$

$$E_{un}^{-3} = \left(1 + \frac{\beta}{2}\right) \left[\left\| \frac{\partial V^0}{\partial x_3} \right\|^2 + \left(\frac{\partial u_3^0}{\partial x_3} \right)^2 \right] \frac{\partial u_3^0}{\partial x_3},$$

$$\Gamma_3^{-3} = 0 \quad \Gamma_n^{-3} = 0.$$

Taking into account the boundary conditions (4.4) and (4.5), equations (4.2) and (4.3) imply the relations

$$\left[\left\| \frac{\partial V^0}{\partial x_3} \right\|^2 + \left(\frac{\partial u_3^0}{\partial x_3} \right)^2 \right] \frac{\partial V^0}{\partial x_3} = 0 \quad \text{in } \Omega_0,$$

$$\left[\left\| \frac{\partial V^0}{\partial x_3} \right\|^2 + \left(\frac{\partial u_3^0}{\partial x_3} \right)^2 \right] \frac{\partial u_3^0}{\partial x_3} = 0 \quad \text{in } \Omega_0,$$

which lead to $\frac{\partial u_3^0}{\partial x_3} = 0$ and $\frac{\partial V^0}{\partial x_3} = 0$ in Ω_0 . Hence, we finally obtain:

$$(4.6) \quad \begin{aligned} V^0 &= V^0(x_1, x_2), \\ u_3^0 &= u_3^0(x_1, x_2). \end{aligned}$$

ii) Determination of u_3^1 and V^1

Considering expression (4.6), we have $Q^{-2} = E_{un}^{-2} = \Gamma_n^{-2} = 0$ so that problems \mathcal{P}_{-1} and \mathcal{P}_0 are evidently satisfied. Equating to zero the coefficient of ε^1 , problem \mathcal{P}_1 can be written in the form

$$(4.7) \quad \frac{\partial}{\partial x_3}(Q^0 + \Gamma_s^0) + \frac{\partial^2 V^1}{\partial x_3^2} = 0 \quad \text{in } \Omega_0,$$

$$(4.8) \quad \frac{\partial}{\partial x_3}(\Gamma_n^0 + E_{un}^0) + (2 + \beta) \frac{\partial^2 u_3^1}{\partial x_3^2} = 0 \quad \text{in } \Omega_0,$$

$$(4.9) \quad Q^0 + \Gamma_s^0 + \frac{\partial V^1}{\partial x_3} + \text{grad } u_3^0 = 0 \quad \text{for } x_3 = \pm 1,$$

$$(4.10) \quad \Gamma_n^0 + E_{un}^0 + \beta \text{div } V^0 + (2 + \beta) \frac{\partial u_3^1}{\partial x_3} = 0 \quad \text{for } x_3 = \pm 1,$$

with

$$\Gamma_s^0 = \frac{\partial u_3^1}{\partial x_3} \text{grad } u_3^0 + \frac{\overline{\partial V^0}}{\partial x} \frac{\partial V^1}{\partial x_3},$$

$$\Gamma_n^0 = \frac{\beta}{2} \text{Tr} \left(\frac{\partial V^0}{\partial x} \frac{\partial V^0}{\partial x} \right) + \frac{\beta}{2} \|\text{grad } u_3^0\|^2 + \left(1 + \frac{\beta}{2} \right) \left[\left\| \frac{\partial V^1}{\partial x_3} \right\|^2 + \left(\frac{\partial u_3^1}{\partial x_3} \right)^2 \right],$$

$$\begin{aligned} Q^0 &= \left[\beta \text{Tr} E_t^0 I_2 + \frac{1 + \beta}{2} \left(\left\| \frac{\partial \psi^1}{\partial x_3} \right\|^2 - 1 \right) I_2 + \frac{\partial V^0}{\partial x} \frac{\overline{\partial V^0}}{\partial x} + \frac{\partial V^0}{\partial x} \right] \frac{\partial V^1}{\partial x_3} \\ &\quad + \frac{\partial V^0}{\partial x} \text{grad } u_3^0 \left(1 + \frac{\partial u_3^1}{\partial x_3} \right), \end{aligned}$$

$$\begin{aligned} E_{un}^0 &= \left[\beta \text{Tr} E_t^0 + \left(\left\| \frac{\partial \psi^1}{\partial x_3} \right\|^2 - 1 \right) \right] \frac{\partial u_3^1}{\partial x_3} + \frac{\overline{\partial V^1}}{\partial x_3} \left[I_2 + \frac{\partial V^0}{\partial x} \right] \text{grad } u_3^0 \\ &\quad + \left(1 + \frac{\partial u_3^1}{\partial x_3} \right) \|\text{grad } u_3^0\|^2, \end{aligned}$$

and

$$E_t^0 = \frac{1}{2} \left(\frac{\partial V^0}{\partial x} + \frac{\overline{\partial V^0}}{\partial x} + \frac{\overline{\partial V^0}}{\partial x} \frac{\partial V^0}{\partial x} + \text{grad } u_3^0 \overline{\text{grad } u_3^0} \right).$$

Let us define

$$\psi^1 = V^1 + (x_3 + u_3^1)e_3 = \begin{pmatrix} V^1 \\ x_3 + u_3^1 \end{pmatrix}_{(e_i)}$$

so we have

$$\left\| \frac{\partial \psi^1}{\partial x_3} \right\|^2 = 1 + \left\| \frac{\partial V^1}{\partial x_3} \right\|^2 + \left(\frac{\partial u_3^1}{\partial x_3} \right)^2 + 2 \frac{\partial u_3^1}{\partial x_3}.$$

The endomorphism E_t^0 represents the membrane strain due to the displacement (V^0, u_3^0) . In particular, we obtain

$$\text{Tr } E_t^0 = \text{div } V^0 + \frac{1}{2} \text{Tr} \left(\frac{\overline{\partial V^0}}{\partial x} \frac{\partial V^0}{\partial x} \right) + \frac{1}{2} \|\text{grad } u_3^0\|^2.$$

REMARK 2

It is natural to introduce the functional mapping ψ^1 to simplify the equations. Indeed, introducing the functional mapping ψ^* , we have

$$\psi^* = X^* + U^* = x_\alpha^* e_\alpha + x_3^* e_3 + U^*. \blacksquare$$

We then obtain the following nondimensional equation:

$$\psi = \frac{\psi^*}{L} = (x_\alpha e_\alpha + U) + \varepsilon x_3 e_3$$

where $U = V + u_3 e_3$. The expansion (4.1) of (V, u_3) with respect to ε implies then the following expansion of ψ :

$$(4.11) \quad \psi = \underbrace{x_\alpha e_\alpha + V^0 + u_3^0 e_3}_{\psi^0} + \varepsilon \underbrace{[(x_3 + u_3^1)e_3 + V^1]}_{\psi^1} + \dots$$

The functional mapping ψ^0 represents the deformed middle surface ω_0 at the leading order of the expansion. With these notations, the membrane strain E_t^0 can also be written

$$(4.12) \quad E_t^0 = \frac{1}{2} \left(\frac{\partial \psi^0}{\partial x} \frac{\partial \psi^0}{\partial x} - I_2 \right) \quad \text{with} \quad \frac{\partial \psi^0}{\partial x} = \begin{pmatrix} I_2 + \frac{\partial V^0}{\partial x} \\ \overline{\text{grad } u_3^0} \end{pmatrix}_{(e_i)}.$$

Considering the boundary conditions (4.9) and (4.10), Eqs. (4.7) and (4.8) lead to

$$(4.13) \quad Q^0 + \Gamma_s^0 + \frac{\partial V^1}{\partial x_3} + \text{grad } u_3^0 = 0 \quad \text{in } \Omega_0,$$

$$(4.14) \quad \Gamma_n^0 + E_{un}^0 + \beta \text{div } V^0 + (2 + \beta) \frac{\partial u_3^1}{\partial x_3} = 0 \quad \text{in } \Omega_0.$$

Since we have

$$\Gamma_n^0 + \beta \text{div } V^0 + (2 + \beta) \frac{\partial u_3^1}{\partial x_3} = \beta \text{Tr } E_t^0 + \left(1 + \frac{\beta}{2}\right) \left(\left\|\frac{\partial \psi^1}{\partial x_3}\right\|^2 - 1\right),$$

equations (4.13) and (4.14) become

$$A \frac{\partial V^1}{\partial x_3} + \left(I_2 + \frac{\partial V^0}{\partial x}\right) B = 0,$$

$$A \left(1 + \frac{\partial u_3^1}{\partial x_3}\right) + \overline{\text{grad } u_3^0} B = 0,$$

with

$$A = \beta \text{Tr } E_t^0 + \left(1 + \frac{\beta}{2}\right) \left(\left\|\frac{\partial \psi^1}{\partial x_3}\right\|^2 - 1\right),$$

$$B = \left(1 + \frac{\partial u_3^1}{\partial x_3}\right) \text{grad } u_3^0 + \left(I_2 + \frac{\partial V^0}{\partial x}\right) \frac{\partial V^1}{\partial x}.$$

Using matrix notations, the last equation can be written as

$$A \begin{pmatrix} \frac{\partial V^1}{\partial x_3} \\ 1 + \frac{\partial u_3^1}{\partial x_3} \end{pmatrix} + \begin{pmatrix} I_2 + \frac{\partial V^0}{\partial x} \\ \overline{\text{grad } u_3^0} \end{pmatrix} B = 0,$$

and considering the expressions of ψ^0 and ψ^1 , we get

$$(4.15) \quad A \frac{\partial \psi^1}{\partial x_3} + \frac{\partial \psi^0}{\partial x} B = 0.$$

Now we finally use the mass conservation law. As ψ^0 is independent of the variable x_3 , the expansion of the condition (3.3) gives

$$\frac{\partial \psi^1}{\partial x_3} \cdot \left(\frac{\partial \psi^0}{\partial x_1} \wedge \frac{\partial \psi^0}{\partial x_2} \right) + O(\varepsilon) \geq a > 0 \quad \forall \varepsilon > 0,$$

where \wedge denotes the cross product in \mathbb{R}^3 . Thus, we must have

$$\frac{\partial\psi^1}{\partial x_3} \cdot \left(\frac{\partial\psi^0}{\partial x_1} \wedge \frac{\partial\psi^0}{\partial x_2} \right) > 0 \quad \text{in } \Omega_0.$$

This condition implies that the vectors $\left(\frac{\partial\psi^0}{\partial x_1}, \frac{\partial\psi^0}{\partial x_2}, \frac{\partial\psi^1}{\partial x_3} \right)$ are linearly independent, and form an \mathbb{R}^3 basis.

Going back to equation (4.15) where $\frac{\partial\psi^0}{\partial x} = \left(\frac{\partial\psi^0}{\partial x_1}, \frac{\partial\psi^0}{\partial x_2} \right)$ represents the local basis of the deformed configuration $\psi^0(\omega_0)$, we have

$$(4.16) \quad A = \beta \text{Tr } E_t^0 + \left(1 + \frac{\beta}{2} \right) \left(\left\| \frac{\partial\psi^1}{\partial x_3} \right\|^2 - 1 \right) = 0$$

and

$$(4.17) \quad B = \left(1 + \frac{\partial u_3^1}{\partial x_3} \right) \text{grad } u_3^0 + \left(I_2 + \frac{\overline{\partial V^0}}{\partial x} \right) \frac{\partial V^1}{\partial x} = 0.$$

The second equation (4.17) can also be written as

$$\frac{\overline{\partial\psi^0}}{\partial x} \frac{\partial\psi^1}{\partial x_3} = 0$$

and implies that

$$\frac{\partial\psi^1}{\partial x_3} = \eta N,$$

where N denotes the unit normal to the surface $\omega = \psi^0(\omega_0)$.

Thanks to equation (4.16) it is now possible to determine the norm $\left\| \frac{\partial\psi^1}{\partial x_3} \right\|$. Indeed

we have

$$(4.18) \quad \left\| \frac{\partial\psi^1}{\partial x_3} \right\|^2 = 1 + \frac{2\beta}{2 + \beta} \text{Tr } E_t^0 = 1 + \frac{2\lambda}{\lambda + 2\mu} \text{Tr } E_t^0.$$

Finally we obtain the following expression of ψ^1 :

$$(4.19) \quad \begin{aligned} \psi^1 &= \eta N + \tilde{u}^1(x_1, x_2), \\ \eta^2 &= 1 + \frac{2\lambda}{\lambda + 2\mu} \text{Tr } E_t^0. \end{aligned}$$

iii) The first membrane equation

The cancellation of the coefficient of ε^2 leads to the problem \mathcal{P}_2 in Ω_0 :

$$(4.20) \quad (1 + \beta)\text{grad}(\text{div } V^0) + \Delta V^0 + \text{div}(E_{ut}^0 + \Gamma_t^0) + (1 + \beta) \text{grad} \frac{\partial u_3^1}{\partial x_3} + \frac{\partial}{\partial x_3}(Q^1 + \Gamma_s^1) + \frac{\partial^2 V^2}{\partial x_3^2} = 0,$$

$$(4.21) \quad \Delta u_3^0 + \text{div}(E_{us}^0 + \Gamma_s^0) + (1 + \beta) \frac{\partial}{\partial x_3} \text{div } V^1 + \frac{\partial}{\partial x_3} (\Gamma_n^1 + E_{un}^1) + (2 + \beta) \frac{\partial^2 u_3^2}{\partial x_3^2} = 0,$$

with the following boundary conditions on the upper and the lower faces:

$$(4.22) \quad \text{grad } u_3^1 + Q^1 + \Gamma_s^1 + \frac{\partial V^2}{\partial x_3} = \pm g_t^\pm \quad \text{for } x_3 = \pm 1,$$

$$(4.23) \quad \beta \text{div } V^1 + \Gamma_n^1 + E_{un}^1 + (2 + \beta) \frac{\partial u_3^2}{\partial x_3} = \pm g_3^\pm \quad \text{for } x_3 = \pm 1.$$

Considering equations (4.17) and (4.18), the expression of E_{ut}^0 , Γ_s^0 , E_{us}^0 and Γ_t^0 can be written as:

$$E_{ut}^0 = \left[\frac{2\beta}{2 + \beta} \text{Tr } E_t^0 I_2 + \frac{\partial V^0}{\partial x} + \frac{\overline{\partial V^0}}{\partial x} + \frac{\overline{\partial V^0}}{\partial x} \frac{\partial V^0}{\partial x} + \text{grad } u_3^0 \overline{\text{grad } u_3^0} \right] \frac{\partial V^0}{\partial x},$$

$$\Gamma_s^0 = \frac{\partial u_3^1}{\partial x_3} \text{grad } u_3^0 + \frac{\overline{\partial V^0}}{\partial x} \frac{\partial V^1}{\partial x_3},$$

$$E_{us}^0 = \left[\frac{2\beta}{2 + \beta} \text{Tr } E_t^0 I_2 + \frac{\partial V^0}{\partial x} + \frac{\overline{\partial V^0}}{\partial x} + \frac{\overline{\partial V^0}}{\partial x} \frac{\partial V^0}{\partial x} + \|\text{grad } u_3^0\|^2 I_2 \right] \text{grad } u_3^0,$$

$$\Gamma_t^0 = \frac{\beta}{2} \text{Tr} \left(\frac{\overline{\partial V^0}}{\partial x} \frac{\partial V^0}{\partial x} \right) I_2 + \frac{\beta}{2} \|\text{grad } u_3^0\|^2 I_2 + \frac{\overline{\partial V^0}}{\partial x} \frac{\partial V^0}{\partial x} + \text{grad } u_3^0 \overline{\text{grad } u_3^0} + \frac{\beta}{2} \left[\left\| \frac{\partial V^1}{\partial x_3} \right\|^2 + \left(\frac{\partial u_3^1}{\partial x_3} \right)^2 \right] I_2.$$

Then using the fact that

$$\text{grad}(\text{div } V^0) = \text{div}(\text{div } V^0 I_2) \quad \text{and} \quad \text{grad} \frac{\partial u_3^1}{\partial x_3} = \text{div} \left(\frac{\partial u_3^1}{\partial x_3} I_2 \right)$$

and taking into account the boundary conditions (4.22), integration by parts of (4.20) leads to

$$(4.24) \quad \int_{-1}^1 \left[\text{grad}(\text{div } V^0) + \Delta V^0 + \text{div} \left(E_{ut}^0 + \Gamma_t^0 + \beta \frac{\partial u_3^1}{\partial x_3} I_2 + \beta \text{div } V^0 I_2 \right) \right] dx_3 = -p_t$$

with $p_t = g_t^+ + g_t^-$.

On the other hand, considering the expression of Γ_t^0 , we get:

$$\Gamma_t^0 + \beta \frac{\partial u_3^1}{\partial x_3} I_2 + \beta \text{div } V^0 I_2 = \frac{2\beta}{2 + \beta} \text{Tr} E_t^0 I_2 + \frac{\overline{\partial V^0}}{\partial x} \frac{\partial V^0}{\partial x} + \text{grad } u_3^0 \overline{\text{grad } u_3^0}.$$

Using the relations

$$\begin{aligned} \text{grad}(\text{div } V^0) &= \text{div}(\text{grad } V^0) = \text{div} \left(\frac{\partial V^0}{\partial x} \right) \quad \text{and} \\ \Delta V^0 &= \text{div}(\overline{\text{grad } V^0}) = \text{div} \left(\frac{\partial V^0}{\partial x} \right), \end{aligned}$$

equation (4.24) becomes

$$\text{div} \left(N_t^0 \left[I_2 + \frac{\partial V^0}{\partial x} \right] \right) = -p_t$$

with

$$\begin{aligned} N_t^0 &= \frac{4\beta}{2 + \beta} \text{Tr} E_t^0 I_2 + 4E_t^0, \\ E_t^0 &= \frac{1}{2} \left(\frac{\partial V^0}{\partial x} + \frac{\overline{\partial V^0}}{\partial x} + \frac{\overline{\partial V^0}}{\partial x} \frac{\partial V^0}{\partial x} + \text{grad } u_3^0 \overline{\text{grad } u_3^0} \right), \\ p_t &= g_t^+ + g_t^-. \end{aligned}$$

iv) The second membrane equation

Considering the boundary conditions (4.23), an integration from -1 to 1 of equation (4.21) leads to

$$(4.25) \quad \int_{-1}^1 \left[\Delta u_3^0 + \text{div} \left(E_{us}^0 + \Gamma_s^0 + \frac{\partial V^1}{\partial x_3} \right) \right] dx_3 = -p_3$$

with $p_3 = g_3^+ - g_3^-$.

Using the property $\Delta u_3^0 = \text{div}(\text{grad} u_3^0)$, we obtain

$$\Delta u_3^0 + \text{div} \left(\Gamma_s^0 + \frac{\partial V^1}{\partial x_3} \right) = \text{div} B = 0$$

thanks to the orthogonality condition (4.17). Finally equation (4.25) becomes:

$$\text{div} \left(N_t^0 \text{grad} u_3^0 \right) = -p_3$$

and the RESULT 1 is then proved.

4.1. Associated variational formulation

Let us define the space of admissible functional mappings

$$\mathcal{Q}(\omega_0) = \left\{ \psi : \omega_0 \mapsto \mathbb{R}^3, \text{ "smooth", } \psi = I_2 \text{ on } \gamma_0 \right\}$$

and the space of kinematically admissible displacements:

$$V(\omega_0) = \left\{ v^0 : \omega_0 \mapsto \mathbb{R}^3, \text{ "smooth", } v = 0 \text{ on } \gamma_0 \right\}.$$

If the displacements are assumed to be smooth enough, then the two-dimensional membrane equations of the RESULT 1 can be written in the following variational form which depends only on ψ^0 (instead of V^0 and u_3^0):

RESULT 2

$\psi^0 \in \mathcal{Q}(\omega_0)$ satisfies the following variational problem:

$$(4.26) \quad \int_{\omega_0} \text{Tr} \left[N_t^0 \frac{\partial \psi^0}{\partial x} \frac{\partial v^0}{\partial x} \right] d\omega_0 = \int_{\omega_0} p \cdot v^0 d\omega_0 \quad \forall v^0 \in V(\omega_0)$$

where

$$N_t^0 = \frac{4\beta}{2 + \beta} \text{Tr} E_t^0(\psi^0) I_2 + 4E_t^0(\psi^0),$$

$$E_t^0(\psi^0) = \frac{1}{2} \left(\frac{\partial \psi^0}{\partial x} \frac{\partial \psi^0}{\partial x} - I_2 \right),$$

$$p = p_t + p_3 e_3.$$

Indeed, considering the expression (4.11) for $\psi^0 = x_\alpha e_\alpha + u_3^0 e_3 + V^0$, the membrane strain E_t^0 can be written as:

$$E_t^0 = \frac{1}{2} \left(\overline{\frac{\partial \psi^0}{\partial x} \frac{\partial \psi^0}{\partial x}} - I_2 \right).$$

Now, if we consider a displacement field of $V(\omega_0)$ of the form $v^0 = v_t^0 + v_3^0 e_3$, the membrane equations of the RESULT 1 becomes:

$$\begin{aligned} \int_{\omega_0} \operatorname{div} \left[N_t^0 \left(I_2 + \frac{\partial V^0}{\partial x} \right) \right] v_t^0 \, d\omega_0 + \int_{\omega_0} \operatorname{div} \left[N_t^0 \operatorname{grad} u_3^0 \right] v_3^0 \, d\omega_0 \\ = - \int_{\omega_0} p_t \cdot v_t^0 \, d\omega_0 - \int_{\omega_0} p_3 v_3^0 \, d\omega_0. \end{aligned}$$

Setting $p = p_t + p_3 e_3$, we get

$$\begin{aligned} \int_{\omega_0} \operatorname{Tr} \left[N_t^0 \left(I_2 + \frac{\partial V^0}{\partial x} \right) \frac{\partial v_t^0}{\partial x} \right] \, d\omega_0 + \int_{\omega_0} \operatorname{Tr} \left[N_t^0 \operatorname{grad} u_3^0 \overline{\operatorname{grad} v_3^0} \right] \, d\omega_0 \\ = \int_{\omega_0} p \cdot v^0 \, d\omega_0. \end{aligned}$$

Finally, using matrix notations, we have:

$$\frac{\partial \psi^0}{\partial x} = \left(\begin{array}{c} I_2 + \frac{\partial V^0}{\partial x} \\ \overline{\operatorname{grad} u_3^0} \end{array} \right) \quad \text{and} \quad \frac{\partial v^0}{\partial x} = \left(\begin{array}{c} \frac{\partial v_t^0}{\partial x} \\ \overline{\operatorname{grad} v_3^0} \end{array} \right).$$

Thus we easily obtain the variational formulation of the RESULT 2.

This variational formulation of membrane equations is identical (more and less the coefficient μ) to the one obtained by D. FOX *et al.* in [9] from the three-dimensional variational formulation of the problem. Nevertheless, the asymptotic approach developed in this paper which leads to the membrane equations of the RESULT 1 presents some advantages:

- we naturally obtain a decomposition of the equilibrium equations into a tangential component (in the plane (O, e_1, e_2) of the initial middle surface ω_0) and a normal component.

- the naturally introduced dimensionless numbers define the domain of validity of the two-dimensional membrane model. Indeed the membrane model is valid for surface forces level such as $G_t = G_3 = \varepsilon$. These forces lead to displacements of order L_0 , i.e to large displacements.

This remark concerning the domain of validity of the membrane model adds some importance to a limitation of Lagrangian approaches where the loads are assumed to be dead. Indeed the membrane model we have obtained is valid for large displacements, and in this case the dead loads hypothesis is not justified. However, we can notice that an Eulerian approach for plates with large displacements has been developed in [15]. With this Eulerian approach, the dead loads hypothesis can be dropped and the Eulerian membrane model we obtain takes into account the real physical forces.

4.2. Back to the dimensional variables

The return to the physical variables in the membrane equations of the RESULT 1 leads to the relations:

$$V^{*0} = V_r V^0 = L_0 V^0,$$

$$u_3^{*0} = u_{3r} u_3^0 = L_0 u_3^0.$$

Therefore, we have the following result:

RESULT 3

For applied forces f^* and g^* such as $\mathcal{F}_3 = \varepsilon^4$ and $\mathcal{G}_t = \mathcal{G}_3 = \varepsilon$, the displacement (V^{*0}, u_3^{*0}) depends only on $x^* = (x_1^*, x_2^*)$ and satisfies the following nonlinear membrane model:

$$h_0 \operatorname{div}^* \left(N_t^{*0} \left[I_2 + \frac{\overline{\partial V^{*0}}}{\partial x^*} \right] \right) = -p_t^*,$$

$$h_0 \operatorname{div}^* (N_t^{*0} \operatorname{grad}^* u_3^{*0}) = -p_3^*,$$

$$V^{*0} = 0 \quad \text{and} \quad u_3^{*0} = 0 \quad \text{on} \quad \gamma_0^* = \partial \omega_0^*,$$

where:

$$N_t^{*0} = \frac{4\lambda\mu}{\lambda + 2\mu} \operatorname{Tr} E_t^{*0} I_2 + 4\mu E_t^{*0},$$

$$E_t^{*0} = \frac{1}{2} \left(\frac{\partial V^{*0}}{\partial x^*} + \frac{\overline{\partial V^{*0}}}{\partial x^*} + \frac{\overline{\partial V^{*0}}}{\partial x^*} \frac{\partial V^{*0}}{\partial x^*} + \operatorname{grad}^* u_3^{*0} \overline{\operatorname{grad}^* u_3^{*0}} \right),$$

$$p_3^* = g_3^{*+} + g_3^{*-}, \quad p_t^* = g_t^{*+} + g_t^{*-}.$$

In the first part of this article we have proved, without any assumption, that the nonlinear membrane model is valid for large surface forces level. Precisely, the dimensional analysis of the three-dimensional equilibrium equations naturally

leads to dimensionless numbers ($\mathcal{F}_3 = h_0 f_{3r}/\mu$, $\mathcal{G}_t = g_{tr}/\mu$ and $\mathcal{G}_3 = g_{3r}/\mu$) which reflect the forces level. Thus the membrane model we have obtained is valid for forces level such as \mathcal{F}_3 is of ε^4 order, \mathcal{G}_t and \mathcal{G}_3 are of ε order.

In the second part of this article, we will consider a plate subjected to moderate forces level. In this case the dimensional analysis and the asymptotic expansions of the equilibrium equations lead to the nonlinear von Kármán model.

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LABORATOIRE DE MODÉLISATION MÉCANIQUE
ET DE MATHÉMATIQUES APPLIQUÉES (L3.M.A)
UNIVERSITÉ DE POITIERS ET ENSMA

SP2MI, Téléport 2, 86960 Futuroscope cedex, France.
e-mail : millet@l3ma.univ-poitiers.fr

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