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On distortion of waves in a nonlinear magnetoelastic conductor

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WAVE DISTORTION and formation of shocks due to the elastic nonlinearity of the medium in the presence of a magnetic field are studied using the multiple scales technique for a one-dimensional travelling longitudinal wave. Condition for formation of shocks has been obtained for a sinusoidal signal.

1. Introduction

TRAVELLING WAVES in a nonlinear elastic medium are studied for the purpose of understanding the phenomena of distortion and formation of shocks. Problems of propagation of one-dimensional longitudinal and transverse waves in nonlinear elasticity have been studied by NAYFEH [1], LARDNER [2, 3]. Growth of amplitude and shock formation were investigated by them using the perturbation and multiple scales technique. The effect of a magnetic field on elastic waves was discussed by MAUGIN in [4], where problems of propagation of harmonic waves in hyperelastic non-linear magnetic dielectrics and shocks and simple waves in a perfectly conducting nonlinear elastic conductor have been considered. HEFNI *et al.* [5] have studied general one-dimensional bulk waves in a non-linear magnetoelastic conductor. They discussed both linear and nonlinear waves, starting from the general formulation of constitutive equations. However, the interesting phenomena of distortion as well as shock formation have not been treated there.

2. Basic equations

We consider a non-linear one-dimensional wave propagating in a perfectly conducting elastic medium in the presence of a uniform magnetic field $\mathbf{H}^0$ transverse to the direction of wave propagation. Maxwell's equations of the electromagnetic field are:

$$
\begin{aligned}
\operatorname{div}\mathbf{B} &= 0,\\
\operatorname{curl}\mathbf{H} &= \mathbf{J},\\
\operatorname{div}\mathbf{D} &= 0,\\
\operatorname{curl}\mathbf{E} &= -\mathbf{B}_t
\end{aligned}
\tag{2.1}
$$

where the displacement current has been neglected.

The constitutive equations are

$$\mathbf{B} = \mu\mathbf{H}, \tag{2.2}$$
$$\mathbf{D} = \varepsilon_1\mathbf{E}.$$

Ohm's law gives

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u}_t \times \mathbf{B}), \tag{2.2$_3$}$$

σ being the conductivity and ε_1 the electric permittivity.

Following BLAND [6], the equations of motion in a conducting medium with a magnetic field **B** are:

$$\mathbf{L}_{ij \cdot j} + (\mathbf{J} \times \mathbf{B})_i = \rho u_{i_{tt}}, \tag{2.3}$$

where

$$\mathbf{L}_{ij} = \frac{\partial W}{\partial u_{i,j}} \tag{2.4}$$

is the Piola-Kirchhoff stress tensor, W being the strain-energy of the material per unit volume.

For a hyperelastic material, W may be taken (correct up to the third power of strain):

$$W = \frac{1}{2}\lambda I_1^2 + GI_2 + \alpha\ I_1^3 + \beta\ I_1 I_2 + \gamma\ I_3. \tag{2.5}$$

I_1, I_2, I_3 are the strain invariants given by

$$I_1 = e_{ii}, \quad I_2 = e_{ij}e_{ij}\ , \quad I_3 = e_{ij}e_{jk}e_{ki}. \tag{2.6}$$

The strain components in terms of displacement u are given by

$$e_{ij} = \left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right)/2. \tag{2.7}$$

$J \times B$ is the Lorentz force per unit volume due to the magnetic field **B** and the current density J.

3. Formulation

Referred to rectangular axes of coordinates (x, y, z), we consider a wave with displacement

$$\mathbf{u} = (u(x,t),\ 0,\ 0) \tag{3.1}$$

propagating in the x-direction in a conducting medium with an initially uniform magnetic field

$$\mathbf{H}^0 = (0,\ 0,\ H^0). \tag{3.2}$$

The perturbations of the electromagnetic field are

$$\begin{aligned} \mathbf{H} &= \mathbf{H}^0 + \mathbf{h}, \\ \mathbf{E} &= 0 + \mathbf{e}, \\ \mathbf{J} &= 0 + \mathbf{j}. \end{aligned} \tag{3.3}$$

For a perfectly conducting medium we have from $(2.2)_3$

$$\mathbf{e} + \mathbf{u}_t \times \mathbf{B} = 0. \tag{3.4}$$

By equations (2.1), (3.3) we get

$$\operatorname{curl} \mathbf{e} = -\mu h_t. \tag{3.5}$$

Using (3.4), (3.5) gives

$$\mathbf{h}_t = \operatorname{curl} (\mathbf{u}_t \times \mathbf{H}). \tag{3.6}$$

Equation (3.6) together with (3.1), (3.3) yields

$$\begin{aligned} h_{1_t} &= 0, \\ h_{2_t} &= -(h_2 u_t)_x, \\ h_{3_t} &= -H^0 u_{xt} - (h_3 u_t)_x, \end{aligned} \tag{3.7}$$

where $\mathbf{h} = (h_1, h_2, h_3)$.

We now use a scaling parameter ε and consider the displacement u to be of order ε. The magnetic field $\mathbf{h}$ being dependent on u is also of order ε.

From $(3.7)_1$ it follows that h_1 is a function of x only. We take $h_1 = 0$ in this wave problem. For the determination of h_2, h_3 and u, we use the perturbation and multiple scales. We introduce only one scale, namely $\xi = \varepsilon x$. The perturbation expansions of h_2, h_3 and u are taken in the form:

$$\begin{aligned} h_2(x,t) &= \varepsilon h_{20}(x,\xi,t) + \varepsilon^2 h_{21}(x,\xi,t) + 0(\varepsilon^3), \\ h_3(x,t) &= \varepsilon h_{30}(x,\xi,t) + \varepsilon^2 h_{31}(x,\xi,t) + 0(\varepsilon^3), \\ u(x,t) &= \varepsilon U_0(x,\xi,t) + \varepsilon^2 U_1(x,\xi,t) + 0(\varepsilon^3). \end{aligned} \tag{3.8}$$

Substituting (3.8) in $(3.7)_2$, $(3.7)_3$ and equating the coefficients of ε, ε^2, we get

$$\begin{aligned} h_{20_t} &= 0, \\ h_{21_t} &= -h_{20_x} U_{0_t} - h_{20} U_{0_{tx}}, \\ h_{30_t} + H^0 U_{0_{xt}} &= 0, \\ h_{31_t} + H^0 U_{1_{xt}} + (h_{30} U_{0_t})_x + H^0 U_{0_{\xi t}} &= 0. \end{aligned} \tag{3.9}$$

Equation $(3.9)_1$ shows h_{20} to be independent of t. We therefore take $h_{20} = 0$ for wave solution. Putting $h_{20} = 0$ in $(3.9)_2$, we get $h_{21_t} = 0$ which implies $h_{21} = 0$. h_2 is therefore zero to within the accuracy of $o(\varepsilon^2)$. Also the Lorentz force components are

$$(3.10)\qquad J \times B = \mu[\text{curl } (0.0.h_3) \times (0, 0, H^0 + h_3)] = [-\mu(H^0 + h_3)h_{3_x}.0.0].$$

Using (3.10) in Eq. (2.3), the only equation of motion not identically satisfied is

$$(3.11)\qquad c_1^2 u_{xx} + 2c_3^2 u_x u_{xx} - \frac{\mu}{\rho}(H^0 + h_3)h_{3_x} = u_{tt},$$

$$c_1^2 = \frac{\lambda + 2G}{\rho}, \qquad c_3^2 = \frac{(3/2)\lambda + 3\gamma + 3\beta + 3G + 3\alpha}{\rho}.$$

Substituting (3.8) in (3.11) and equating the terms of order ε, ε^2 separately to zero, we get

$$(3.12)\qquad c_1^2 U_{0_{xx}} - \frac{\mu H^0}{\rho} h_{30_x} - U_{0_{tt}} = 0,$$

$$(3.13)\qquad c_1^2 U_{1_{xx}} - U_{1_{tt}} + 2c_1^2 U_{0_{x\xi}} + 2c_3^2 U_{0_x} U_{0_{xx}} - \frac{\mu H^0}{\rho}\left(h_{31_x} + h_{30_\xi}\right) - \frac{\mu h_{30}}{\rho} h_{30_x} = 0.$$

Integrating $(3.9)_3$ and $(3.9)_4$ partially with respect to time and neglecting the time-independent term, we obtain

$$(3.14)\qquad h_{30} = -H^0 U_{0_x},$$

$$(3.15)\qquad h_{31} = -H^0 U_{1x} - H^0 U_{0\xi} + H^0 \int (U_{0x} U_{0t})_x \, dt.$$

From Eqs. (3.12), (3.14) we get

$$(3.16)\qquad c^2 U_{0_{xx}} = U_{0_{tt}},$$

where

$$(3.17)\qquad c^2 = c_1^2 + c_H^2,$$

$$(3.18)\qquad c_H^2 = \mu H^{0^2}/\rho,$$

c_H being the Alfvén wave velocity, and c_1 the P-wave velocity in linear elastic solids.

4. Travelling wave solution

Solution of (3.16) suitable for a wave progressing in the positive x-direction is

$$U_0 = F(\theta, \xi), \tag{4.1}$$

where

$$\theta = t - \frac{x}{c}. \tag{4.2}$$

From equations (3.15) and (4.1),

$$h_{31_x} = H^0 U_{1_{xx}} + \frac{H^0}{c} F_{\xi\theta} - \frac{2H^0}{c^3} F_\theta F_{\theta\theta}. \tag{4.3}$$

Substituting from (3.14), (3.15). (4.1), and (4.3) in equation (3.13), one obtains

$$c^2 U_{1_{xx}} - U_{1_{tt}} = 2cF_{\theta\xi} + \frac{2c_3^2 - 3c_H^2}{c^3} F_\theta F_{\theta\theta}. \tag{4.4}$$

On using the transformation $\theta = t - \frac{x}{c}$, $\phi = t + \frac{x}{c}$ in Eq. (4.4), it takes the form

$$4U_{1_{\theta\phi}} = -2cF_{\theta\xi} - \frac{2c_3^2 - 3c_H^2}{c^3} F_\theta F_{\theta\theta}. \tag{4.5}$$

Hence from (4.5), it follows

$$4U_1 = -\left(2cF_\xi + \frac{2c_3^2 - 3c_H^2}{2c^3} F_\theta^2\right)\phi + \text{complementary function}. \tag{4.6}$$

For U_1 to be finite for large t, the coefficient of ϕ must be zero. Thus

$$2cF_\xi + \frac{2c_3^2 - 3c_H^2}{2c^3} F_\theta^2 = 0. \tag{4.7}$$

On differentiating (4.7) with respect to θ and on substituting $F_\theta = cf$, the equation satisfied by f is (WHITHAM [8])

$$cf_\xi + Mff_\theta = 0, \tag{4.8}$$

where

$$M = (c_3/c)^2 - 1.5\,(c_H/c)^2. \tag{4.9}$$

The solution of the quasilinear equation (4.8) is

$$f(\theta, \xi) = Z(\theta_1), \tag{4.10}$$

where $Z(\theta_1)$ is a function of θ_1 and

$$\theta_1 = \theta - M\xi Z(\theta_1)/c. \tag{4.11}$$

The main wave form $U_0(x, t, \xi) = F(\theta, \xi)$ propagates with a velocity c which is dependent on both the elastic and Alfvén wave velocities. It is also distorted for large x and a shock wave is formed (LARDNER [3]). The presence of the magnetic field changes the elastic non-linear effect. If the elastic field is linear, a non-linear effect due to the magnetic field persists.

A shock is formed for a value of θ for which $d\theta/d\theta_1 = 0$, i.e. when

$$Z'(\theta_1) = -c/(M\xi). \tag{4.12}$$

For an initially sinusoidal pulse $Z(\theta_1) = \sin(p\theta_1)$, the shock is formed if $\cos p\theta_1 = -c/(Mp\xi)$.

A shock is therefore formed in this case if $0 < (c/Mp\xi) \leqslant 1$ and the corresponding value of θ is

$$\theta = (1/p)\cos^{-1}(-c/Mp\xi) + (M\xi/c)(1 - c^2/M^2p^2\xi^2)^{1/2}. \tag{4.13}$$

To have an idea of the non-linear effect on the wave form, $f(\theta, \xi)$ is plotted against θ in Fig. 1 for different ξ, corresponding to the sinusoidal signal

$$f(\theta, 0) = \sin(\pi\theta/5).$$

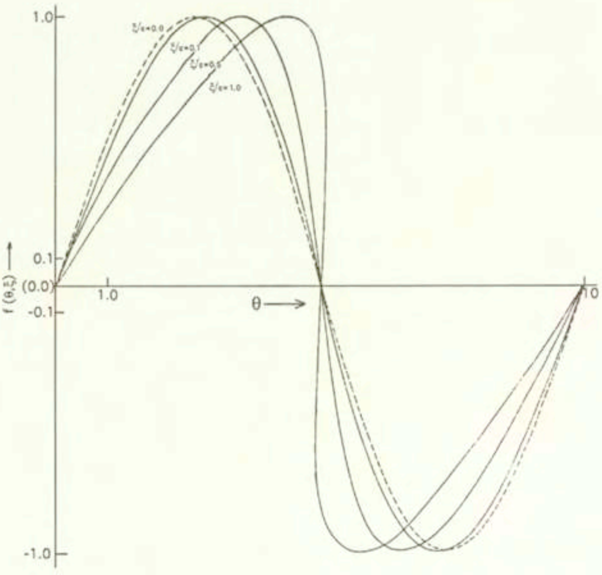

FIG. 1. $f(\theta, \xi)$ against θ for different values of ξ.

5. Conclusion

It is seen from the figure that, as the slow distance scale increases, the asymmetry grows and the possibility of shock increases.

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