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FIFTY YEARS OF THE ARCHIVES OF MECHANICS

Objective corotational rates and unified work-conjugacy relation between Eulerian and Lagrangean strain and stress measures

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BY VIRTUE OF OBJECTIVE corotational rates and related corotating frames, a unified work-conjugacy relation between Eulerian and Lagrangean strain and stress measures is established, which is a natural extension of Hill's work-conjugacy relation between Lagrangean strain and stress measures. It turns out that the latter is the particular case of the former one when a corotating frame with the well-known spin $\Omega^R = \dot{R}R^T$ is concerned, where R is the rotation tensor defined by the polar decomposition of the deformation gradient. The work-conjugate stress measure of an arbitrary Hill's strain measure (either Eulerian or Lagrangean) with regard to any kind of objective corotational rate is determined in the sense of the introduced unified work-conjugacy relation. The result is presented both in the principal component form and explicit basis-free form valid for all cases of the principal stretches. In particular, the intrinsic, unique relationship between Hencky's logarithmic strain measures $\ln V$ and $\ln U$ and the fundamental mechanical quantities, i.e. the Eulerian and Lagrangean stretching tensors D and $R^T DR$ and Eulerian and Lagrangean Kirchhoff stress measures σ and $R^T \sigma R$, are disclosed. It is shown that there are objective corotational rates of $\ln V$ and $\ln U$ that are identical with the Eulerian and Lagrangean stretching tensors D and $R^T DR$ respectively, and further that only $\ln V$ and $\ln U$ enjoy the just-stated favourable properties. As a result, the two pairs of strain and stress measures, $(\ln V, \sigma)$ and $(\ln U, R^T\sigma R)$, form a work-conjugate Eulerian strain-stress pair and a work-conjugate Lagrangean strain-stress pair, respectively, in the sense of the introduced work-conjugacy relation. Finally, application of the unified work-conjugacy notion in formulating the rate – type constitutive relations is indicated.

1. Introduction

IN SOLID MECHANICS and other related fields, there is a variety of strain and stress measures (actually infinitely many). It is well-known that strain measures and stress measures can be associated with each other via the stress power per unit volume, in a manner independent of any material behaviours. According to HILL [21 – 23] (see also WANG and TRUESDELL [53], OGDEN [37–38], *et al.*), a Lagrangean strain-stress pair (E, T) forms a work-conjugate pair if the inner product of the Lagrangean stress measure T and the material time rate $\dot{E}$ of the Lagrangean strain measure E furnishes the stress power $\dot{w}$ (cf. the formulas (2.13) – (2.14) given later):

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$$\dot{w} = \mathbf{T} : \dot{\mathbf{E}} = T_{ij}\dot{E}_{ij}. \tag{1.1}$$

The just-stated Hill's work-conjugacy notion for Lagrangean strain and stress measures has found applications in constitutive modeling and proved to be fruitful (e.g., see HILL [21 – 23], RICE [42], HUTCHINSON and NEALE [27], NEMAT-NASSER [36], PALGEN and DRUCKER [39], OGDEN [38], *et al.*). However, such notion in general does not apply to Eulerian stress and strain measures, as proved by OGDEN [37 – 38] and HOGER [26], *et al.* Moreover, even within the scope of Lagrangean strain and stress measures, the aforementioned work-conjugacy notion excludes the possibility of associating certain significant stress measures with strain measures. Indeed, it is known that the *rotated Kirchhoff stress measure* $\mathbf{R}^T\boldsymbol{\sigma}\mathbf{R}$, which is a Lagrangean stress measure useful in formulation of elastic and elastoplastic constitutive relations (e.g., see GREEN and NAGHDI [9], RICE [42], SIMO and MARSDEN [49]), can not be related to any strain measure via (1.1), as noted by HILL [21 – 23], RICE [42], PALGEN and DRUCKER [39], and OGDEN [38], *et al.*

The main objective of this article is to investigate the work-conjugacy notion in both the Lagrangean and Eulerian strain and stress measures in a broader sense and in a unified manner. It is shown that by virtue of the objective corotational rates and the related corotating frames, a unified work-conjugacy relation can be established for both the Eulerian and Lagrangean strain and stress measures. This unified work-conjugacy relation may be visualized as a natural extension of Hill's work-conjugacy notion in the sense that there exists a certain class of corotating frames relative to each of which the stress power $\dot{w}$ can be expressed as the inner product of a stress measure and the time rate of a strain measure. It turns out that Hill's work-conjugacy relation is the particular case when a corotating frame with the well-known spin $\boldsymbol{\Omega}^R = \dot{\mathbf{R}}\mathbf{R}^T$ is concerned, where $\mathbf{R}$ is the rotation tensor defined by the polar decomposition of the deformation gradient (see (2.1) given later). By applying a general expression for the spin tensors defining objective corotational rates derived in XIAO, BRUHNS and MEYERS [60], the conjugate stress measure of an arbitrary Hill's generalized strain measure (either Eulerian or Lagrangean) with regard to any kind of objective corotational rate is determined in the sense of the introduced unified work-conjugacy relation. The results are presented in both the principal component form and the explicit basis-free form. In particular, the intrinsic, unique relationship between Hencky's logarithmic strain measures $\ln\mathbf{V}$ and $\ln\mathbf{U}$ and the fundamental mechanical quantities, i.e. the Eulerian and Lagrangean stretching tensors $\mathbf{D}$ and $\mathbf{R}^T\mathbf{D}\mathbf{R}$ and Eulerian and Lagrangean Kirchhoff stress measures $\boldsymbol{\sigma}$ and $\mathbf{R}^T\boldsymbol{\sigma}\mathbf{R}$, are disclosed. It is shown that there exist objective corotational rates of $\ln\mathbf{V}$ and $\ln\mathbf{U}$ that are identical with the Eulerian and Lagrangean stretching tensors $\mathbf{D}$ and $\mathbf{R}^T\mathbf{D}\mathbf{R}$, respectively, and furthermore that only $\ln\mathbf{V}$ and $\ln\mathbf{U}$ enjoy the just-stated favourable properties. As a result, the two pairs $(\ln\mathbf{V}, \boldsymbol{\sigma})$ and $(\ln\mathbf{U}, \mathbf{R}^T\boldsymbol{\sigma}\mathbf{R})$ form a work-conjugate Eulerian strain-stress pair and a work-conjugate Lagrangean strain-stress pair respectively, where $\boldsymbol{\sigma}$ is the Kirchhoff stress measure. The fact concerning the Eulerian logarithmic strain measure $\ln \mathbf{V}$ has been established recently by various authors independently of their different points of view (see XIAO, BRUHNS and MEYERS [57 – 60]; see also LEHMANN, GUO and LIANG [29], REINHARDT and DUBEY [40 – 41], DUBEY and REINHARDT [7]). However, the fact concerning the Lagrangean logarithmic strain measure $\ln \mathbf{U}$ has been unknown until very recently (see XIAO, BRUHNS and MEYERS [61]), since for a long time it has been believed that the rotated stretching tensor $\mathbf{R}^T\mathbf{D}\mathbf{R}$ is not a direct flux of a strain measure (see HILL [21 – 23], RICE [42], PALGEN and DRUCKER [39], OGDEN [38], *et al.*). Finally, application of the unified work-conjugacy notion in formulating rate-type constitutive relations is indicated.

It should be pointed out that other extended work-conjugacy notions are possible and useful. For example, we refer to ZIEGLER and MACVEAN [64] and MACVEAN [32] for a discussion of work-conjugacy relation from a general point of view, and to HAUPT and TSAKMAKIS [18 – 19] and SVENDSEN and TSAKMAKIS [51] for a comprehensive account of associating strain and stress measures via the concept of dual variables, etc. In this article, we shall confine ourselves to the objective mentioned before.

2. Preliminaries in kinematics

To facilitate the succeeding account, in this section we recapitulate some related facts and results in kinematics of finite deformations of continua. For details, refer to TRUESDELL and TOUPIN [52], WANG and TRUESDELL [53], GURTIN [16], MARSDEN and HUGHES [34], and OGDEN [37 – 38], *et al.*

In this article, vector and tensor mean vectors and tensors over a three-dimensional Euclidean space.

2.1. Some fundamental kinematical quantities

Consider a material body experiencing finite deformation over a time interval $I \subset R$. A typical particle of this body is identified with a position vector $\mathbf{X}$ relative to a fixed reference state. The motion of the body is described by the current position vector $\mathbf{x} = \tilde{\mathbf{x}}(\mathbf{X}, t)$, $t \in I$. The velocity vector of the particle $\mathbf{X}$ is given by $\mathbf{v} = \dot{\mathbf{x}}$.

The state of the local rotation and deformation in a neighbourhood of a particle $\mathbf{X}$ at any instant $t \in I$ is characterized by the *deformation gradient*

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}, \tag{2.1}$$

while the rate-of-change of state of the rotation and deformation in a neighbourhood of a particle $\mathbf{X}$ at any instant $t \in I$ is described by the *velocity gradient*

$$\mathbf{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \dot{\mathbf{F}}\mathbf{F}^{-1}. \tag{2.2}$$

For the former, the following unique left and right polar decompositions hold:

$$\begin{aligned} \mathbf{F} &= \mathbf{VR} = \mathbf{RU}, \\ \mathbf{V}^2 &= \mathbf{B} = \mathbf{FF}^T, \\ \mathbf{U}^2 &= \mathbf{C} = \mathbf{F}^T\mathbf{F}, \\ \mathbf{RR}^T &= \mathbf{I}, \qquad \mathbf{R}^T\mathbf{R} = \hat{\mathbf{I}}, \end{aligned} \tag{2.3}$$

where the two tensors $\mathbf{V}$ and $\mathbf{B}$ and the two tensors $\mathbf{B}$ and $\mathbf{C}$ are known as the *left* and *right stretch tensors* and the *left* and *right Cauchy-Green tensors* respectively, each of which is symmetric, positive definite; the proper orthogonal tensor $\mathbf{R}$ is the *rotation tensor*. Throughout, $\mathbf{I}$ and $\hat{\mathbf{I}}$ are used to represent the *metric tensors* in the current configuration and the fixed reference configuration, respectively. On the other hand, the following unique additive decomposition holds:

$$\begin{aligned} \mathbf{L} &= \mathbf{D} + \mathbf{W}, \\ \mathbf{D} &= \frac{1}{2}(\mathbf{L} + \mathbf{L}^T), \\ \mathbf{W} &= \frac{1}{2}(\mathbf{L} - \mathbf{L}^T), \end{aligned} \tag{2.4}$$

where the tensor $\mathbf{D}$, the symmetric part of the velocity gradient $\mathbf{L}$, is known as the *stretching tensor*, and the tensor $\mathbf{W}$, the antisymmetric part of the velocity gradient $\mathbf{L}$, as the *vorticity tensor*.

In addition to those given above, there are other basic tensor quantities, some of which will be given in the next subsection.

2.2. Eulerian and Lagrangean tensors and their rotated correspondence

In a deforming material body, several types of tensor quantities are involved due to the different ways by which a fixed reference configuration and a current configuration are related, refer to, e.g., OGDEN [37 – 38] for details. There are three types of second order tensors: *Eulerian, Lagrangean and mixed-type*, for which the current configuration only, the reference configuration only, and both the reference and current configurations are related, respectively (see OGDEN [37 – 38]). In the tensor quantities mentioned before, the deformation gradient $\mathbf{F}$, the rotation tensor $\mathbf{R}$ and its transpose $\mathbf{R}^T$ are mixed-type, the right stretch tensor $\mathbf{U}$ and the right Cauchy–Green tensor $\mathbf{C}$ are Lagrangean, and the others, including the left stretch tensor $\mathbf{V}$, the left Cauchy-Green tensor $\mathbf{B}$, the velocity gradient $\mathbf{L}$, the vorticity tensor $\mathbf{W}$ and the stretching tensor $\mathbf{D}$ etc., are Eulerian. In this article, we are mainly concerned with Eulerian and Lagrangean second order tensors. Henceforth, *tensor* means *second order tensor*, if not otherwise indicated.

It is known that the transformation between the reference configuration and the current configuration can be effectuated by virtue of the rotation tensor $\mathbf{R}$. As a result, a natural correspondence between Eulerian and Lagrangean tensors can be established via the rotation tensor $\mathbf{R}$. Let $\mathbf{G}$ be an Eulerian tensor. Then the *Lagrangean counterpart* of $\mathbf{G}$, denoted by $\hat{\mathbf{G}}$, is defined as

$$\hat{\mathbf{G}} = \mathbf{R}^T\mathbf{G}\mathbf{R}\,. \tag{2.5}$$

Conversely, we call $\mathbf{G}$ the *Eulerian counterpart of the Lagrangean tensor* $\hat{\mathbf{G}}$ and we have

$$\mathbf{G} = \mathbf{R}\hat{\mathbf{G}}\mathbf{R}^T\,. \tag{2.6}$$

The above correspondence is called the *rotated correspondence* between Eulerian and Lagrangean tensors. It is evident that any two Eulerian and Lagrangean tensors $\mathbf{G}$ and $\hat{\mathbf{G}}$ associated by the rotated correspondence represent the same tensor quantity.

From (2.3) it follows

$$\mathbf{U} = \hat{\mathbf{V}} = \mathbf{R}^T\mathbf{V}\mathbf{R}\,, \tag{2.7}$$

$$\mathbf{C} = \hat{\mathbf{B}} = \mathbf{R}^T\mathbf{B}\mathbf{R}\,, \tag{2.8}$$

$$\mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^T\,, \tag{2.9}$$

$$\mathbf{B} = \mathbf{R}\mathbf{C}\mathbf{R}^T\,. \tag{2.10}$$

Moreover, the Lagrangean counterparts of the stretching tensor $\mathbf{D}$ and the Kirchhoff stress tensor $\boldsymbol{\sigma}$, called the *Lagrangean stretching tensor* and the *Lagrangean Kirchhoff stress* (or the rotated Kirchhoff stress tensor according to SIMO and MARSDEN [49]) respectively, are given by

$$\hat{\mathbf{D}} = \mathbf{R}^T\mathbf{D}\mathbf{R}\,, \tag{2.11}$$

$$\hat{\boldsymbol{\sigma}} = \mathbf{R}^T\boldsymbol{\sigma}\mathbf{R}\,. \tag{2.12}$$

The following Eqs. represent two standard formulas for the stress power $\dot{w}$ *per unit reference state volume*

$$\dot{w} = \boldsymbol{\sigma} : \mathbf{D}\,, \tag{2.13}$$

$$\dot{w} = \hat{\boldsymbol{\sigma}} : \hat{\mathbf{D}}\,. \tag{2.14}$$

2.3. Rotating frames and objectivity

Let $\mathbf{Q}^*$ be an Eulerian time-dependent proper orthogonal tensor. Then a rotating frame $*$ relative to a fixed background frame is defined as follows[(2)]:

$$\mathbf{x}^*(\mathbf{X},t) = \mathbf{Q}^*\tilde{\mathbf{x}}(\mathbf{X},t) + \mathbf{x}_0(t)\,. \tag{2.15}$$

It is evident that the rotating frame $*$ is defined by the proper orthogonal tensor $\mathbf{Q}^*$. On the other hand, given the spin $\boldsymbol{\Omega}^*$ of the frame $*$, the latter is in turn determined by the first order tensorial differential equation

$$\dot{\mathbf{Q}}^{*T}\mathbf{Q}^* = -\mathbf{Q}^{*T}\dot{\mathbf{Q}}^* = \boldsymbol{\Omega}^* \tag{2.16}$$

up to a constant initial rotation. Thus, a rotating frame $*$ can also be defined by its spin. The latter definition serves our purpose and will be adopted. Let $\boldsymbol{\Omega}^*$ be a spin, i.e. an Eulerian time-dependent skewsymmetric tensor. Henceforth, by an $\boldsymbol{\Omega}^*$-frame we mean a rotating frame defined by (2.15) – (2.16).

Let $\mathbf{G}$ and $\hat{\mathbf{H}}$ be, respectively, an Eulerian and a Lagrangean tensors defined in a deforming material body. Following OGDEN [37 – 38], we say that the Eulerian tensor $\mathbf{G}$ and the Lagrangean tensor $\hat{\mathbf{H}}$ are *objective*, respectively, if they obey the following transformation rules with respect to any change of frame indicated by (2.15):

$$\mathbf{G}^* = \mathbf{Q}^*\mathbf{G}\mathbf{Q}^{*T}\,, \tag{2.17}$$

$$\hat{\mathbf{H}}^* = \hat{\mathbf{H}}\,, \tag{2.18}$$

where the superscript $*$ indicates the association with a rotating frame defined by any continuously time-dependent rotation tensor $\mathbf{Q}^* = \mathbf{Q}^*(t)$, i.e. an $\boldsymbol{\Omega}^*$-frame, where $\boldsymbol{\Omega}^* = \dot{\mathbf{Q}}^{*T}\mathbf{Q}^*$.

The left stretch tensor $\mathbf{V}$, the left Cauchy–Green tensor $\mathbf{B}$ and the stretching tensor $\mathbf{D}$ are objective Eulerian tensors. The right stretch tensor $\mathbf{U}$ and the right Cauchy–Green tensor $\mathbf{C}$ are objective Lagrangean tensors. The velocity gradient $\mathbf{L}$ and the vorticity tensor $\mathbf{W}$ and their Lagrangean counterparts $\hat{\mathbf{L}}$ and $\hat{\mathbf{W}}$ provide, respectively, two examples of Eulerian and Lagrangean tensors which are not objective. For detail, refer to OGDEN [37 – 38].

2.4. Hill's generalized strain measures and their alternative expressions

A general class of Eulerian and Lagrangean strain measures, called Hill's *generalized strain measures*, was introduced by HILL [21 – 23] (see also WANG

(2) Generally, there is a time difference between the two frames, i.e. $t^* = t + t_0$, which is irrelevant to our purpose. Here we assume $t_0 = 0$ for the sake of simplicity.

and TRUESDELL [53] and OGDEN [38]). Let λ_1, λ_2 and λ_3 be the three principal stretches, i.e. the three eigenvalues (possibly repeated) of the stretch tensor **V** or **U**. A set of three orthonormal eigenvectors of **V** (resp. **U**) is called an *Eulerian triad* (resp. a *Lagrangean triad*), denoted by $\{\mathbf{n}_i\}$ (resp. $\{\mathbf{N}_i\}$). Hill's generalized strain measures are of the forms

$$\mathbf{e} = \mathbf{f}(\mathbf{V}) = \sum_{i=1}^{3} f(\lambda_i)\mathbf{n}_i \otimes \mathbf{n}_i\,, \tag{2.19}$$

$$\mathbf{E} = \mathbf{f}(\mathbf{U}) = \sum_{i=1}^{3} f(\lambda_i)\mathbf{N}_i \otimes \mathbf{N}_i\,. \tag{2.20}$$

In the above, the function $f: R^+ \to R$ is a smooth monotonic increasing function with the property $f(1) = f'(1) - 1 = 0$, which defines the strain measures **e** and **E** and is called the *scale function*. Since the tensor functions $\mathbf{f}(\mathbf{V})$ and $\mathbf{f}(\mathbf{U})$ are isotropic and the left and right stretch tensors **V** and **U** are objective, both the strain measures **e** and **E** defined through the scale function $f(\lambda)$, which are Eulerian and Lagrangean respectively, are objective. Moreover, they can be related to each other via the rotated correspondence indicated by (2.5) and (2.6), i.e.

$$\mathbf{e} = \mathbf{R}\mathbf{E}\mathbf{R}^T\,, \tag{2.21}$$

$$\mathbf{E} = \hat{\mathbf{e}} = \mathbf{R}^T\mathbf{e}\mathbf{R}\,. \tag{2.22}$$

It is known (see DOYLE and ERICKSEN [5], TRUESDELL and TOUPIN [52], SETH [48], HILL [21 – 23], OGDEN [38], *et al.*) that by choosing the scale function $f(\lambda)$ in the particular form

$$f(\lambda) = \frac{1}{m}(\lambda^m - 1) \tag{2.23}$$

and assigning several integers to the number m, Hill's generalized strain measures, i.e.

$$\mathbf{e}^{(m)} = \frac{1}{m}(\mathbf{V}^m - \mathbf{I})\,, \tag{2.24}$$

$$\mathbf{E}^{(m)} = \frac{1}{m}(\mathbf{U}^m - \hat{\mathbf{I}})\,, \tag{2.25}$$

supply all commonly-known objective Eulerian and Lagrangean strain measures. In particular, the limiting process $m \to 0$ or the logarithmic scale function $f(\lambda) = \ln\lambda$ results in Hencky's *logarithmic strain measures* (see HENCKY [20])

$$\ln\mathbf{V} = \sum_{i=1}^{3}(\ln\lambda_i)\mathbf{n}_i \otimes \mathbf{n}_i\,, \tag{2.26}$$

(2.27)
$$\ln \mathbf{U} = \sum_{i=1}^{3} (\ln \lambda_i) \mathbf{N}_i \otimes \mathbf{N}_i ,$$

which have received much attention (e.g., see TRUESDELL and TOUPIN [52], HILL [21 – 23], RICE [42], FITZGERALD [8], GURTIN and SPEAR [17], HOGER [25 – 26], and LEHMANN and LIANG [30], *et al.*) and will be discussed in Sec. 6 of this article.

Since each principal stretch λ_i is always positive, we can give an alternative definition of Hill's strain measures $\mathbf{e}$ and $\mathbf{E}$ in terms of the left and right Cauchy–Green tensors $\mathbf{B}$ and $\mathbf{C}$. Let $g: R^+ \rightarrow R$ be a new scale function defined by

(2.28)
$$g(\chi) = f(\sqrt{\chi}) \quad (\forall \chi > 0) .$$

Moreover, let $\chi_1, \cdots, \chi_r$ be the distinct eigenvalues of the left Cauchy–Green tensor $\mathbf{B}$ or, equivalently, the right Cauchy–Green tensor $\mathbf{C}$, and $\mathbf{B}_\sigma$ and $\mathbf{C}_\sigma$ be the corresponding subordinate eigenprojections of $\mathbf{B}$ and $\mathbf{C}$ respectively, where $\sigma = 1, \cdots, r$. Then we have

(2.29)
$$\mathbf{e} = \mathbf{g}(\mathbf{B}) = \sum_{\sigma=1}^{r} g(\chi_\sigma) \mathbf{B}_\sigma ,$$

(2.30)
$$\mathbf{E} = \mathbf{g}(\mathbf{C}) = \sum_{\sigma=1}^{r} g(\chi_\sigma) \mathbf{C}_\sigma .$$

Henceforth we shall adopt the latter definition. In so doing, the main results that will be derived in Sec. 5 can be expressed in terms of the Cauchy–Green tensors $\mathbf{B}$ and $\mathbf{C}$ instead of the stretch tensors $\mathbf{V}$ and $\mathbf{U}$. Here we would mention the fact that once the deformation gradient $\mathbf{F}$ is known, it is much easier to calculate the Cauchy–Green tensors $\mathbf{B} = \mathbf{F}^T\mathbf{F}$ and $\mathbf{C} = \mathbf{F}\mathbf{F}^T$ than to calculate the stretch tensors $\mathbf{V}$ and $\mathbf{U}$, the latter being the complicated square roots $\sqrt{\mathbf{F}^T\mathbf{F}}$ and $\sqrt{\mathbf{F}\mathbf{F}^T}$, respectively. Moreover, the use of eigenprojections instead of eigenvectors will prove to be crucial. The use of eigenprojections has certain advantages over the use of eigenvectors. Here we mention the following three aspects only.

1. The eigenprojections $\mathbf{B}_\sigma$ and $\mathbf{C}_\sigma$ are unique and this applies to all the cases for eigenvalues of $\mathbf{B}$ and $\mathbf{C}$. In fact, the following Sylvester's formulas hold

(2.31)
$$\mathbf{C}_\sigma = \delta_{1r} \hat{\mathbf{I}} + \prod_{\tau \neq \sigma}^{r} \frac{\mathbf{C} - \chi_\tau \hat{\mathbf{I}}}{\chi_\sigma - \chi_\tau} ,$$

(2.32)
$$\mathbf{B}_\sigma = \delta_{1r} \mathbf{I} + \prod_{\tau \neq \sigma}^{r} \frac{\mathbf{B} - \chi_\tau \mathbf{I}}{\chi_\sigma - \chi_\tau} ;$$

2. The explicit basis-free form of the main results for strain rates and conjugate stresses can easily be derived with the aid of Sylvester's formulas (2.31) – (2.32); and

3. All the procedure can be fulfilled merely by means of the following simple manipulation properties concerning the eigenprojections $\mathbf{B}_\sigma$ and $\mathbf{C}_\sigma$

$$\mathbf{C}_\sigma \mathbf{C}_\tau = \delta_{\sigma\tau} \mathbf{C}_\tau \,, \tag{2.33}$$

$$\sum_{\sigma=1}^{r} \mathbf{C}_\sigma = \hat{\mathbf{I}} \,, \tag{2.34}$$

and

$$\mathbf{B}_\sigma \mathbf{B}_\tau = \delta_{\sigma\tau} \mathbf{B}_\tau \,, \tag{2.35}$$

$$\sum_{\sigma=1}^{r} \mathbf{B}_\sigma = \mathbf{I} \,, \tag{2.36}$$

with no summation over repeated indices. Here, $\delta_{\sigma\tau}$ is used to denote the Kronecker delta.

The advantage of using the Cauchy–Green tensors and the eigenprojections was known by HOGER and CARLSON [24], CARLSON and HOGER [1], SCHEIDLER [46] and XIAO [56] *et al.*, and was further exploited by these authors (see XIAO, BRUHNS and MEYERS [57 – 63], esp. [65]).

Finally, it should be pointed out that once the deformation gradient $\mathbf{F}$ is given in a coordinate system other than the particular ones formed by the principal bases, usually it is not easy to calculate the generalized strain measures $\mathbf{e}$ and $\mathbf{E}$ for any nonpolynomial scale function $g(\chi)$, especially for any transcendental scale function $g(\chi)$ such as the logarithmic function $g(\chi) = \frac{1}{2}\ln\chi$ etc. This difficulty may be circumvented by using the formula (5.24) – (5.25) given later and by the explicit basis-free formulas derived from (2.29) – (2.32).

3. Material spins, corotating material frames and objective corotational rates

3.1. Corotational rates of Eulerian and Lagrangean tensors

Let $\mathbf{\Omega}^*$ be an Eulerian spin relative to a fixed background frame, i.e. a continuous time-dependent skewsymmetric Eulerian tensor, and let $\mathbf{G}$ be an objective Eulerian tensor. The Eulerian tensor defined by

$$\overset{\circ}{\mathbf{G}}{}^{*} = \dot{\mathbf{G}} + \mathbf{G}\mathbf{\Omega}^* - \mathbf{\Omega}^*\mathbf{G} \tag{3.1}$$

is called the *corotational rate of the tensor* $\mathbf{G}$ *defined by the Eulerian spin* $\mathbf{\Omega}^*$. To see what the term *corotational rate* means, let us consider an $\mathbf{\Omega}^*$-frame $*$ (see Sec. 2.3) and let $\mathbf{Q}^*$ be a continuously differentiable time-dependent proper orthogonal tensor determined by the spin $\mathbf{\Omega}^*$ (cf. (2.16)). In such an $\mathbf{\Omega}^*$-frame $*$, the Eulerian tensor $\mathbf{G}$ is of the form $\mathbf{Q}^*\mathbf{G}\mathbf{Q}^{*T}$. We have

$$\begin{aligned}\overline{(\mathbf{Q}^*\mathbf{G}\mathbf{Q}^{*T})}^{\boldsymbol{\cdot}} &= \mathbf{Q}^*\dot{\mathbf{G}}\mathbf{Q}^{*T} + \dot{\mathbf{Q}}^*\mathbf{G}\mathbf{Q}^{*T} + \mathbf{Q}^*\mathbf{G}\dot{\mathbf{Q}}^{*T} \\ &= \mathbf{Q}^* \overset{\circ}{\mathbf{G}}{}^* \mathbf{Q}^{*T}.\end{aligned} \tag{3.2}$$

Note that the latter is the counterpart of $\overset{\circ}{\mathbf{G}}{}^*$ in the $\mathbf{\Omega}^*$-frame. Thus, *the corotational rate of an objective Eulerian tensor defined by an Eulerian spin* $\mathbf{\Omega}^*$ *(cf. (3.1)) is a material time derivative in an* $\mathbf{\Omega}^*$*-frame.* It should be noted that this interpretation can be made only when the tensor $\mathbf{G}$ is an *objective Eulerian tensor.*

The Lagrangean counterpart of the corotational rate $\overset{\circ}{\mathbf{G}}{}^*$ provided by the tensor $\mathbf{R}^T \overset{\circ}{\mathbf{G}}{}^* \mathbf{R}$. From (2.5) we derive

$$\dot{\mathbf{G}} = \overline{(\mathbf{R}\hat{\mathbf{G}}\mathbf{R}^T)}^{\boldsymbol{\cdot}} = \mathbf{R}\dot{\hat{\mathbf{G}}}\mathbf{R}^T + \dot{\mathbf{R}}\hat{\mathbf{G}}\mathbf{R}^T + \mathbf{R}\hat{\mathbf{G}}\dot{\mathbf{R}}^T = \mathbf{R}(\dot{\hat{\mathbf{G}}} - \hat{\mathbf{G}}\hat{\mathbf{\Omega}}^R + \hat{\mathbf{\Omega}}^R\hat{\mathbf{G}})\mathbf{R}^T,$$

where $\hat{\mathbf{G}}$ and $\hat{\mathbf{\Omega}}^R = \mathbf{R}^T\dot{\mathbf{R}}$ are the Lagrangean counterparts of the Eulerian tensors $\mathbf{G}$ and $\mathbf{\Omega}^R = \dot{\mathbf{R}}\mathbf{R}^T$ respectively. Applying the result just derived and the identity

$$\mathbf{Q}(\mathbf{H}_1\mathbf{H}_2)\mathbf{Q}^T = (\mathbf{Q}\mathbf{H}_1\mathbf{Q}^T)(\mathbf{Q}\mathbf{H}_2\mathbf{Q}^T)$$

for any two tensors $\mathbf{H}_1$ and $\mathbf{H}_2$ and for any orthogonal tensor $\mathbf{Q}$, we arrive at the following formula

$$\mathbf{R}^T \overset{\circ}{\mathbf{G}}{}^* \mathbf{R} = \dot{\hat{\mathbf{G}}} + \hat{\mathbf{G}}(\hat{\mathbf{\Omega}}^* - \hat{\mathbf{\Omega}}^R) - (\hat{\mathbf{\Omega}}^* - \hat{\mathbf{\Omega}}^R)\hat{\mathbf{G}}, \tag{3.3}$$

where

$$\begin{aligned}\hat{\mathbf{G}} &= \mathbf{R}^T\mathbf{G}\mathbf{R}, \\ \hat{\mathbf{\Omega}}^* &= \mathbf{R}^T\mathbf{\Omega}^*\mathbf{R}, \\ \hat{\mathbf{\Omega}}^R &= \mathbf{R}^T\mathbf{\Omega}^R\mathbf{R} = \mathbf{R}^T\dot{\mathbf{R}} = -\dot{\mathbf{R}}^T\mathbf{R},\end{aligned} \tag{3.4}$$

are the Lagrangean counterparts of the Eulerian tensors $\mathbf{G}$ and $\mathbf{\Omega}^*$ and $\mathbf{\Omega}^R$ respectively.

The Lagrangean counterpart of each corotational rate $\overset{\circ}{\mathbf{G}}{}^*$ of an Eulerian tensor $\mathbf{G}$ provides a rate measure of the Lagrangean counterpart $\hat{\mathbf{G}}$ of this tensor. Since the right-hand side of the formula (3.3) is of the same structure as that of

(3.1), we call the rate measure $\mathbf{R}^T \overset{\circ}{\mathbf{G}}{}^* \mathbf{R}$, i.e. the right-hand side of (3.3), the *corotational rate of the Lagrangean tensor* $\hat{\mathbf{G}}$ *defined by the spin* $\hat{\boldsymbol{\Omega}}^*$, *denoted by* $\overset{o}{\hat{\mathbf{G}}}{}^*$. Thus, we have

$$\overset{\circ}{\hat{\mathbf{G}}}{}^* = \mathbf{R}^T \overset{\circ}{\mathbf{G}}{}^* \mathbf{R} = \dot{\hat{\mathbf{G}}} + \hat{\mathbf{G}}(\hat{\boldsymbol{\Omega}}^* - \hat{\boldsymbol{\Omega}}^R) - (\hat{\boldsymbol{\Omega}}^* - \hat{\boldsymbol{\Omega}}^R)\hat{\mathbf{G}}\,. \tag{3.5}$$

Note the difference between the two definitions (3.1) and (3.5): the latter is using the additional spin $\hat{\boldsymbol{\Omega}}^R = \mathbf{R}^T\dot{\mathbf{R}}$, while the former is not concerned with any other spin except the spin $\boldsymbol{\Omega}^*$. This difference arises from the fact that there is a relative rotation between the Eulerian triad and the Lagrangean triad, which is just given by the rotation tensor $\mathbf{R}$.

Let $\boldsymbol{\Omega}^* = \boldsymbol{\Omega}^R$, i.e. $\hat{\boldsymbol{\Omega}}^* = \hat{\boldsymbol{\Omega}}^R$ and introduce the *polar rate* of a symmetric Eulerian tensor $\mathbf{G}$ (see GREEN and NAGHDI [9], DIENES [3 – 4] and SCHEIDLER [45], *et al.*)

$$\overset{\circ}{\mathbf{G}}{}^R = \dot{\mathbf{G}} + \mathbf{G}\boldsymbol{\Omega}^R - \boldsymbol{\Omega}^R\mathbf{G}\,. \tag{3.6}$$

Then the formula (3.5) yields

$$\dot{\hat{\mathbf{G}}} = \mathbf{R}^T \overset{\circ}{\mathbf{G}}{}^R \mathbf{R}, \qquad \overset{\circ}{\mathbf{G}}{}^R = \mathbf{R}\dot{\hat{\mathbf{G}}}\mathbf{R}^T. \tag{3.7}$$

It turns out that *the Lagrangean counterpart of the polar rate of an objective Eulerian tensor is just the material time rate of the Lagrangean counterpart of this tensor* and *vice versa*, i.e. *the Eulerian counterpart of the material time rate of an objective Lagrangean tensor is the polar rate of the Eulerian counterpart of this tensor.* This fact indicates that the material time rate of an objective Lagrangean tensor is merely a particular kind of corotational rate of this tensor, which is defined by the spin $\hat{\boldsymbol{\Omega}}^R = \mathbf{R}^T\dot{\mathbf{R}}$.

We would emphasize that the formula (3.5) and hence the above fact apply to objective Eulerian and Lagrangean tensors only.

3.2. Material spins and objective corotational rates

More essentially, it is required that corotational rates of objective Eulerian and Lagrangean tensors be *objective rate measures* so that any superimposed rigid rotation motion has no effect on it. Moreover, to establish the extended work-conjugacy relation, this requirement is just what is needed, as will be shown in the next section. It can be readily proved that if an Eulerian tensor is objective, then its Lagrangean counterpart via the rotated correspondence (2.5) is also objective. The opposite is true, i.e. if a Lagrangean tensor is objective, then its Eulerian counterpart via the rotated correspondence (2.6) is also objective. In view of this fact and the rotated correspondence relationship between the Eulerian and

Lagrangean corotational rates as indicated in the last subsection, it suffices to consider objective corotational rates of objective Eulerian tensors.

Let $\mathbf{G}$ be an objective Eulerian tensor. Not every corotational rate $\overset{\circ}{\mathbf{G}}{}^*$ is objective. For instance, let $\boldsymbol{\Omega}^* = c\mathbf{W}$ with c – any given constant and $\mathbf{W}$ being the vorticity tensor (cf. $(2.4)_3$). Then, (3.1) defines infinitely many corotational rates of the Eulerian tensor $\mathbf{G}$, when the constant c runs over all the reals. However, only the one with $c = 1$, i.e. the Zaremba–Jaumann rate $\dot{\mathbf{G}} + \mathbf{GW} - \mathbf{WG}$, is objective. Generally, whether the corotational rate $\overset{\circ}{\mathbf{G}}{}^*$ is objective or not depends on its defining spin $\boldsymbol{\Omega}^*$. To arrive at objective corotational rates, the defining spin $\boldsymbol{\Omega}^*$ should be associated with the deformation and motion of the deforming material body under consideration in an appropriate manner, as has been shown for several well-known examples: $\boldsymbol{\Omega}^* = \mathbf{W}$ (Zaremba–Jaumann rate), $\boldsymbol{\Omega}^* = \dot{\mathbf{R}}\mathbf{R}^T$ (the polar rate or Green–Naghdi–Dienes rate), $\boldsymbol{\Omega}^* = \boldsymbol{\Omega}^E$, etc. Here the latter is the twirl tensor of the Eulerian triad $\{\mathbf{n}_i\}$, i.e. $\dot{\mathbf{n}}_i = \boldsymbol{\Omega}^E\mathbf{n}_i$.

Since the deformation gradient $\mathbf{F}$ and the velocity gradient $\mathbf{L}$ characterize the local deformation state and the rate-of-change of the local deformation state at a generic material particle, the most general form of the spin tensor $\boldsymbol{\Omega}^*$ associated with the deformation and rotation of a deforming body may be assumed as

$$\boldsymbol{\Omega}^* = \boldsymbol{\Upsilon}(\mathbf{F}, \mathbf{L}) \tag{3.8}$$

with $\boldsymbol{\Upsilon}(\mathbf{F}, \mathbf{L})$ being an antisymmetric tensor-valued function of the deformation gradient $\mathbf{F}$ and the stretching $\mathbf{D}$. Of course, such a general form is of little use. To make the corotational rate defined by the above spin a reasonable objective rate measure, the spin $\boldsymbol{\Omega}^*$ must fulfill certain necessary requirements. The latter place restrictions on the form of the tensor function $\boldsymbol{\Upsilon}(\mathbf{F}, \mathbf{L})$. Recently, these authors (see XIAO, BRUHNS and MEYERS [60]) have introduced the following necessary requirements for $\boldsymbol{\Omega}^*$:

(i) any superimposed constant rigid rotation has no effect on $\boldsymbol{\Omega}^*$, and, moreover, any superimposed constant uniform dilational deformation has also no effect on $\boldsymbol{\Omega}^*$;

(ii) the corotational rate of an Eulerian tensor defined by the spin $\boldsymbol{\Omega}^*$ depends linearly on the change of time scale,

(iii) the corotational rate of each time-differentiable objective Eulerian tensor field defined by the spin $\boldsymbol{\Omega}^*$ is objective, and

(iv) the tensor function $\boldsymbol{\Upsilon}(\mathbf{F}, \mathbf{L})$ is continuously differentiable at $\mathbf{L} = \mathbf{O}$.

From these requirements, a general form of spin $\boldsymbol{\Omega}^*$ has been derived (see XIAO, BRUHNS and MEYERS [60]):

$$\boldsymbol{\Omega}^* = \mathbf{W} + \sum_{\sigma,\tau=1}^{r} h\left(\frac{\chi_\sigma}{I}, \frac{\chi_\tau}{I}\right)\mathbf{B}_\sigma\mathbf{D}\mathbf{B}_\tau\,, \tag{3.9}$$

where

(3.10) $$I = \mathrm{tr}\mathbf{B} = \mathrm{tr}\mathbf{C} = \mathrm{tr}\mathbf{F}\mathbf{F}^{\mathrm{T}}\,.$$

In the above, the function $h(x,y)$: $R^+ \times R^+ \to R$, which defines the spin tensor $\mathbf{\Omega}^*$ and is hence called the *spin function*, is antisymmetric, i.e.

$$h(x,y) = -h(y,x).$$

Each spin given by (3.9), which is the same kind of kinematical quantity as the vorticity tensor $\mathbf{W}$, is called a *material strain* in XIAO, BRUHNS and MEYERS [60]. In particular, a subclass of the above material spins is as follows

(3.11)
$$\mathbf{\Omega}^* = \mathbf{W} + \sum_{\sigma,\tau=1}^{r} \tilde{h}\left(\frac{\chi_\sigma}{\chi_\tau}\right)\mathbf{B}_\sigma \mathbf{D}\mathbf{B}_\tau\,,$$
$$\tilde{h}(z^{-1}) = -\tilde{h}(z) \quad (\forall z > 0)\,.$$

It has been shown (see XIAO, BRUHNS and MEYERS [60]) that all commonly-known spins, including the vorticity tensor $\mathbf{\Omega}^J = \mathbf{W}$, the *polar spin* $\mathbf{\Omega}^R = \dot{\mathbf{R}}\mathbf{R}^T$, the twirl tensors $\mathbf{\Omega}^E$ and $\mathbf{\Omega}^L$ of the Eulerian and Lagrangean triads (for these spins, refer to, e.g., HILL [23], OGDEN [38] and MEHRABADI and NEMAT-NASSER [35], for detail) and the newly discovered *logarithmic spin* $\mathbf{\Omega}^{\log}$ (see XIAO, BRUHNS and MEYERS [57 – 60]; see also LEHMANN, GUO and LIANG [29] and REINHARDT and DUBEY [40 – 41]), are incorporated into the above subclass by taking the simplified spin function $\tilde{h}(z)$ in the particular forms

(3.12) $$\tilde{h}(z) = \tilde{h}^J(z) = 0\,,$$

(3.13) $$\tilde{h}(z) = \tilde{h}^R(z) = \frac{1-\sqrt{z}}{1+\sqrt{z}}\,,$$

(3.14) $$\tilde{h}(z) = \tilde{h}^E(z) = \frac{1+z}{1-z}\,,$$

(3.15) $$\tilde{h}(z) = \tilde{h}^L(z) = \frac{2\sqrt{z}}{1-z}\,,$$

(3.16) $$\tilde{h}(z) = \tilde{h}^{\log}(z) = \frac{1+z}{1-z} + \frac{2}{\ln z}\,,$$

respectively.

Henceforth, the objective corotational rates of the objective Eulerian tensor $\mathbf{G}$ and the objective Lagrangean tensor $\hat{\mathbf{G}}$ defined by the above five material spins and their Lagrangean counterparts, respectively, are denoted by

(3.17) $$\overset{\circ}{\mathbf{G}}{}^M = \dot{\mathbf{G}} + \mathbf{G}\mathbf{\Omega}^M - \mathbf{\Omega}^M\mathbf{G}, \qquad M \in \{J, R, E, L, \log\}\,,$$

$$\overset{\circ}{\hat{\mathbf{G}}}{}^{M} = \mathbf{R}^T \overset{\circ}{\mathbf{G}}{}^{M}\mathbf{R} = \dot{\hat{\mathbf{G}}} + \hat{\mathbf{G}}(\hat{\boldsymbol{\Omega}}^M - \hat{\Omega}^R) - (\hat{\boldsymbol{\Omega}}^M - \hat{\Omega}^R)\hat{\mathbf{G}}, \quad M \in \{J, R, E, L, \log\}. \tag{3.18}$$

Note that for any material spin $\boldsymbol{\Omega}^*$ (cf. (3.9)), the Lagrangean spin $\hat{\boldsymbol{\Omega}}^* - \hat{\boldsymbol{\Omega}}^R$ appearing in (3.5) and in particular in (3.18), i.e.

$$\hat{\boldsymbol{\Omega}}^* - \hat{\boldsymbol{\Omega}}^R = \sum_{\sigma,\tau=1} (h(\frac{\chi_\sigma}{I}, \frac{\chi_\tau}{I}) - \tilde{h}^R(\frac{\chi_\sigma}{\chi_\tau}))\mathbf{C}_\sigma \hat{\mathbf{D}} \mathbf{C}_\tau,$$

is independent of the vorticity tensor $\mathbf{W}$. In the above, the spin function $\tilde{h}^R(z)$ is given by $(3.13)_2$.

In particular, from (3.7) it follows

$$\overset{\circ}{\hat{\mathbf{G}}}{}^{R} = \dot{\hat{\mathbf{G}}} \tag{3.19}$$

for any objective Lagrangean tensor $\hat{\mathbf{G}}$.

Rates of various strain measures (mainly the material time rate) and the material spins $\mathbf{W}$, $\boldsymbol{\Omega}^R$, $\boldsymbol{\Omega}^E$, $\boldsymbol{\Omega}^L$ and $\boldsymbol{\Omega}^{\log}$ have been studied by many authors and many results are available, refer to, e.g., HILL [21 – 23], DIENES [3 – 4], FITZGERALD [8], GURTIN and SPEAR [17], OGDEN [38], GUO and DUBEY [12], GUO [11], HOGER and CARLSON [24], CARLSON and HOGER [1], HOGER [25], MEHRABADI and NEMAT-NASSER [35], DUBEY [6], STICKFORTH and WEGENER [50], WHEELER [55], GUO, LEHMANN and LIANG [13], SCHEIDLER [44 – 47], WANG and DUAN [54], GUO, LEHMANN, LIANG and MAN [14], MACMILLAN [31], CHEN and WHEELER [2], MAN and GUO [33], XIAO [56], REINHARDT and DUBEY [40], and XIAO, BRUHNS and MEYERS [57 – 58, 60 – 61, 65], *et al.*

A unified study of time derivatives of tensor fields via Lie derivatives, which incorporates corotational rates as particular cases, was given earlier by GUO [10] and later by MARSDEN and HUGHES [34].

3.3. Material corotating frames

Each material spin $\boldsymbol{\Omega}^*$ of the form (3.9), which defines a kind of objective corotational rates of objective tensors, is associated with the rotation and deformation of a deforming material body in a suitable manner. Thus, an $\boldsymbol{\Omega}^*$-frame is a rotating frame that is embedded in a deforming material body in a suitable manner, and hence it traces the rotation and deformation of the deforming material body in an intrinsic way. In view of this, we call a rotating frame defined by a material spin through (2.15) – (2.16) a *corotating material frame*. Evidently, the Eulerian and Lagrangean triads are two corotating material frames when they are well-defined. Another two important examples are provided by the rotating frames defined by the vorticity tensor $\boldsymbol{\Omega}^J = \mathbf{W}$ and the polar spin $\boldsymbol{\Omega}^R = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}$, respectively.

A significant fact for corotating material frames is as follows:
The time rate of an objective Eulerian tensor in a corotating material frame is an objective corotational rate and vice versa, i.e. an objective corotational rate of an objective Eulerian tensor is the time rate of this tensor in a corotating material frame.

This fact is essential for the subsequent considerations, as will be seen in the next section.

4. Unified work-conjugacy relation between Eulerian and Lagrangean strain and stress measures

Let $(\mathbf{e}, \mathbf{t})$ be a pair of objective Eulerian strain and stress measures, both being symmetric, and let $\boldsymbol{\Omega}^*$ be an Eulerian spin. In an $\boldsymbol{\Omega}^*$-frame (cf. (2.15) – (2.16)) relative to a fixed background frame, this pair becomes $(\mathbf{Q}^*\mathbf{e}\mathbf{Q}^{*T}, \mathbf{Q}^*\mathbf{t}\mathbf{Q}^{*T})$. Then an observer in the just-mentioned $\boldsymbol{\Omega}^*$-frame forms the inner product

$$(\mathbf{Q}^*\mathbf{t}\mathbf{Q}^{*T}) : \dot{\overline{(\mathbf{Q}^*\mathbf{e}\mathbf{Q}^{*T})}},$$

just as an observer in a fixed background frame does for a Lagrangean strain-stress pair. Following the same argument as that used in Hill's work-conjugacy notion (cf. (1.1)), which is concerned with a fixed background frame, the observer in the $\boldsymbol{\Omega}^*$-frame judges that the pair $(\mathbf{t}, \mathbf{e})$ is an $\boldsymbol{\Omega}^*$-*work-conjugate pair* if the just-mentioned inner product furnishes the stress power $\dot{w}$, i.e.

$$\dot{w} = (\mathbf{Q}^*\mathbf{t}\,\mathbf{Q}^{*T}) : \dot{\overline{(\mathbf{Q}^*\mathbf{e}\,\mathbf{Q}^{*T})}}\,, \tag{4.1}$$

or, equivalently,

$$\dot{w} = \mathbf{t} : \overset{\circ}{\mathbf{e}}{}^*\,, \tag{4.2}$$

where $\overset{\circ}{\mathbf{e}}{}^*$ is the corotational rate of the strain measure $\mathbf{e}$ defined by the spin $\boldsymbol{\Omega}^*$, i.e.

$$\overset{\circ}{\mathbf{e}}{}^* = \dot{\mathbf{e}} + \mathbf{e}\boldsymbol{\Omega}^* - \boldsymbol{\Omega}^*\mathbf{e}\,. \tag{4.3}$$

As has been shown, the above work-conjugacy relation is defined in a rotating frame. However, the definition itself does not mean that such relation is well-defined for every rotating frame, i.e. for every kind of corotational rates. Now we are in a position to find out in what rotating frames the aforementioned work-conjugacy relation can be defined. Since both $\dot{w}$ and $\mathbf{t}$ are objective and both $\mathbf{t}$ and $\mathbf{e}$ are symmetric, from (4.2) we conclude that *the relation (4.1), i.e. (4.2), may be defined only if the corotational rate* $\overset{\circ}{\mathbf{e}}{}^*$ *of the strain measure* $\mathbf{e}$ *is objective.*

The above fact justifies the introduction of objective corotational rates in Sec. 3.2. Furthermore, let the spin $\boldsymbol{\Omega}^*$ be associated with the rotation and deformation of the deforming material body at issue in a manner indicated by (3.8). Then, by applying the general result proved in Xiao, Bruhns and Meyers [60] we infer that the spin $\boldsymbol{\Omega}^*$ must be of the form given by (3.9), i.e., the spin $\boldsymbol{\Omega}^*$ must be a material spin. Accordingly, *the work-conjugacy relation (4.1), i.e. (4.2), may be defined only in a corotating material frame.*
However, we still can not say that the work-conjugacy relation (4.1), i.e. (4.2), can be defined in all possible corotating material frames. In fact, each objective corotational strain rate $\overset{\circ}{\mathbf{e}}{}^*$ defined by a material spin $\boldsymbol{\Omega}^*$ is of the form (see (5.1) given in Sec. 5)

$$\overset{\circ}{\mathbf{e}}{}^* = \mathcal{L}[\mathbf{D}],$$

where $\mathcal{L} = \tilde{\mathcal{L}}(\mathbf{B})$ is a fourth order tensor depending on the left Cauchy–Green tensor $\mathbf{B}$, i.e. a linear transformation between second order tensors, with the index symmetry properties

$$\mathcal{L}_{ijkl} = \mathcal{L}_{jikl} = \mathcal{L}_{ijlk} = \mathcal{L}_{klij}.$$

Hence, (4.2) may be recast in

$$\dot{w} = \mathbf{t} : (\mathcal{L}[\mathbf{D}]).$$

From the latter and the formula (2.13) we deduce that the equality

$$\mathbf{t} : (\mathcal{L}[\mathbf{D}]) = \boldsymbol{\sigma} : \mathbf{D}$$

must hold for each $\mathbf{D}$ and each $\boldsymbol{\sigma}$. Since $\mathcal{L}$ is a symmetric linear transformation, we derive

$$\mathcal{L}[\mathbf{t}] = \boldsymbol{\sigma}\,. \tag{4.4}$$

Thus, if the stress $\boldsymbol{\sigma}$ is not allowed to be restricted in any manner, as it should be, the fourth order tensor $\mathcal{L} = \tilde{\mathcal{L}}(\mathbf{B})$ must be a nonsingular linear transformation between symmetric second order tensors, and therefore the objective corotational strain rate $\overset{\circ}{\mathbf{e}}{}^*$ must be a *complete strain rate measure*. By the latter we mean that the strain rate $\overset{\circ}{\mathbf{e}}{}^*$ and the stretching tensor $\mathbf{D}$ constitutes a one-to-one correspondence for any given left Cauchy–Green tensor $\mathbf{B}$. By applying the expression for the strain rate $\overset{\circ}{\mathbf{e}}{}^*$ in terms of $\mathbf{B}$ and $\mathbf{D}$ (cf. (5.1)), from (4.4) we can derive the $\boldsymbol{\Omega}^*$-work-conjugate stress measure $\mathbf{t}$ of the strain measure $\mathbf{e}$ in terms of $\mathbf{B}$ and $\mathbf{D}$. We postpone the further discussion in this aspect to the next section.

Now we consider objective Lagrangean strain and stress measures. By virtue of the rotated correspondence (2.5), we can convert (4.2) to

$$\dot{w} = \hat{\mathbf{t}} : (\mathbf{R}^T \overset{\circ}{\mathbf{e}}{}^* \mathbf{R})\,, \tag{4.5}$$

where the identity

$$\mathrm{H}_1 : \mathrm{H}_2 = (\mathbf{Q}\mathrm{H}_1\mathbf{Q}^{\mathrm{T}}) : (\mathbf{Q}\mathrm{H}_2\mathbf{Q}^{\mathrm{T}})$$

for any two tensors H_1 and H_2 and for any orthogonal tensor $\mathbf{Q}$, is used. Applying the formula $(3.5)_1$, we further arrive at

$$\dot{w} = \mathrm{T} : \overset{\circ}{\mathrm{E}}{}^{*}, \tag{4.6}$$

where $\mathrm{T} = \hat{\mathrm{t}}$ and $\mathrm{E} = \hat{\mathrm{e}}$ are the Lagrangean counterparts of the Eulerian stress and strain measures t and e through the rotated correspondence (2.5), and $\overset{\circ}{\mathrm{E}}{}^{*}$ is the objective corotational rate of the objective Lagrangean strain measure E defined by the Lagrangean spin $\hat{\boldsymbol{\Omega}}^{*} = \mathbf{R}^{T}\boldsymbol{\Omega}^{*}\mathbf{R}$ (cf. (3.5)), i.e.

$$\overset{\circ}{\mathrm{E}}{}^{*} = \dot{\mathrm{E}} + \mathrm{E}(\hat{\boldsymbol{\Omega}}^{*} - \hat{\boldsymbol{\Omega}}^{R}) - (\hat{\boldsymbol{\Omega}}^{*} - \hat{\boldsymbol{\Omega}}^{R})\mathrm{E}. \tag{4.7}$$

Observing that (4.6) has the same structure as that of (4.2), we say that a pair of Lagrangean strain and stress measures, (T, E), is an $\hat{\boldsymbol{\Omega}}^{*}$-*work-conjugate pair* if (4.6) holds.

Let $(\boldsymbol{\Omega}^{*}, \hat{\boldsymbol{\Omega}}^{*})$, $(\mathrm{t}, \hat{\mathrm{t}})$ and $(\mathrm{e}, \hat{\mathrm{e}})$ be, respectively, the Eulerian-Lagrangean spin pair, stress pair and strain pair related by the rotated correspondence (2.5) and (2.6). Then, it is evident that

$$(\mathrm{e}, \mathrm{t}) \text{ is an } \boldsymbol{\Omega}^{*}\text{-work-conjugate pair} \Longleftrightarrow (\hat{\mathrm{e}}, \hat{\mathrm{t}}) \text{ is an } \hat{\boldsymbol{\Omega}}^{*}\text{-work-conjugate pair.}$$

Thus, via (4.2) and (4.6) we have established a unified work-conjugacy relation between Eulerian and Lagrangean strain and stress measures. The just-stated fact indicates that in the sense of this unified work-conjugacy relation, the rotated correspondence relationship via the rotation tensor R remains true for work-conjugate Eulerian strain-stress pairs and work-conjugate Lagrangean strain-stress pairs.

The introduced unified work-conjugacy relation is much broader than the Hill's work-conjugacy relation, even within the scope of objective Lagrangean strain and stress measures. In fact, via various kinds of corotating material frames, or, equivalently, via various kinds of objective corotational rates, a given strain measure may be related to different stress measures in the sense of the introduced work-conjugacy relation. It turns out that the introduced unified work-conjugacy relation incorporates the Hill's work-conjugacy relation into a particular case when an $\boldsymbol{\Omega}^{R}$-frame is concerned. In fact, by utilizing (3.19) we have

$$\dot{w} = \mathrm{T} : \dot{\mathrm{E}} = \mathrm{T} : \overset{\circ}{\mathrm{E}}{}^{R} = \mathrm{t} : \overset{\circ}{\mathrm{e}}{}^{R}$$

where $\mathrm{t} = \mathbf{R}\mathrm{T}\mathbf{R}^{T}$ and $\mathrm{e} = \mathbf{R}\mathrm{E}\mathbf{R}^{T}$ are the Eulerian counterparts of the objective Lagrangean stress and strain measures T and E, and, moreover, $\overset{\circ}{\mathrm{E}}{}^{R}$ and $\overset{\circ}{\mathrm{e}}{}^{R}$

are the polar rates of E and e (cf. (3.17) and (3.18) with $M = R$), respectively. Thus,

$$(4.8)\quad \begin{cases} (\mathbf{E},\mathbf{T}) \text{ is a work-conjugate Lagrangean strain-stress pair} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in Hill's sense}, \\ \Longleftrightarrow (\mathbf{e},\mathbf{t}) \text{ is an } \mathbf{\Omega}^R - \text{work-conjugate Eulerian strain-stress pair}. \end{cases}$$

In the above, the two strain-stress pairs are related to each other by the rotated correspondence (2.5) and (2.6).

ZIEGLER and MACVEAN [64] introduced a more general work-conjugacy notion. MACVEAN [32] studied work-conjugacy relation between certain commonly-known Eulerian and Lagrangean strain and stress measures. The general work-conjugacy notion was also adopted in HAUPT and TSAKMAKIS [19]. These studies allow for general objective rate measures including objective corotational rates. Moreover, the definition (4.2) was used by LEHMANN [28] in a thermodynamical setting and later used in some particular cases by LEHMANN, GUO and LIANG [29] and LEHMANN and LIANG [30]. It seems that the interpretation (cf. (4.1)) of this notion in terms of corotating material frames, and hence the fact that the introduced work-conjugacy relation is a natural extension of Hill's work-conjugacy relation, are disclosed first by these authors in XIAO, BRUHNS and MEYERS [57 – 58]. Moreover, it seems that the unified work-conjugacy relation for both Eulerian and Lagrangean strain and stress measures in a general sense is established here for the first time.

5. Work-conjugate stresses of generalized Eulerian and Lagrangean strain measures

Let $\mathbf{e} = \mathbf{g}(\mathbf{B})$ be any given Hill's Eulerian strain measure defined by the scale function $g(\chi)$ (cf. (2.28)–(2.29)) and moreover, let $\mathbf{\Omega}^*$ be any given Eulerian material spin characterized by the spin function $h(x, y)$ (cf. (3.9)). According to the formulas (31a) and (30) given in XIAO [56], we have

$$\dot{\mathbf{e}} = \sum_{\sigma,\tau=1}^{r} g_{\sigma\tau}\mathbf{B}_\sigma\dot{\mathbf{B}}\mathbf{B}_\tau,$$

where

$$g_{\sigma\tau} = \frac{g(\chi_\sigma) - g(\chi_\tau)}{\chi_\sigma - \chi_\tau}$$

with the limiting process $\lim_{\sigma\to\tau} g_{\sigma\tau} = g'(\chi_\sigma)$ understood when $\sigma = \tau$ in the summation. On the other hand, by using $(2.3)_2$ and (2.2) and $(2.4)_1$ we infer

$$\dot{\mathbf{B}} = \dot{\mathbf{F}}\mathbf{F}^T + \mathbf{F}\dot{\mathbf{F}}^T = (\mathbf{DB} + \mathbf{BD}) + (\mathbf{WB} - \mathbf{BW}).$$

Then, utilizing the above results and the equalities

$$\mathbf{B}\mathbf{B}_\theta = \mathbf{B}_\theta\mathbf{B} = \chi_\theta\mathbf{B}_\theta; \quad \mathbf{e}\mathbf{B}_\theta = \mathbf{B}_\theta\mathbf{e} = \mathrm{g}(\chi_\theta)\mathbf{B}_\theta,$$

for the material spin $\boldsymbol{\Omega}^*$ given by (3.9) we derive

$$\begin{aligned}\overset{\circ}{\mathbf{e}}{}^* = & \left(\sum_{\sigma,\tau=1}^{r} g_{\sigma\tau}\mathbf{B}_\sigma\dot{\mathbf{B}}\mathbf{B}_\tau\right) + \mathbf{e}\left(\sum_{\sigma,\tau=1}^{r} \mathrm{h}\left(\frac{\chi_\sigma}{I},\frac{\chi_\tau}{I}\right)\mathbf{B}_\sigma\mathbf{D}\mathbf{B}_\tau\right) \\ & - \left(\sum_{\sigma,\tau=1}^{r} h\left(\frac{\chi_\sigma}{I},\frac{\chi_\tau}{I}\right)\mathbf{B}_\sigma\mathbf{D}\mathbf{B}_\tau\right)\mathbf{e} \\ = & \sum_{\sigma,\tau=1}^{r}\left((\chi_\sigma+\chi_\tau)g_{\sigma\tau} + (g(\chi_\sigma)-g(\chi_\tau))h\left(\frac{\chi_\sigma}{I},\frac{\chi_\tau}{I}\right)\right)\mathbf{B}_\sigma\mathbf{D}\mathbf{B}_\tau .\end{aligned}$$

Hence, the objective corotational strain rate $\overset{\circ}{\mathbf{e}}{}^*$ defined by the material spin $\boldsymbol{\Omega}^*$ is given by

$$\overset{\circ}{\mathbf{e}}{}^* = \tilde{\mathcal{L}}(\mathbf{B})[\mathbf{D}] = \sum_{\sigma,\tau=1}^{\mathbf{r}} \rho(\chi_\sigma,\chi_\tau)\mathbf{B}_\sigma\mathbf{D}\mathbf{B}_\tau , \tag{5.1}$$

where

$$\rho(x,y) = \left((x+y)+(x-y)h\left(\frac{x}{I},\frac{y}{I}\right)\right)\frac{g(x)-g(y)}{x-y} \tag{5.2}$$

for any $x, y > 0$, where the invariant I is given by (3.10). In (5.1), $\chi_1,\cdots,\chi_r$ are the distinct eigenvalues of $\mathbf{B}$ and $\mathbf{B}_1,\cdots,\mathbf{B}_r$ are the corresponding subordinate eigenprojections of $\mathbf{B}$, and the fourth order tensor $\tilde{\mathcal{L}}(\mathbf{B})$ is defined by $(5.1)_2$. In order to obtain the conjugate stress $\mathbf{t}$ from (4.4), it is required to judge whether or not the strain rate $\overset{\circ}{\mathbf{e}}{}^*$ is a complete one, i.e. whether or not the fourth order tensor $\tilde{\mathcal{L}}(\mathbf{B})$ as a linear transformation between second order tensors is nonsingular for all $\mathbf{B}$, and, moreover, to work out the inverse of $\tilde{\mathcal{L}}(\mathbf{B})$. At first sight, it seems not easy to solve the just-mentioned two problems, since we have to deal with a fourth order tensor depending on a second order tensor. Fortunately, utilizing (2.35) – (2.36), from (5.1) we can derive a *spectral representation* of the fourth order tensor $\tilde{\mathcal{L}}(\mathbf{B})$ and hence the aforementioned tough problems become tractable.

Let $\mathbf{H}_1$ and $\mathbf{H}_2$ be two given second order tensors. We introduce the *Kronecker product* $\mathbf{H}_1 * \mathbf{H}_2$ of the tensors $\mathbf{H}_1$ and $\mathbf{H}_2$ by

$$(\mathbf{H}_1 * \mathbf{H}_2)[\mathbf{X}] = \mathbf{H}_1\mathbf{X}\mathbf{H}_2 \tag{5.3}$$

for any second order tensor $\mathbf{X}$. It is evident that the Kronecker product $\mathbf{H}_1 * \mathbf{H}_2$ defined above is a linear transformation between second order tensors, i.e. a fourth order tensor. With the help of the Kronecker product introduced, from (5.1) we derive

$$\tilde{\mathcal{L}}(\mathrm{B}) = \sum_{\sigma,\tau=1}^{r} \rho(\chi_\sigma, \chi_\tau)\mathrm{B}_\sigma * \mathrm{B}_\tau \,. \tag{5.4}$$

The crucial point is that *the above expression is exactly a spectral representation of the fourth order tensor $\tilde{\mathcal{L}}(\mathrm{B})$ as a linear transformation between second order tensors, in which each $\rho(\chi_\sigma, \chi_\tau)$ is an eigenvalue.* In fact, let $\mathcal{L}_1 \circ \mathcal{L}_2$ designate the composition of the two fourth order tensors $\mathcal{L}_1$ and $\mathcal{L}_2$ as two tranformations. Then, by utilizing (2.35) and (2.36) and the definition (5.3), we deduce

$$\begin{aligned}((\mathbf{B}_\alpha * \mathbf{B}_\beta) \circ (\mathbf{B}_\sigma * \mathbf{B}_\tau))[\mathbf{X}] &= (\mathbf{B}_\alpha * \mathbf{B}_\beta)[(\mathbf{B}_\sigma * \mathbf{B}_\tau)[\mathbf{X}]] \\ &= (\mathbf{B}_\alpha * \mathbf{B}_\beta)[\mathbf{B}_\sigma \mathbf{X} \mathbf{B}_\tau] \\ &= \mathbf{B}_\alpha \mathbf{B}_\sigma \mathbf{X} \mathbf{B}_\tau \mathbf{B}_\beta \\ &= \begin{cases} (\mathbf{B}_\sigma * \mathbf{B}_\tau)[\mathbf{X}] & \text{if} \quad \alpha = \sigma, \beta = \tau\,, \\ \mathbf{O} \quad \text{otherwise}\,, \end{cases}\end{aligned}$$

for any $1 \leqslant \alpha, \beta, \sigma, \tau \leqslant r$ and for any second order tensor $\mathbf{X}$, and, moreover, we have

$$\sum_{\sigma,\tau=1}^{r} \mathbf{B}_\sigma * \mathbf{B}_\tau = \mathbf{I} * \mathbf{I}.$$

The former yields

$$(\mathbf{B}_\alpha * \mathbf{B}_\beta) \circ (\mathbf{B}_\sigma * \mathbf{B}_\tau) = \begin{cases} \mathbf{B}_\sigma * \mathbf{B}_\tau & \text{if} \quad \alpha = \sigma, \beta = \tau\,, \\ \mathbf{O} \otimes \mathbf{O} & \text{otherwise}\,, \end{cases}$$

for any $1 \leqslant \alpha, \beta, \sigma, \tau \leqslant r$, where $\mathbf{O} \otimes \mathbf{O}$ is the fourth order null tensor. Moreover, the tensor $\mathbf{I} * \mathbf{I}$ gives the identity transformation between second order tensors, since

$$(\mathbf{I} * \mathbf{I})[\mathbf{X}] = \mathbf{X}$$

for any second order tensor $\mathbf{X}$. Thus, from the above facts and a well-known fact from the linear transformations we conclude that (5.4) is a spectral representation of $\tilde{\mathcal{L}}(\mathrm{B})$ and hence each $\rho(\chi_\sigma, \chi_\tau)$ is an eigenvalue.

From the fact just proved, we can derive the desired results immediately. First, we assert that $\tilde{\mathcal{L}}(\mathrm{B})$ is nonsingular if and only if each eigenvalue of it is nonzero, i.e. $\rho(\chi_\sigma, \chi_\tau) \neq 0$. The latter produces

$$(x + y) + (x - y)h(x, y) \neq 0, \qquad \forall x, y > 0, \qquad x \neq y\,. \tag{5.5}$$

In deriving the above, the condition $(g(x) - g(y))/(x - y) \neq 0$ is used. For the latter, we would mention that the scale function $g(z)$ is a monotonic increasing function. Then, combining the condition (5.5) and the related result derived in Sec. 4, we conclude that

the Ω^*-work-conjugacy relation (4.1), i.e. (4.2) can be defined if and only if the spin Ω^* is a material spin (cf. (3.9)) fulfilling the condition (5.5).

Next, for any given material spin Ω^* (cf. (3.9)) fulfilling the condition (5.5), from (4.4) we derive the Ω^*-work-conjugate stress measure $\mathbf{t}$ of an arbitrary Hill's Eulerian strain measure $\mathbf{e}$ as follows

$$\mathbf{t} = (\tilde{\mathcal{L}}(\mathbf{B}))^{-1}[\sigma] = \sum_{\sigma,\tau=1}^{r} \rho(\chi_\sigma, \chi_\tau)^{-1}\mathbf{B}_\sigma \boldsymbol{\sigma} \mathbf{B}_\tau\,, \tag{5.6}$$

where the symmetric function $\rho(x, y)$ is given by (5.4).

The formula (5.6), which provides the Ω^*-work-conjugate stress of the Eulerian strain measure $\mathbf{e}$ in terms of the related basic quantities, i.e. the left Cauchy–Green tensor B and the Kirchhoff stress measure $\boldsymbol{\sigma}$, is valid for any given Eulerian material spin Ω^* (cf. (3.9)) fulfilling the condition (5.5) and for any given Eulerian strain measure $\mathbf{e}$ (cf. (2.19) or (2.29)). Moreover, by means of the rotated correspondence relationship as indicated by (4.8), we obtain the $\hat{\Omega}^*$-work-conjugate stress measure of any given Lagrangean strain measure $\mathbf{E}$ (cf. (2.20) or (2.30)) as follows:

$$\mathbf{T} = \sum_{\sigma,\tau=1}^{r} \rho(\chi_\sigma, \chi_\tau)^{-1}\mathbf{C}_\sigma \hat{\boldsymbol{\sigma}} \mathbf{C}_\tau\,. \tag{5.7}$$

We would point out that in the formulas (5.6) – (5.7), the limiting process $\lim\limits_{\chi_\sigma\to\chi_\tau} \dfrac{g(\chi_\sigma) - g(\chi_\tau)}{\chi_\sigma - \chi_\tau} = g'(\chi_\sigma)$ is meant when $\sigma = \tau$.

Substituting the four spin functions $h(x, y) = \tilde{h}^M\left(\dfrac{x}{y}\right)$ for $M \in \{J, R, L, \log\}$(3) given by (3.12) – (3.13) and (3.15) – (3.16) into (5.6) and (5.7) respectively, one can obtain the work-conjugate stress measures of any given Eulerian and Lagrangean strain measures $\mathbf{e}$ and $\mathbf{E}$ with regard to the material spins Ω^M and $\hat{\Omega}^M$ for $M \in \{J, R, L, \log\}$, respectively. In particular, substituting the spin function $h(x, y) = \tilde{h}^R\left(\dfrac{x}{y}\right)$ (cf. (3.13)) defining the spin $\Omega^R = \dot{\mathbf{R}}\mathbf{R}^T$ into the formula (5.7), we derive the work-conjugate stress measure of any given Lagrangean strain measure $\mathbf{E}$ (cf. (2.20) or (2.30)) in Hill's work-conjugacy sense (cf. (1.1)), i.e. the $\hat{\Omega}^R$-work-conjugate stress measure of the Lagrangean strain measure $\mathbf{E}$, as follows:

$$\mathbf{T}^R = \sum_{\sigma,\tau=1}^{r} (2\sqrt{\chi_\sigma\chi_\tau})^{-1}\frac{\chi_\sigma - \chi_\tau}{g(\chi_\sigma) - g(\chi_\tau)}\mathbf{C}_\sigma \hat{\boldsymbol{\sigma}} \mathbf{C}_\tau\,. \tag{5.8}$$

(3)The spin function given by (3.14) is excluded, since it fails to meet the condition (5.5), i.e., the strain rate $\overset{\circ}{\mathbf{e}}{}^E$ for any $\mathbf{e}$ is not a complete strain rate measure.

The above result is presented in terms of the right Cauchy–Green tensor $\mathbf{C}$ and the Lagrangean Kirchhoff stress measure or the *rotated Kirchhoff stress*, $\hat{\boldsymbol{\sigma}} = \mathbf{R}^T\boldsymbol{\sigma}\mathbf{R}$. Other forms of expressions for $\mathbf{T}^R$ can be found in HILL [21 – 23], WANG and TRUESDELL [53], OGDEN [38], WANG and DUAN [54], and XIAO [56], *et al.*; see also HOGER [26], LEHMANN and LIANG [30], GUO and MAN [15], *et al.*, for some particular cases.

The formulas (5.6) – (5.7) are expressed in terms of the eigenprojections of the Cauchy–Green tensors B and C, respectively. Applying Sylvester's formulas (2.31) and (2.32), from (5.6) and (5.7) one can derive explicit basis-free expressions for the conjugate stresses t and T. In fact, we have

$$\mathbf{t} = \sum_{i,j=0}^{r-1} \varrho_{ij}\mathbf{B}^i\boldsymbol{\sigma}\mathbf{B}^j\,, \tag{5.9}$$

$$\mathbf{T} = \sum_{i,j=0}^{r-1} \varrho_{ij}\mathbf{C}^i\hat{\boldsymbol{\sigma}}\mathbf{C}^j\,, \tag{5.10}$$

where each coefficient $\varrho_{ij} = \varrho_{ji}$ is a symmetric function of the distinct eigenvalues $\chi_1, \cdots, \chi_r$ of B or C, i.e. an invariant of the latter. The expressions for the coefficients ϱ_{ij} are given as follows.

(i) $r = 1:\ \chi_1 = \chi_2 = \chi_3 = \chi$.

$$\varrho_{00} = (2\chi g'(\chi))^{-1}\,. \tag{5.11}$$

(ii) $r = 2:\ \chi_1 \neq \chi_2 = \chi_3$.

The eigenprojections are of the forms

$$\begin{aligned} \mathbf{B}_1 &= \mathbf{R}\mathbf{C}_1\mathbf{R}^T = (\chi_1 - \chi_2)^{-1}(\mathbf{B} - \chi_2\mathbf{I})\,,\\ \mathbf{B}_2 &= \mathbf{R}\mathbf{C}_1\mathbf{R}^T = (\chi_2 - \chi_1)^{-1}(\mathbf{B} - \chi_1\mathbf{I})\,. \end{aligned} \tag{5.12}$$

The coefficients ϱ_{ij}, $i, j = 0, 1$, are given by

$$\varrho_{00} = \frac{1}{2}(\chi_2^2(\chi_1 g_1')^{-1} + \chi_1^2(\chi_2 g_2')^{-1} - 4\chi_1\chi_2\rho_{12}^{-1})(\chi_1 - \chi_2)^{-2}\,, \tag{5.13}$$

$$\varrho_{01} = \varrho_{10} = -\frac{1}{2}(\chi_2(\chi_1 g_1')^{-1} + \chi_1(\chi_2 g_2')^{-1}g' \\ -2(\chi_1 + \chi_2)\rho_{12}^{-1})(\chi_1 - \chi_2)^{-2}\,, \tag{5.14}$$

$$\varrho_{11} = \frac{1}{2}((\chi_1 g_1')^{-1} + (\chi_2 g_2')^{-1} - 4\rho_{12}^{-1})(\chi_1 - \chi_2)^{-2}\,. \tag{5.15}$$

(iii) $r = 3$: $\chi_1 \neq \chi_2 \neq \chi_3 \neq \chi_1$.

The eigenprojections are of the forms

$$
(5.16)\quad
\begin{aligned}
&\mathbf{B}_1 = \mathbf{R}\mathbf{C}_1\mathbf{R}^T = \frac{\chi_2-\chi_3}{\Delta}(\mathbf{B}-\chi_2\mathbf{I})(\mathbf{B}-\chi_3\mathbf{I}),\\
&\mathbf{B}_2 = \mathbf{R}\mathbf{C}_2\mathbf{R}^T = \frac{\chi_3-\chi_1}{\Delta}(\mathbf{B}-\chi_3\mathbf{I})(\mathbf{B}-\chi_1\mathbf{I}),\\
&\mathbf{B}_3 = \mathbf{R}\mathbf{C}_3\mathbf{R}^T = \frac{\chi_1-\chi_2}{\Delta}(\mathbf{B}-\chi_1\mathbf{I})(\mathbf{B}-\chi_2\mathbf{I}),\\
&\Delta = (\chi_3-\chi_2)(\chi_2-\chi_1)(\chi_1-\chi_3)\,.
\end{aligned}
$$

The coefficients ϱ_{ij}, $i,j=0,1,2$, are given by

$$
(5.17)\quad \varrho_{00} = \frac{1}{2}\Delta^{-2}\sum_{(ijk)}\big(({\chi_i}^2\chi_j-\chi_i{\chi_j}^2)^2(\chi_k g_k')^{-1} - 4III(\chi_i-\chi_k)(\chi_j-\chi_k)\chi_k\rho_{ij}^{-1}\big),
$$

$$
(5.18)\quad \varrho_{11} = \frac{1}{2}\Delta^{-2}\sum_{(ijk)}\big(({\chi_i}^2-{\chi_j}^2)^2(\chi_k g_k')^{-1} - 4({\chi_i}^2-{\chi_k}^2)({\chi_j}^2-{\chi_k}^2)\rho_{ij}^{-1}\big),
$$

$$
(5.19)\quad \varrho_{22} = \frac{1}{2}\Delta^{-2}\sum_{(ijk)}\big((\chi_i-\chi_j)^2(\chi_k g_k')^{-1} - 4(\chi_i-\chi_k)(\chi_j-\chi_k)\rho_{ij}^{-1}\big),
$$

$$
(5.20)\quad \varrho_{01} = -\frac{1}{2}\Delta^{-2}\sum_{(ijk)}\big((\chi_i-\chi_j)^2({\chi_i}^2\chi_j+\chi_i{\chi_j}^2)(\chi_k g_k')^{-1} - 2(III+II\chi_k)(\chi_i-\chi_k)(\chi_j-\chi_k)\rho_{ij}^{-1}\big),
$$

$$
(5.21)\quad \varrho_{02} = \frac{1}{2}\Delta^{-2}\sum_{(ijk)}\big(\chi_i\chi_j(\chi_i-\chi_j)^2(\chi_k g_k')^{-1} - 2(\chi_i+\chi_j)(\chi_i-\chi_k)(\chi_j-\chi_k)\chi_k\rho_{ij}^{-1}\big),
$$

$$
(5.22)\quad \varrho_{12} = -\frac{1}{2}\Delta^{-2}\sum_{(ijk)}\big((\chi_i+\chi_j)(\chi_i-\chi_j)^2(\chi_k g_k')^{-1} - 2(I+\chi_k)(\chi_i-\chi_k)(\chi_j-\chi_k)\rho_{ij}^{-1}\big),
$$

where the notation $\sum_{(ijk)}$ means the summation for $(ijk) = (123), (231), (312)$. In the above, we denote

$$
(5.23)\quad g_k' = g'(\chi_k), \qquad \rho_{ij} = \rho(\chi_i,\chi_j)\,.
$$

Moreover, I, II and III are the three principal invariants of $\mathbf{B}$ or $\mathbf{C}$, i.e. for $\mathbf{G} = \mathbf{B},\ \mathbf{C}$,

$$
\begin{aligned}
I &= \chi_1 + \chi_2 + \chi_3 = \mathrm{tr}\mathbf{G}\,,\\
II &= \chi_1\chi_2 + \chi_2\chi_3 + \chi_3\chi_1 = \frac{1}{2}(\mathrm{tr}\mathbf{G})^2 - \frac{1}{2}\mathrm{tr}\mathbf{G}^2\,,\\
III &= \det\mathbf{G} = \chi_1\chi_2\chi_3 = \frac{1}{6}(\mathrm{tr}\mathbf{G})^3 - \frac{1}{2}(\mathrm{tr}\mathbf{G})(\mathrm{tr}\mathbf{G})^2 + \frac{1}{3}\mathrm{tr}\mathbf{G}^3\,.
\end{aligned}
\tag{5.24}
$$

The three eigenvalues (possibly repeated) of $\mathbf{B}$ or $\mathbf{C}$ are determined by the formula (see SAWYERS [43])

$$
\begin{aligned}
\chi_i &= \frac{1}{3}(I + 2\sqrt{I^2 - 3II}\cos\frac{1}{3}(\theta - 2\pi i)), \qquad i = 1, 2, 3\,,\\
\cos\theta &= \frac{2I^3 - 9I\cdot II + 27III}{2(I^2 - 3II)^{3/2}}, \qquad 0 \leqslant \theta < \pi\,.
\end{aligned}
\tag{5.25}
$$

In the above results, the formulas (5.6) and (5.7) in a principal component form are simple, but to apply them actually, one needs to calculate the eigenvalues and eigenvectors of $\mathbf{B}$ at each material particle. However, in a deforming material body, the Eulerian and Lagrangean triads may vary with the time and the spacial position of material particle during the course of deformation in a rather complicated manner. Thus, the just-mentioned eigenvalue/eigenvector calculation may become cumbersome, or even intractable. The explicit basis-free formulas, which have rather complicated forms, are just what are needed to avoid this unsatisfactory situation. They are valid for all coordinate system and hence independent of any particular coordinate system. Once the deformation gradient $\mathbf{F}$ is known under any coordinate system, by means of the explicit basis-free formulas given, one can directly calculate the desired results for conjugate stresses without recourse to the complicated eigenvalue/eigenvector calculation.

6. On Hencky's logarithmic strain measures

The logarithmic strain measures $\ln\mathbf{V}$ and $\ln\mathbf{U}$ (cf. (2.26) and (2.27)), introduced by HENCKY [20], have long been popular and enjoyed favoured treatment in solid mechanics, metallurgy and materials science, etc. In constitutive modeling, these measures and their rates are often chosen as basic strain measures and basic strain rate measures and have been shown to possess certain intrinsic advantages (e.g., see HILL [21 – 23], RICE [42], HUTCHINSON and NEALE [27], and OGDEN [38], *et al.*). Earlier, the usefulness of them was confined to some particular cases due to their complicated transcendental form, as pointed out by TRUESDELL and TOUPIN [52]. This situation was later improved by FITZGERALD [8], GURTIN and SPEAR [17], HOGER [25 – 26], and SCHEIDLER [44 – 46], *et al.*

The main objective of this section is to disclose intrinsic, unique relationship between the logarithmic strain measures and the fundamental mechanical quantities, i.e., the stretching tensors $\mathbf{D}$ and $\hat{\mathbf{D}}$ and the Kirchhoff stress measures $\boldsymbol{\sigma}$ and $\hat{\boldsymbol{\sigma}}$. (2.11) (2.12)

It has been known for a long time that neither of the Eulerian and Lagrangean stretching tensors $\mathbf{D}$ and $\hat{\mathbf{D}}$ can be written in a direct flux of a strain measure (cf. HILL [23]), although they are frequently referred to as the rate of deformation tensor, the Eulerian strain rate, etc. As a result, neither of the Eulerian and Lagrangean Kirchhoff stress tensors $\boldsymbol{\sigma}$ and $\hat{\boldsymbol{\sigma}}$ is conjugate to any strain measure. Recently, these authors have proved (see XIAO, BRUHNS and MEYERS [57 – 60]) that an objective corotational rate of the Hencky's Eulerian logarithmic strain measure $\ln\mathbf{V}$ can be identical with the Eulerian stretching tensor $\mathbf{D}$, i.e., in a corotating material frame the latter is a *true* time rate of the former, and furthmore that in all strain measures only $\ln\mathbf{V}$ enjoys this property. Specifically, we have

$$\overset{\circ}{\mathbf{e}}{}^{*} = \dot{\mathbf{e}} + \mathbf{e}\boldsymbol{\Omega}^{*} - \boldsymbol{\Omega}^{*}\mathbf{e} = \mathbf{D} \Longleftrightarrow \mathbf{e} = \ln\mathbf{V} \;\&\; \boldsymbol{\Omega}^{*} = \boldsymbol{\Omega}^{\log}, \tag{6.1}$$

where $\boldsymbol{\Omega}^{\log}$ is the logarithmic spin (cf. (3.11) and (3.16)). As a result, from (2.13) and (4.2) and

$$\mathbf{D} = (\ln\overset{\circ}{\mathbf{V}})^{\log} \tag{6.2}$$

it follows that the pair $(\ln\mathbf{V}, \boldsymbol{\sigma})$ is an $\boldsymbol{\Omega}^{\log}$-work-conjugate Eulerian strain-stress pair. A particular case of the above fact was known first by LEHMANN, GUO and LIANG [29] and later by REINHARDT and DUBEY [40 – 41] and DUBEY and REINHARDT [7], where only $\mathbf{e} = \ln\mathbf{V}$, i.e.

$$\mathbf{e} = \ln\mathbf{V} \;\&\; \boldsymbol{\Omega}^{*} = \boldsymbol{\Omega}^{\log} \Longrightarrow \overset{\circ}{\mathbf{e}}{}^{*} = \dot{\mathbf{e}} + \mathbf{e}\boldsymbol{\Omega}^{*} - \boldsymbol{\Omega}^{*}\mathbf{e} = \mathbf{D} \tag{6.3}$$

was considered, and hence the aforementioned unique property of $\ln\mathbf{V}$ was not realized. The aforementioned intrinsic, unique property of the Hencky's Eulerian strain measure $\ln\mathbf{V}$ has proved to be far-reaching and found applications in constitutive modeling (see REINHARDT and DUBEY [41], DUBEY and REINHARDT [7], BRUHNS, XIAO and MEYERS [62], XIAO, BRUHNS and MEYERS [58 – 59, 63]).

On the other hand, the corresponding question concerning the Lagrangean stretching tensor $\hat{\mathbf{D}}$ and the Lagrangean Kirchhoff stress measure $\hat{\boldsymbol{\sigma}}$ have been discussed recently by these authors (see XIAO, BRUHNS and MEYERS [61]). Here we supply a short alternative proof for the main results in the latter.

In fact, from the notion of objective corotational rate and the formula (3.5), the following fact follows immediately:

$$\stackrel{\circ}{\mathbf{E}}{}^{*} = \dot{\mathbf{E}} + \mathbf{E}(\hat{\mathbf{\Omega}}^{*} - \hat{\mathbf{\Omega}}^{\mathrm{R}}) - (\hat{\mathbf{\Omega}}^{*} - \hat{\mathbf{\Omega}}^{\mathrm{R}})\mathbf{E} = \hat{\mathbf{D}} \Longleftrightarrow \mathbf{E} = \ln \mathbf{U} \ \& \ \hat{\mathbf{\Omega}}^{*} = \hat{\mathbf{\Omega}}^{\log} . \tag{6.4}$$

Namely, an objective corotational rate of the Hencky's Lagrangean logarithmic strain measure $\ln \mathbf{U}$ is identical with the Lagrangean stretching tensor $\hat{\mathbf{D}}$, and furthermore that in all strain measures only $\ln \mathbf{U}$ enjoys this property. Thus, from (2.14) and (4.6) and

$$\hat{\mathbf{D}} = (\ln \stackrel{\circ}{\mathbf{U}})^{\log} \tag{6.5}$$

it follows that the pair $(\ln \mathbf{U}, \hat{\boldsymbol{\sigma}})$ is an $\hat{\mathbf{\Omega}}^{\log}$-work-conjugate Lagrangean strain-stress pair.

7. On Eulerian and Lagrangean formulations of rate-type constitutive relations

Let $\mathbf{E}$ be any given Lagrangean strain measure, defined by a scale function $g(\chi)$ (see (2.30)). The conjugate stress $\mathbf{T}^{R}$ of $\mathbf{E}$ in Hill's work-conjugacy sense (see (1.1) and (2.14)) is given by (5.8). Suppose the response of a material to incremental loading is rate-independent, i.e., either linear or piecewise linear. Following HILL [23], relative to a reference configuration we write down the Lagrangean rate-type constitutive relation

$$\dot{\mathbf{T}}^{R} = \hat{\mathcal{M}}^{R}[\dot{\mathbf{E}}] . \tag{7.1}$$

The fourth-order tensor of moduli, $\hat{\mathcal{M}}^{R}$, may depend on the stress and the deformation state, but not on $\dot{\mathbf{E}}$. In particular, $\hat{\mathcal{M}}^{R}$ depends on the choice of reference configuration and of measure, i.e., scale function $g(\chi)$. In HILL [23], certain significant properties of rate-type constitutive relation (7.1) are exploited by means of the class of generalized strain measures characterized by the scale function $g(\chi)$ as well as the work-conjugacy notion (1.1), such as the dependence of the moduli $\hat{\mathcal{M}}^{R}$ on the change of scale function $g(\chi)$, constitutive inequalities in terms of the scale function $g(\chi)$, the measure invariance, etc.

The unified work-conjugacy notion introduced enables us to broaden the scope of the foregoing study. Indeed, let $\mathbf{\Omega}^{*} \neq \mathbf{\Omega}^{L}$ be a material spin of the form (3.9), and $\mathbf{e}$ – any given Eulerian strain measure with the scale function $g(\chi)$. Then, the $\mathbf{\Omega}^{*}$-work-conjugate stress $\mathbf{t}$ of $\mathbf{e}$ is given by $(5.6)_2$ and (5.2). We propose the following Eulerian rate-type constitutive relation for a rate-independent elastoplastic material:

$$\stackrel{\circ}{\mathbf{t}}{}^{*} = \mathcal{M}^{*}[\stackrel{\circ}{\mathbf{e}}{}^{*}] , \tag{7.2}$$

where $\stackrel{\circ}{\mathbf{t}}{}^{*}$ and $\stackrel{\circ}{\mathbf{e}}{}^{*}$ are the objective corotational stress and strain rates given by (3.1) with $\mathbf{G} = \mathbf{e}$, $\mathbf{t}$ and (3.11). The corresponding Lagrangean formulation of the above relation can be obtained by using the rotated correspondence relation (2.5)–(2.6). We have

$$\overset{\circ}{\mathbf{T}}{}^{*} = \hat{\mathcal{M}}^{*}[\overset{\circ}{\mathbf{E}}{}^{*}], \tag{7.3}$$

where the Lagrangean strain-stress pair $(\mathbf{E}, \mathbf{T})$ is the counterpart of the Eulerian strain-stress pair $(\mathbf{e}, \mathbf{t})$ via the rotated correspondence relation (see (2.5)–(2.6)); the objective corotational rates $\overset{\circ}{\mathbf{T}}{}^{*}$ and $\overset{\circ}{\mathbf{E}}{}^{*}$ are the Lagrangean counterparts of $\overset{\circ}{\mathbf{t}}{}^{*}$ and $\overset{\circ}{\mathbf{e}}{}^{*}$ (see (3.5)); and finally

$$\hat{\mathcal{M}}^{*} = \mathbf{R} \star \mathcal{M}^{*}; \quad (\mathbf{R} \star \mathcal{M}^{*})_{ijkl} = \mathbf{R}_{ip}\mathbf{R}_{jq}\mathbf{R}_{kr}\mathbf{R}_{ls}\mathcal{M}^{*}_{pqrs}\,. \tag{7.4}$$

The Eulerian and Lagrangean fourth order tensors of moduli, $\mathcal{M}^{*}$ and $\hat{\mathcal{M}}^{*}$, may depend on the stress and the deformation state, as well as certain internal variables[(4)] characterizing the internal state of material, etc. In particular, either of them depends on both the choice of strain measure and the choice of spin tensor, i.e. both the choice of scale function $g(\chi)$ and the choice of spin function $h(x, y)$. It should be pointed out that the just-mentioned double choices are arbitrary and independent of each other. Thus, the proposed Eulerian and Lagrangean formulations of rate-type constitutive models broaden the usual Lagrangean formulation with the material time rate and Eulerian formulations with several known objective corotational rates. The former allow for the double choices of scale function $g(\chi)$ and spin function $h(x, y)$, while the latter are concerned merely with the choice of scale function $g(\chi)$. In fact, the usual Lagrangean formulation (7.1) is incorporated into a particular case of the proposed, more general Lagrangean formulation (7.3) when $h(x, y) = \tilde{h}^{R}(y/x)$ (see (3.14)), i.e. $\mathbf{\Omega}^{*} = \mathbf{\Omega}^{R}$.

Since the stretching $\mathbf{D}$ is a simple, natural measure for the rate-of-change of deformation state, it is hopeful that the strain rate measure $\overset{\circ}{\mathbf{e}}{}^{*}$ is replaced by $\mathbf{D}$, as is most often done. This results in the noticeable fact (see XIAO, BRUHNS and MEYERS [58] or last section): the strain measure $\mathbf{e}$ must be the logarithmic strain measure $\ln \mathbf{V}$, the stress $\mathbf{t}$ must be the Kirchhoff stress $\boldsymbol{\sigma}$ and the spin $\mathbf{\Omega}^{*}$ must be the logarithmic spin $\mathbf{\Omega}^{\log}$. Such uniqueness yields the following formulations based on the logarithmic rate:

$$\overset{\circ}{\boldsymbol{\sigma}}{}^{\log} = \mathcal{M}^{\log}[\mathbf{D}]\,, \tag{7.5}$$

$$\overset{\circ}{\hat{\boldsymbol{\sigma}}}{}^{\log} = \hat{\mathcal{M}}^{\log}[\hat{\mathbf{D}}]\,. \tag{7.6}$$

For hypoelasticity and finit deformation elastoplasticity etc., further study shows (see XIAO, BRUHNS and MEYERS [58 – 59, 63] and BRUHNS, XIAO and MEYERS [62]) that the above formulations possess certain unique, far-reaching properties.

(4) For the sake of simplicity, the evolution equations of internal variables are not discussed here.

In general, by virtue of objective corotational stress rates and the unified work-conjugacy notion introduced, the structure and property of rate-type constitutive relations may be further exploited, such as the dependence of the moduli on both the scale function $g(\chi)$ and the spin function $h(x, y)$, constitutive inequalities in terms of both the scale function $g(\chi)$ and the spin function $h(x, y)$, the broader invariance relative to both strain measure and strain rate measure, etc. This line of investigation will be pursued elsewhere.

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References

1. D.E. CARLSON, and A. HOGER, *The derivative of a tensor-valued function of a tensor*, Q. Appl. Math., 44, 409–423, 1986.
2. Y.C. CHEN and L. WHEELER, *Derivatives of the stretch and rotation tensors*, J. Elasticity, 32, 175–185, 1993.
3. J.K. DIENES, *On the analysis of rotation and stress rate in deforming bodies*, Acta Mechanica, 32, 217–232, 1979.
4. J.K. DIENES, *A discussion of material rotation and stress rate*, Acta Mechanica, 65, 1–11, 1987.
5. T.C. DOYLE and J.L. ERICKSEN, *Nonlinear elasticity*, Advances in Appl. Mech., 4, 53–115, 1956.
6. R.N. DUBEY, *Choice of tensor-rates – a methodology*, SM Archives, 12, 233–244, 1987.
7. R.N. DUBEY and W.D. REINHARDT, *Large deformation description of rate-dependent plasticity*, [in:] Plasticity and Impact Mechanics, N.K. GUPTA [Ed.], 79–99, New Age Intern. Ltd. Publ., New Delhi etc. 1996.
8. J.E. FITZGERALD, *A tensorial Hencky measure of strain and strain rate for finite deformation*, J. Appl. Phy., 51, 5111–5115, 1980.
9. A.E. GREEN and P.M. NAGHDI, *A general theory of an elastic-plastic continuum*, Arch. Rat. Mech. Anal., 18, 251–281, 1965.
10. Z.H. GUO, *Time derivatives of tensor fields in nonlinear continuum mechanics*, Arch. Mech., **15**, 131–163, 1963.
11. Z.H. GUO, *Rates of stretch tensors*, J. Elasticity, 14, 263–267, 1984.
12. Z.H. GUO and R.N. DUBEY, *Basic aspects of Hill's method in solid mechanics*, SM Archives, 9, 353–380, 1984.
13. Z.H. GUO, TH. LEHMANN, and H.Y. LIANG, *Further remarks on rates of stretch tensors*, Trans. CSME, 15, 161–172, 1991.
14. Z.H. GUO, Th. LEHMANN, H.Y. LIANG, and C.S. MAN, *Twirl tensors and the tensor equation* $\mathbf{AX} - \mathbf{XA} = \mathbf{C}$, J. Elasticity, 27, 227–242, 1992.

15. Z.H. GUO and C.S. MAN, *Conjugate stresses and the tensor equation* $\sum_{r=1}^{m} \mathbf{U}^{m-r}\mathbf{X}\mathbf{U}^{r-1} = \mathbf{C}$, Int. J. Solids Struct., 29, 2063–2076, 1992.

16. M.E. GURTIN, *An introduction to continuum mechanics*, Academic Press, New York 1981.

17. M.E. GURTIN and K. SPEAR, *On the relationship between the logarithmic strain rate and the stretching tensor*, Int. J. Solids Struct., 19, 437–444, 1983.

18. P. HAUPT and CH. TSAKMAKIS, *On the application of dual variables in continuum mechanics*, Continuum Mech. Thermodyn., 1, 165–196, 1989.

19. P. HAUPT and CH. TSAKMAKIS, *Stress tensors associated with deformation tensors via duality*, Arch. Mech., 48, 347–384, 1996.

20. H. HENCKY, *Über die Form des Elastizitätsgesetzes bei ideal elastischen Stoffen*, Z. Techn. Phys., 9, 214–247, 1928.

21. R. HILL, *On constitutive inequalitites for simple materials-I*, J. Mech. Phys. Solids, 16, 229–242, 1968.

22. R. HILL, *Constitutive inequalities for isotropic elastic solids under finite strain*, Proc. R. Soc. London, A326, 131–147, 1970.

23. R. HILL, *Aspects of invariance in solid mechanics*, Advances in Appl. Mech., 18, 1–75, 1978.

24. A. HOGER and D.E. CARLSON, *On the derivative of the square root of a tensor and Guo's rate theorem*, J. Elasticity, 14, 329–336, 1984.

25. A. HOGER, *The material time derivative of logarithmic strain tensor*, Int. J. Solids Struct., 22, 1019–1032, 1986.

26. A. HOGER, *The stress conjugate to logarithmic strain*, Int. J. Solids Struct., 23, 1645–1656, 1987.

27. J.W. HUTCHINSON and K.W. NEALE, *Finite deformation J_2 deformation theory*, [in:] Finite Elasticity, D.E. CARLSON and R.T. SHIELD [Eds.], Martinus Nijhoff, The Hague 1982.

28. TH. LEHMANN, *On the balance of energy and entropy at inelastic deformations of solid bodies*, Eur. J. Mech. A/Solids, 8, 235–251, 1989.

29. TH. LEHMANN, Z.H. GUO and H.Y. LIANG, *The conjugacy between Kirchhoff stress and logarithm of the left stretch tensor*, Eur. J. Mech., A/Solids, 10, 395–404, 1991.

30. TH. LEHMANN and H.Y. LIANG, *The stress conjugate to logarithmic strain* $\ln \mathbf{V}$, Zeits. Angew. Math. Mech., 73 , 357–363, 1993.

31. E.H. MACMILLAN, *On the spin of tensors*, J. Elasticity, 27, 69–84, 1992.

32. D.B. MACVEAN, *Die Elementararbeit in einem Kontinuum und die Zuordnung von Spannungs- und Verzerrungstensoren*, ZAMP, 19, 157–185, 1968.

33. C.S. MAN and Z.H. GUO, *A basis-free formula for time rate of Hill's strain tensors*, Int. J. Solids Struct., 30, 2819–2842, 1993.

34. J.E. MARSDEN and T.J.R. HUGHES, *Mathematical foundations of elasticity*, Prentice-Hall, Englewood Cliffs, New Jersey 1983.

35. M.M. MEHRABADI and S. NEMAT-NASSER, *Some basic kinematical relations for finite deformations of continua*, Mech. Mater., 6, 127–138, 1987.

36. S. NEMAT-NASSER, *On finite deformation elasto-plasticity*, Int. J. Solids Struct., 18, 857–872, 1982.

37. R.W. OGDEN, *On Eulerian and Lagrangean objectivity in continuum mechanics*, Arch. Mech., 36, 207–218, 1984.

38. R.W. OGDEN, *Non-linear elastic deformations*, Ellis Horwood, Chichester 1984.

39. L. PALGEN and D.C. DRUCKER, *The structure of stress-strain relations in finite elasto-plasticity*, Int. J. Solids Struct., 19, 519–531, 1983.

40. W.D. REINHARDT and R.N. DUBEY, *Coordinate-independent representation of spins in continuum mechanics*, J. Elasticity, 42, 133–144, 1996.

41. W.D. REINHARDT and R.N. DUBEY, *Application of objective rates in mechanical modelling of solids*, ASME J. Appl. Mech., 63, 692–698, 1996.

42. J.R. RICE, *Continuum mechanics and thermodynamics of plasticity in relation to micro-scale deformation mechanisms*, [in:] Constitutive Equations in Plasticity, A.S. ARGON [Ed.], 23–75, MIT Press, Cambridge, Mass. 1975.

43. K. SAWYERS, *Comments on the paper determination of the stretch and rotation in the polar decomposition of the deformation gradient*, Q. Appl. Math., 44, 309–311, 1986.

44. M. SCHEIDLER, *Time rates of generalized strain tensors. Part I: Component formulas*, Mech. Mater., 11, 199–210, 1991.

45. M. SCHEIDLER, *Time rates of generalized strain tensors. Part II: Approximate basis-free formulas*, Mech. Mater., 11, 211–219, 1991.

46. M. SCHEIDLER, *Time rates of generalized strain tensors with applications to elasticity*, [in:] Proc. 12th Army Symp. on Solid Mechanics, S.C. CHOU [Ed.], 59–71, Plymouth. 1992.

47. M. SCHEIDLER, *The tensor equation* $\mathrm{AX}+\mathrm{XA}=\Phi(\mathrm{A},\mathrm{H})$, *with applications to kinematics of continua*, J. Elasticity, 36, 117–153, 1994.

48. B.R. SETH, *Generalized strain measures with applications to physical problems*, [in:] Second-Order Effects in Elasticity, Plasticity and Fluid Dynamics, M. REINER and D. ABIR [Eds.], 162–172, Pergamon Press, Oxford 1964.

49. J.C. SIMO and J.E. MARSDEN, *On the rotated stress tensor and the material version of the Doyle-Ericksen formula*, Arch. Rat. Mech. Anal., 86, 213–231, 1984.

50. J. STICKFORTH and K. WEGENER, *A note on Dienes' and Aifantis' co-rotational derivatives*, Acta Mechanica, 74, 227–234, 1988.

51. B. SVENDSEN and CH. TSAKMAKIS, *A local differential geometric formulation of dual stress-strain variables and time derivatives*, Arch. Mech., 46, 49–91, 1994.

52. C. TRUESDELL and R. TOUPIN, *The classical field theories*, [in:] Handbuch der Physik, S. Flügge [Ed.], Vol. III/1, 226–858, Springer-Verlag, Berlin, Göttingen, Heidelberg 1960.

53. C.C. WANG and C. TRUESDELL, *Introduction to rational elasticity*, Noordhoff, Leyden 1973.

54. W.B. WANG and Z.P. DUAN, *On the invariant representation of spin tensors with applications*, Int. J. Solids Struct., 27, 329–341, 1991.

55. L. WHEELER, *On the derivatives of the stretch and rotation with respect to the deformation gradient*, J. Elasticity, 24, 129–133, 1990.

56. H. XIAO, *Unified explicit basis-free expressions for time rate and conjugate stress of an arbitrary Hill's strain*, Int. J. Solids Struct., 32, 3327–3340, 1995.

57. H. XIAO, O.T. BRUHNS and A. MEYERS, *A new aspect in the kinematics of large deformations*, [in:] Plasticity and Impact Mechanics, N.K. GUPTA [Ed.], 100–109, New Age Intern. Ltd. Publ., New Delhi etc. 1996.

58. H. XIAO, O.T. BRUHNS and A. MEYERS, *Logarithmic strain, logarithmic spin and logarithmic rate,* Acta Mechanica, 124, 89–105, 1997.

59. H. XIIAO, O.T. BRUHNS and A. MEYERS, *Hypo-elasticity model based upon the logarithmic stress rate,* J. Elasticity, 47, 51–68, 1997.

60. H. XIAO, O.T. BRUHNS and A. MEYERS, *On objective corotational rates and their defining spin tensors,* Int. J. Solids Struct., **35**, 4001–4014, 1998.

61. H. XIAO, O.T. BRUHNS and A. MEYERS, *Direct relationship between the Lagrangean logarithmic strain and the Lagrangean stretching and the Lagrangean Kirchhoff stress,* Mech. Res. Commun., 25, 59–67, 1998.

62. O.T. BRUHNS, H. XIAO and A. MEYERS, *Self-consistent Eulerian rate type elasto-plastic models based upon the logarithmic stress rate,* Int. J. Plasticity [in press].

63. H. XIAO, O.T. BRUHNS and A. MEYERS, *On existence and uniqueness of the integrable-exactly hypoelastic equation* $\overset{\circ}{\boldsymbol{\sigma}}{}^{*} = \lambda(\mathrm{tr}\mathbf{D})\mathbf{I} + 2\mu\mathbf{D}$ *and its significance to self-consistent rate-type formulations of finite inelasticity models,* Acta Mech. [in press].

64. H. ZIEGLER and D.B. MACVEAN, *On the notion of an elastic solid,* [in:] Recent Progress in Applied Mechanics. The Folke Odquist Volume, B. BROBERG, J. HULT and F. NIORDSEN [Eds.], 561–572, Almquist and Wiksell, Stockholm 1967.

65. H. XIAO, O.T. BRUHNS and A. MEYERS, *Strain rates and material spins,* J. Elasticity, **52**, 1–41, 1998.

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