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FIFTY YEARS OF THE ARCHIVES OF MECHANICS

On waves due to a line source in front of a vertical wall with a gap

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IN THE PRESENT PAPER waves due to presence of a line source in front of a vertical wall with a gap are studied. A simple expression for amplitude of radiated waves at infinity is obtained by application of Green's integral theorem.

1. Introduction

WATER WAVE PROPAGATION in presence of a vertical barrier form an important class of problems within the framework of linearised theory. Among the various types of problems in this class, the study of wave motion due to presence of line source in front of an obstacle has been made by various researchers.

EVANS [2], while studying the wave motion produced by small oscillations of a partially immersed vertical plate, obtained as a special case the amplitude of radiated waves due to presence of a line source in front of a vertical plate partially immersed in deep water by simple application of Green's integral theorem. Later BASU and MANDAL [3] and MANDAL [4] used the same technique to find the amplitude of radiated waves when the vertical barrier is completely submerged and extends infinitely downwards, or is submerged up to a finite depth below the mean free surface.

In the present paper, the wave motion due to a line source present in front of a vertical wall with a gap in deep water is studied. These problems have relevance in manoeuvring of a ship near a wall (cf. [7]). In general, a study of wave motion in presence of a vertical wall with a gap has practical application in construction of breakwaters. Here the amplitude of radiated waves at infinity is obtained by applying Green's integral theorem in the fluid region to two suitably chosen functions. One of the functions represents the velocity potential which is the solution of the corresponding problem of scattering of a normally incident wave train by a vertical wall with a gap. This solution is given in [6]. However, we have obtained it here by a different method using an integral equation formulation based on Havelock's expansion of the water wave potential. The other function is chosen in appropriate form, the unknown velocity potential describing the motion in the given problem. From the results thus obtained, it is observed that, when the source is situated within the gap in the wall, then the wall has no effect on the source.

2. Statement and formulation of the problem

We consider a vertical wall extending from above the mean free surface and having a gap given by $x = 0$ and $y \in L \equiv (0, a) \cup (b, \infty)$ in deep water occupying the region $y \geq 0$ with $y = 0$ as the mean free surface (cf. Fig. 1). The motion is generated in water due to a harmonically oscillating line source of unit strength and circular frequency σ, acting at the point (ξ, η), $(\xi > 0, \eta > 0)$ in front of the wall.

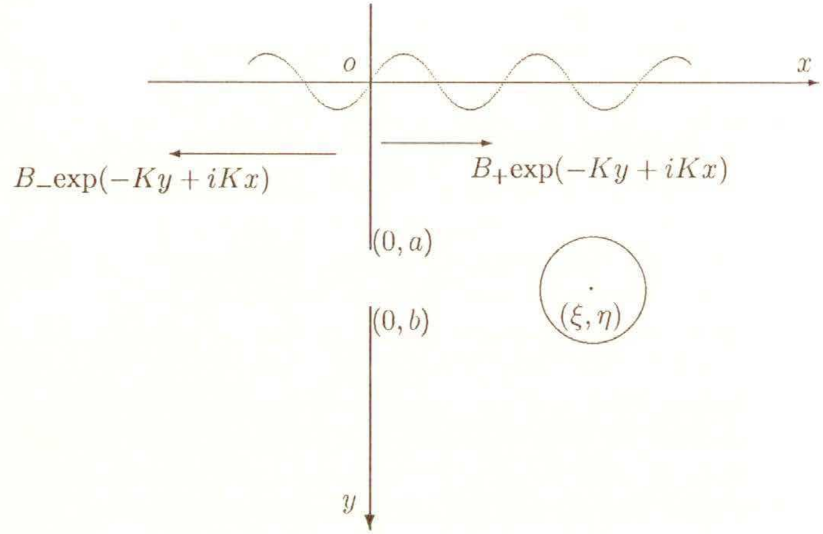

FIG. 1.

Assuming the linearised theory, the motion is described by the velocity potential $\text{Re}\{\Phi(x, y)\exp(-i\sigma t)\}$ where Φ satisfies the following boundary value problem:

$$\nabla^2\Phi = 0 \ \text{ in the fluid region except at } \ (\xi, \eta), \tag{2.1}$$

$$K\Phi + \Phi_y = 0 \ \text{ on } \ y = 0, \tag{2.2}$$

where $K = \sigma^2/g$, g being acceleration of gravity,

$$\Phi_x = 0, \qquad x = 0, \qquad y \in L, \tag{2.3}$$

$$\Phi \sim \ln\rho \ \text{ as } \rho \to 0 \qquad \text{where} \qquad \rho = \{(x - \xi)^2 + (y - \eta)^2\}^{1/2}, \tag{2.4}$$

$$r^{1/2}\nabla\Phi \ \text{ is bounded as } r \to 0, \qquad r = \{(x)^2 + (y - c)^2\}^{1/2}, \quad c = a \text{ or } b \tag{2.5}$$

$$\nabla\Phi \to 0 \ \text{ as } y \to \infty, \tag{2.6}$$

$$\Phi \sim \begin{cases} B_+ \quad \exp(-Ky + iKx) & \text{as } x \to \infty, \\ B_- \quad \exp(-Ky - iKx) & \text{as } x \to -\infty, \end{cases} \tag{2.7}$$

where $B_\pm$ (unknown) are (complex) amplitudes of radiated waves at infinity on either side of the wall. Let $G(x, y; \xi, \eta)$ denote the potential due to a line source of unit strength at $(\xi, \eta), (\eta > 0)$ in the absence of the barrier which is given by (cf. [1]),

$$G(x, y\,; \xi, \eta) = -2 \int_0^\infty \frac{M(k, \eta) M(k, y)}{k(K^2 + k^2)} \exp(-k \mid x - \xi \mid) dk \\ - 2\pi i \exp(-K(y + \eta) + iK \mid x - \xi \mid), \tag{2.8}$$

where $M(k, \eta) = k \cos k\eta - K \sin k\eta$.

We express the potential function Φ as

$$\Phi = G + \phi, \tag{2.9}$$

where ϕ is the correction of G due to the presence of the barrier. Then ϕ satisfies the equations:

$$\nabla^2 \phi = 0, \qquad y > 0, \tag{2.10}$$

$$K\phi + \phi_y = 0 \qquad \text{on} \qquad y = 0, \tag{2.11}$$

$$\phi_x(0, y) = f(y) = -G_x(0, y; \xi, \eta), \qquad x = 0, \quad y \in L \equiv (0, a) \cup (b, \infty), \tag{2.12}$$

$$r^{1/2} \nabla \phi \ \text{ is bounded as } \ r \to 0, \tag{2.13}$$

$$\nabla \phi \to 0, \qquad \text{as} \quad y \to \infty, \tag{2.14}$$

$$\phi \sim \begin{cases} B \quad \exp(-Ky + iKx), & x \to \infty \\ -B \quad \exp(-Ky - iKx), & x \to -\infty \end{cases} \tag{2.15}$$

where B (unknown) is the complex amplitude of scattered field. It may be noted here that because of (2.12), ϕ is odd in x.

3. Method of solution

Let $\psi(x, y)$ denote the potential describing the motion due to normal incidence of a progressive wave $\exp(-Ky + iKx)$ from negative infinity upon the vertical wall $x = 0$, $y \in L \equiv (0,a) \cup (b,\infty)$ present in deep water. The explicit form for $\psi(x,y)$ can be obtained as (see Appendix and also [6]):

$$
\psi(x,y) = \begin{cases} \exp(-Ky + iKx) + R\,\exp(-Ky - iKx) \\ \qquad + \int\limits_0^\infty D(k)M(k,y)\exp(kx)dk, \quad x < 0 \\ T\exp(-Ky + iKx) \\ \qquad + \int\limits_0^\infty C(k)\; M(k,y)\; \exp(-kx)dk, \quad x > 0, \end{cases} \tag{3.1}
$$

where $M(k,y)$ is given by (2.8)

$$
\begin{aligned}
R &= A_1 I = \frac{Ii}{J + iI}, \\
T &= 1 - R = -iJA_1 = \frac{J}{(J + Ii)}, \\
A_1 &= \frac{i}{(J + Ii)}, \\
J &= \frac{\exp(-Ka)}{K} + \delta\alpha_2(K) - \frac{2\alpha_2(K,F_1)}{\pi}, \qquad (3.2)\\
I &= \delta\{\alpha_1(K) - \alpha_3(K)\} - \frac{2}{\pi}\{\alpha_1(K,F_1) - \alpha_3(K,F_1)\}, \\
\delta &= \frac{\{K^{-1}\;\exp(Ka) + \frac{2}{\pi}\alpha_2(-K,F_1)\}}{\alpha_2(-K)},
\end{aligned}
$$

$$
\alpha_i(K) \equiv \alpha_i(K,1), \qquad \alpha_i(K,F_1) = \int\limits_{t_i} \frac{uF_1(a,b,u)}{R_0(u)}\exp(-Ku)du,
$$

where $R_0(u) = | u^2 - a^2 |^{1/2} \;\; | u^2 - b^2 |^{1/2}$

$$
t_i = \begin{cases} (-a,a), & i = 1, \\ (a,b), & i = 2, \\ (b,\infty), & i = 3, \end{cases}
$$

$$
F_1(a,b,u) = \int\limits_0^a \frac{R_0(v)}{v^2 - u^2}dv.
$$

Therefore,

$$-C(k) = D(k) = \frac{2}{\pi}\frac{A_1}{k(k^2+K^2)}\left[-\sin ka + k\int_a^b \frac{uS(u)}{R_0(u)}\cos ku du\right]$$

$$s(u) = \left[\delta - \frac{2}{\pi}F_1(a,b,u)\right].$$

Applying Green's integral theorem to the harmonic functions ϕ, ψ within the region bounded by the lines

$$y = 0, \quad 0 < x \le X; \quad x = 0^+, \quad 0 \le y < a; \quad x = 0^-, \quad 0 \le y < a;$$

$$y = 0, \quad -X \le x < 0; \quad x = -X, \quad 0 \le y \le Y; \quad y = Y, \quad -X \le x < 0;$$

$$x = 0^-, \quad b < y < \infty; \quad x = 0^+, \quad b < y < \infty; \quad y = Y, \quad 0 \le x \le X;$$

$$x = X, \quad 0 \le y \le Y;$$

for $X,\ Y \to \infty$ we obtain

$$iB = \int_0^a g(y)f(y)dy + \int_b^\infty g(y)f(y)dy \tag{3.3}$$

where

$$g(y) = \psi(0^+, \psi(0^+, y), y) - \psi(0^-, y).$$

Using the expression for $g(y)$ from (B.9), the following simplifications can be made.

$$\int_0^a f(y)g(y)dy = -\int_0^a \frac{2y\ \exp(Ky)}{R_0(y)}s(y)h_1(y)dy,$$

(3.4)

$$\int_b^\infty f(y)g(y)dy = \int_b^\infty \frac{2ys(y)h_2(y)}{R_0(y)}\exp(Ky)dy,$$

where

$$h_1(y) = A_1\int_0^y f(t)\exp(-Kt)dt,$$

$$h_2(y) = A_1\int_\infty^y f(t)\exp(-Kt)dt,$$

$s(y)$ is given by (3.2) and $f(y)$ can be obtained from (2.12) and (2.8) as

$$f(y) = -G_x(0, y;\ \xi, \eta) = 2 \int_0^\infty \frac{M(k,y)M(k,\eta)}{k^2+K^2}\ \exp(-k\xi)dk$$

$$+2\pi K\ \exp(-(y+\eta)+iK\xi).$$

Thus, using (3.4) and (B.9) in (3.3), we get B in the form

$$B = -2\pi i \left[R\ \exp(-K\eta + iK\xi) - \int_0^\infty C(k)M(k,\eta)\exp(-k\xi)dk \right] \tag{3.5}$$

where R and $C(k)$ are given by Eqs. (3.2).

Now $B_\pm$ can be obtained by assuming $|\ x\ | \to \infty$ in (2.9) after using (2.7), (2.8), (2.15).
Thus as $x \to \infty$, we have

$$B_+ = -2\pi i\ \exp(-K\eta - iK\xi) + B = -2\pi i\psi(-\xi, \eta). \tag{3.6}$$

Also as $x \to -\infty$,

$$B_- = -B - 2\pi i\ \exp(-K\eta + iK\xi) = -2\pi i\psi(\xi, \eta). \tag{3.7}$$

It is obvious that

$$B_+ + B_- = -4\pi i\ \exp(-K\eta)\ \cos K\xi. \tag{3.8}$$

This shows that if $K\xi$ is an odd multiple of $\pi/2$ and $K\eta$ is arbitrary, then the wave amplitudes at either infinity are the same, the surface elevation being exactly 180° out of phase with each other. Similar conclusion were also drawn by EVANS [2] and BASU and MANDAL [3].

Again,

$$\psi(0, \eta) = T\ \exp(-K\eta) + \int_0^\infty C(k)M(k,\eta)dk.$$

Using $(\mathrm{B}.4)_1$ we have for $\eta \in (a, b)$

$$\psi(o, \eta) = (T + R)\ \exp(-K\eta)$$

and immediately it follows from (B.3) that

$$\psi(0, \eta) = \exp(-K\eta). \tag{3.9}$$

Therefore,

$$B_+(0,\eta) = B_-(0,\eta) = -2\pi i \exp(-K\eta). \tag{3.10}$$

This shows that the wall has no effect on the source if the source is situated within the gap in the wall.

4. Appendix

Let us consider a wall $x = 0, y \in L,\ L \equiv (0,a) \cup (b,\infty)$ immersed in deep water with $y = 0$ as a mean free surface. A train of surface waves $\exp(-Ky+iKx)$ of frequency σ is incident on the wall from negative infinity, then it is partially reflected and partially transmitted. If $Re\{\psi(x,y)\ \exp(-i\sigma t)\}$ denotes the velocity potential, then ψ satisfies the following boundary value problem:

(i) $\nabla^2\psi = 0, \qquad y \geq 0,$

(ii) $K\psi + \psi_y = 0 \quad \text{on} \quad y = 0,$

(iii) $\psi_x = 0, \qquad y \in L \equiv (0,a) \cup (b,\infty),$

(iv) $r^{1/2}\nabla\psi$ is bounded as $r \to 0,$

r being the distance from the sharp edges of the plate,

(v) $\nabla\psi \to 0 \quad \text{as} \quad y \to \infty,$

(vi) $\psi = \begin{cases} \exp(-Ky + iKx) + R\exp(-Ky - iKx), & \text{as} \quad x \to -\infty, \\ T\ \exp(-Ky + iKx), \quad x \to \infty, \end{cases}$

where R and T are reflection and transmision co-efficients, respectively, to be determined. Using Havelock's expansion of water wave potential, $\psi(x,y)$ can be expressed by

$$\psi(x,y) = \begin{cases} \exp(-Ky \quad +iKx) + R\exp(-Ky - iKx) \\ \qquad + \displaystyle\int_0^\infty D(k)M(k,y)\ \exp(kx)dk, \quad x < 0 \\ T\exp(-Ky \ +iKx) \\ \qquad + \displaystyle\int_0^\infty C(k)M(k,y)\exp(-kx)dk, \quad x > 0 \end{cases} \tag{B.1}$$

where $M(k,y)$ is given by (2.8), and $C(k)$ and $D(k)$ are unknown.

Let

$$\psi(0,y) = \begin{cases} 0, & y \in L \\ F(y), & y \in (0,\infty) - L, \end{cases} \tag{B.2}$$

where by (iv)

$$F(y) = \begin{cases} O(\mid y-a\mid^{-1/2}) & y \to a, \\ O(\mid y-b\mid^{-1/2}) & \text{as} \quad y \to b. \end{cases}$$

Then by Havelock's inversion theorem,

$$\text{(B.3)} \qquad \begin{aligned} T = 1 - R &= -2i \int\limits_a^b F(y) \exp(-Ky) dy, \\ -C(k) = D(k) &= \frac{2}{\pi} \frac{1}{k(K^2+k^2)} \int\limits_a^b F(y) M(k,y) dy, \end{aligned}$$

Now an integral equation for $F(y)$ can be obtained from the fact that $\psi(x, y)$ is continuous across the gap in the wall. Thus,

$$\psi(+0, y) = \psi(-0, y), \qquad y \in (a, b).$$

Using (B.1) and noting (B.3) we have,

$$\text{(B.4)}_1 \qquad R \exp(-Ky) = \int\limits_0^\infty M(k,y) C(k)\, dk, \qquad y \in (a, b).$$

Substituting $C(k)$ from (B.3) we get

$$\text{(B.4)}_2$$

$$-\frac{\pi}{2} R \exp(-Ky) = \int\limits_0^\infty \frac{M(k,y)}{k(k^2+K^2)} \int\limits_a^b F(t) M(k,t) dt\, dk, \quad y \in (a, b).$$

Applying the operator $(\dfrac{d}{dy} + K)$ to (B.3) we have the following integral equation:

$$\text{(B.5)} \qquad \int\limits_a^b F(t) \left[K \ln \left| \frac{y-t}{y+t} \right| + \frac{1}{y-t} + \frac{1}{y+t} \right] dt = 0, \qquad y \in (a, b).$$

The solution of integral equation (B.5) is given by (cf. [5])

$$\text{(B.6)} \qquad F(x) = \frac{d}{dx} \exp(-Kx) \int\limits_b^x \exp(Ku) \lambda(u) du$$

where

$$\lambda(u) = \frac{uA_1}{R_0(u)} \left[\delta - \frac{2}{\pi} F_1(a, b, u) \right],$$

$R_0(u)$, $F_1(a,b,u)$, δ and A_1 are given in (3.2). One relation connecting A_1 and R can be obtained by substituting $F(x)$ in $(\mathrm{B}.4)_2$. After some simplification we obtain

$$R = \delta A_1 \left[\alpha_1(K) - \alpha_3(K)\right] - \frac{2A_1}{\pi}[[\alpha_1(K,F_1) - \alpha_3(K,F_1)], \tag{B.7}$$

where $\alpha_i(K)$ and $\alpha_i(K,F_1)$ are given by (3.2).

Also substituting $F(t)$ in the first equation of (B.3), we get another relation connecting R, A_1 which is given by

$$1 - R = \left[-\delta\ \alpha_2(K) + \frac{2}{\pi}\ \ \alpha_2(K,F_1) - \frac{1}{K}\exp(-Ka)\right](i.A_1). \tag{B.8}$$

Thus from (B.7) and (B.8), R and A_1 can be obtained. Again, $C(k)$ is obtained by substituting $F(t)$ in the second equation of (B.3). After simplifications, $C(k)$ can be obtained as given in (3.2).

Let $g(y) = \psi(+0,y) - \psi(-0,y)$. Using (B.1) we get

$$g(y) = -2R\exp(-Ky) + 2\int_0^\infty C(k)M(k,y)dk.$$

Therefore,

$$Kg + g_y(y) = -2\int_0^\infty C(k)(K^2+k^2)\sin ky\ dk.$$

Substituting $C(k)$ from (B.3) and making simplification we have,

$$Kg(y) + g_y(y) = \begin{cases} 0, & a < y < b, \\ \dfrac{2y\ A_1\ S(y)}{R_0(y)}, & 0 < y < a, \\ -\dfrac{2y\ A_1\ S(y)}{R_0(y)}, & b < y < \infty, \end{cases}$$

which gives after integration

$$g(y) = \begin{cases} 0, & a < y < b, \\ \exp(-Ky)\displaystyle\int_a^y \frac{2t\ S(t)\ \exp(Kt)}{R_0(t)}dt, & 0 < y < a, \\ -\exp(-Ky)\displaystyle\int_b^y \frac{2t\ S(t)\ \exp(Kt)}{R_0(t)}dt, & b < y < \infty, \end{cases} \tag{B.9}$$

where the constant of integration can be chosen to be zero, and $s(y)$, A_1 is given by (3.4).

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