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FIFTY YEARS OF THE ARCHIVES OF MECHANICS

On elastic energy of structures under proportional loading

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THE PAPER CONCERNS the proportional loading of structures made of time-independent materials. It has been shown that the elastic energy can be a decreasing function of the load multiplier if unilateral constraints are introduced into an elastic-plastic structure. Results obtained in the work seem to be of importance for the theory of structures and may have some theoretical implications. An exhaustive example illustrates the theory.

1. Introduction

THE PRESENT PAPER CONCERNS the problems of energy in structural systems. The energy, being a scalar quantity, is a diagnostic measure of the current mechanical state of the system and is of importance for theoretical considerations.

The elastic energy of structures made of the elastic-perfectly plastic materials will be evaluated. The load is assumed to be proportional and the problem is to establish whether the elastic energy is a monotone function of the load multiplier or not. It seems that the answer is "yes", but there is no theorem concerning this question known to the author. However, the problem is not trivial in general cases of time-independent systems. A case will be shown when the elastic energy can decrease while the proportional load increases.

The distortion approach has been applied in our considerations. The essence of this approach consists in the observation that all deformations due to nonlinearity of the material and/or boundary conditions are caused by the presence of distortions imposed on the linear elastic structure. Distortions are defined as enforced deformations which are not kinematically admissible, in general. The concept of distortions was introduced in the last years of the 19th century and, among others, was used in the papers of V. VOLTERRA [1] and G. COLONNETTI [2]. The distortion approach allowed us to obtain many valuable results, particularly in the thermoelasticity and shakedown theory of elastic-plastic structures. Some information concerning this topic can be found in the monographs of W. NOWACKI [3] and J. A. KÖNIG [4].

All considerations presented herein are carried out in the framework of the kinematically linear theory. The FEM-oriented matrix description, worked out by G. MAIER [5] and his co-workers, is used.

The elastic energy will be estimated for elastic (E), elastic-perfectly plastic (EpP), slackened-elastic (SE) and slackened-elastic-perfectly plastic ($SEpP$)

structures. "Slackening" is a structural property, consisting in the presence of gaps (clearances) at structural joints. Thus, on the macro scale, the slackened structure behaviour exhibits the locking effects. Deformations of slackened systems are due to elastic ε_E, plastic ε_P and also concentrated clearance strains ε_L (i.e. relative displacements of members and connection elements). The plastic and clearance strains can be treated as distortions imposed on the linear elastic structure. It should be pointed out however that clearance strains are "load-dependent" distortions, because they can vary during the deformation processes. More details concerning the slackened systems can be found in [7, 8].

2. Mathematical description of elastic systems with distortions

Consider an linear elastic system subjected to external loads $\mathbf{p}$ and distortions ε_R. The elasticity coefficients are assumed to be constant and independent of distortions. A current mechanical state, independently of the deformation history, can then be described by the following system of matrix relations:

$$\begin{aligned} \mathbf{Cu} &= \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_E + \boldsymbol{\varepsilon}_R, \\ \mathbf{C}^T\boldsymbol{\sigma} &= \mathbf{p}, \\ \boldsymbol{\sigma} &= \mathbf{E}\boldsymbol{\varepsilon}_E. \end{aligned} \tag{2.1}$$

In Eqs. (2.1) $\mathbf{p}$, $\mathbf{u}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ denote the vectors of loads (or generalized loads), displacements (or generalized displacements), stresses (or generalized stresses) and strains (or generalized strains), respectively. All these state variables are consistent in the sense of the virtual work equation:

$$\mathbf{p}^T\mathbf{u} = \boldsymbol{\sigma}^T\boldsymbol{\varepsilon}, \tag{2.2}$$

where T denotes the transpose. $\mathbf{C}$ is the geometric compatibility matrix, which depends only on the geometry and boundary conditions of the system. $\mathbf{E}$ denotes the strictly positive definite, square and symmetric matrix of elasticity. Since the kinematically linear approach is used, the strain vector $\boldsymbol{\varepsilon}$ can be split into elastic $\boldsymbol{\varepsilon}_E$ and distortion $\boldsymbol{\varepsilon}_R$ parts.

From (2.1) the following matrix relations can be derived, [8]:

$$\begin{aligned} \mathbf{p} &= \mathbf{Ku} - \mathbf{C}^T\mathbf{E}\boldsymbol{\varepsilon}_R, & \mathbf{u}_e &= \mathbf{K}^{-1}\mathbf{p}, & \mathbf{u}_r &= \mathbf{K}^{-1}\mathbf{C}^T\mathbf{E}\boldsymbol{\varepsilon}_R, \\ \mathbf{u} &= \mathbf{u}_e + \mathbf{u}_r, & \boldsymbol{\sigma}_e &= \mathbf{ECK}^{-1}\mathbf{p}, & \boldsymbol{\sigma}_r &= \mathbf{Z}\boldsymbol{\varepsilon}_R, \\ \boldsymbol{\sigma} &= \boldsymbol{\sigma}_e + \boldsymbol{\sigma}_r, & \mathbf{K} &= \mathbf{C}^T\mathbf{EC}, & \mathbf{Z} &= \mathbf{ECK}^{-1}\mathbf{C}^T\mathbf{E} - \mathbf{E}. \end{aligned} \tag{2.3}$$

where $\mathbf{K}$ is the square, symmetric and strictly positive definite stiffness matrix. In Eqs. (2.3) subscript e relates to the linear elastic structure without distortions, subjected to load $\mathbf{p}$, and subscript r indicates all the quantities due to the presence of distortions.

The distortion influence matrix $\mathbf{Z}$ is square and symmetric. It is well-known that the same stress state can be induced by various distortions, but any difference between these distortions is kinematically admissible. Thus, the matrix $\mathbf{Z}$ has to be singular. It is easy to show that

$$\mathbf{ZC} \equiv \mathbf{0} \quad \text{and} \quad \mathbf{C}^T\mathbf{Z} \equiv \mathbf{0}. \tag{2.4}$$

From (2.4) we can formulate the following properties of distortions, namely:

• any kinematically admissible distortion field (i.e. $\boldsymbol{\varepsilon}_R = \mathbf{Cu}_r$) does not induce self-stresses $\boldsymbol{\sigma}_r$:

$$\boldsymbol{\sigma}_r = \mathbf{Z}\boldsymbol{\varepsilon}_R = \mathbf{ZCu}_r \equiv \mathbf{0}; \tag{2.5}$$

• the self-stresses due to the presence of distortions ($\boldsymbol{\sigma}_r = \mathbf{Z}\boldsymbol{\varepsilon}_R$) are in equilibrium with zero-valued external loads:

$$\mathbf{p}_r = \mathbf{C}^T\boldsymbol{\sigma}_r = \mathbf{C}^T\mathbf{Z}\boldsymbol{\varepsilon}_R \equiv \mathbf{0}. \tag{2.6}$$

Compute now the total elastic energy W_E of a load-free ($\mathbf{p} = \mathbf{0}$) elastic structure subjected to steady distortions $\boldsymbol{\varepsilon}_R$:

$$W_E = \frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{\varepsilon}_E = \frac{1}{2}\boldsymbol{\sigma}^T(\mathbf{Cu} - \boldsymbol{\varepsilon}_R) = \frac{1}{2}\mathbf{p}^T\mathbf{u} - \frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{\varepsilon}_R = -\frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{\varepsilon}_R. \tag{2.7}$$

The elastic energy is positive definite unless the distortions $\boldsymbol{\varepsilon}_R$ are kinematically admissible. Hence

$$\boldsymbol{\sigma}^T\boldsymbol{\varepsilon}_R = \boldsymbol{\varepsilon}_R^T\mathbf{Z}\boldsymbol{\varepsilon}_R \leq 0. \tag{2.7$'$}$$

From (2.7)′ it is clearly seen that matrix $\mathbf{Z}$ is negative semi-definite.

In order to avoid a possible confusion, it should be mentioned that the distortion description used herein corresponds to the standard approach which is slightly different from the Colonnetti's one where the total strain vector is divided into three parts (for details see [9]), namely

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_e^{(p)} + (\boldsymbol{\varepsilon}_e^{(R)} + \boldsymbol{\varepsilon}_R). \tag{2.8}$$

In Eq. (2.8) $\boldsymbol{\varepsilon}_e^{(p)}$ denotes the compatible strain vector due to the load vector $\mathbf{p}$ in bvarepe pure elastic structure, while $\boldsymbol{\varepsilon}_e^{(R)}$ is the elastic strain vector induced by the distortions $\boldsymbol{\varepsilon}_R$ in the absence of the load $\mathbf{p}$. Thus, the sum $\boldsymbol{\varepsilon}_e^{(p)} + \boldsymbol{\varepsilon}_R$ is kinematically admissible. Consequently, the relations between the standard and Colonnetti's descriptions take the form:

$$\begin{aligned} \boldsymbol{\varepsilon}_E &= \boldsymbol{\varepsilon}_e^{(p)} + \boldsymbol{\varepsilon}_e^{(R)}, \\ \boldsymbol{\sigma}_e &= \mathbf{E}\boldsymbol{\varepsilon}_e^{(p)} = \mathbf{ECu}_e = \mathbf{E}\boldsymbol{\varepsilon}_E - \boldsymbol{\sigma}_r, \\ \boldsymbol{\sigma}_r &= \mathbf{E}\boldsymbol{\varepsilon}_e^{(R)} = \mathbf{E}(\mathbf{Cu}_r - \boldsymbol{\varepsilon}_R) = \mathbf{E}\boldsymbol{\varepsilon}_E - \boldsymbol{\sigma}_e = \mathbf{Z}\boldsymbol{\varepsilon}_R. \end{aligned} \tag{2.9}$$

3. Bounds on the elastic energy

Assume that an elastic structure is subjected to two load and distortion systems $\mathbf{p}_1$, $\boldsymbol{\varepsilon}_{R1}$ and $\mathbf{p}_2$, $\boldsymbol{\varepsilon}_{R2}$, respectively. The difference of the elastic energies of both the systems can be expressed as

$$\Delta W_E = W_{E2} - W_{E1} = \frac{1}{2}\boldsymbol{\sigma}_2^T \boldsymbol{\varepsilon}_{E2} - \frac{1}{2}\boldsymbol{\sigma}_1^T \boldsymbol{\varepsilon}_{E1}. \tag{3.1}$$

Turning now to the general case of deformable systems we use the positive definiteness of the elasticity matrix $\mathbf{E}$ in order to formulate the following inequality:

$$(\boldsymbol{\varepsilon}_{E2} - \boldsymbol{\varepsilon}_{E1})^T \mathbf{E}(\boldsymbol{\varepsilon}_{E2} - \boldsymbol{\varepsilon}_{E1}) = (\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)^T(\boldsymbol{\varepsilon}_{E2} - \boldsymbol{\varepsilon}_{E1}) \geq 0, \tag{3.2}$$

where the equality sign occurs if both the elastic strain vectors are equal to each other. Inequality (3.2), using Eqs. (2.1), can be rewritten in the form

$$(\mathbf{p}_2 - \mathbf{p}_1)^T(\mathbf{u}_2 - \mathbf{u}_1) - (\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)^T(\boldsymbol{\varepsilon}_{R2} - \boldsymbol{\varepsilon}_{R1}) \geq 0. \tag{3.2$'$}$$

On the other hand, inequality (3.2) leads to

$$(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)^T \boldsymbol{\varepsilon}_{E1} \leq (\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)^T \boldsymbol{\varepsilon}_{E2}. \tag{3.2$''$}$$

It can be easily shown that if $a \leq b$ then $a \leq (a+b)/2 \leq b$. Using this result in inequality $(3.2)''$ we obtain

$$(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)^T \boldsymbol{\varepsilon}_{E1} \leq \frac{1}{2}(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)^T(\boldsymbol{\varepsilon}_{E2} + \boldsymbol{\varepsilon}_{E1}) \leq (\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)^T \boldsymbol{\varepsilon}_{E2}. \tag{3.3}$$

Since $\boldsymbol{\sigma}_1^T \boldsymbol{\varepsilon}_{E2} = \boldsymbol{\sigma}_2^T \boldsymbol{\varepsilon}_{E1}$, we can conclude that the intermediate term of (3.3) represents the difference between the elastic energies of two systems of loads and distortions, namely:

$$\begin{aligned} \frac{1}{2}(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)^T(\boldsymbol{\varepsilon}_{E1} + \boldsymbol{\varepsilon}_{E2}) &= \frac{1}{2}\boldsymbol{\sigma}_2^T \boldsymbol{\varepsilon}_{E2} - \frac{1}{2}\boldsymbol{\sigma}_1^T \boldsymbol{\varepsilon}_{E1} \\ &= W_{E2} - W_{E1} = \Delta W_E. \end{aligned} \tag{3.4}$$

Thus, Ineq. (3.3) takes the form

$$(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)^T \boldsymbol{\varepsilon}_{E1} \leq \Delta W_E \leq (\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)^T \boldsymbol{\varepsilon}_{E2}. \tag{3.4$'$}$$

The left-hand side of $(3.4)'$ can be modified as follows:

$$(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)^T \boldsymbol{\varepsilon}_{E1} = (\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)^T(\mathbf{C}\mathbf{u}_1 - \boldsymbol{\varepsilon}_{R1}) = (\mathbf{p}_2 - \mathbf{p}_1)^T \mathbf{u}_1 + (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)^T \boldsymbol{\varepsilon}_{R1},$$

or, using the reciprocal principle (cf. [8])

$$(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)^T \boldsymbol{\varepsilon}_{E1} = (\mathbf{u}_2 - \mathbf{u}_1)^T \mathbf{p}_1 + \boldsymbol{\sigma}_1^T(\boldsymbol{\varepsilon}_{R1} - \boldsymbol{\varepsilon}_{R2}).$$

Similar transformations of the right-hand side of $(3.4)'$ allow us to construct the following inequalities, [8]:

$$
\begin{aligned}
L_1 &\le \Delta W_E \le R_1, \\
L_2 &\le \Delta W_E \le R_2,
\end{aligned}
\tag{3.5}
$$

where

$$
\begin{aligned}
L_1 &= (\mathbf{p}_2 - \mathbf{p}_1)^T \mathbf{u}_1 + (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)^T \boldsymbol{\varepsilon}_{R1}, \\
R_1 &= (\mathbf{p}_2 - \mathbf{p}_1)^T \mathbf{u}_2 + (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)^T \boldsymbol{\varepsilon}_{R2}, \\
L_2 &= (\mathbf{u}_2 - \mathbf{u}_1)^T \mathbf{p}_1 + (\boldsymbol{\varepsilon}_{R1} - \boldsymbol{\varepsilon}_{R2})^T \boldsymbol{\sigma}_1, \\
R_2 &= (\mathbf{u}_2 - \mathbf{u}_1)^T \mathbf{p}_2 + (\boldsymbol{\varepsilon}_{R1} - \boldsymbol{\varepsilon}_{R2})^T \boldsymbol{\sigma}_2.
\end{aligned}
\tag{3.6}
$$

$$\Delta W_E = (L_1 + R_1)/2; \quad \Delta W_E = (L_2 + R_2)/2; \quad L_1 = L_2 \text{ and } R_1 = R_2.$$

The equality signs relate to the particular cases of kinematically admissible distortions which do not induce any additional stresses.

It should be pointed out that inequalities (3.5) hold true for any unspecified loading paths. These inequalities will be used to evaluate the elastic energy for various types of structures under proportional loads.

4. Elastic energy changes during proportional loading

4.1. Definitions and assumptions

The proportional loading can be defined as follows:

$$\mathbf{p} = \mu_0 \mathbf{p}_0, \tag{4.1}$$

where μ_0 is a positive definite scalar multiplier, and $\mathbf{p}_0$ denotes a reference load vector. Consider two levels of proportional loads $\mathbf{p}_1$ and $\mathbf{p}_2$, which are associated with two load multipliers μ_1 and μ_2, respectively. If $\mathbf{p}_1 = \mu_1 \mathbf{p}_0$ and $\mathbf{p}_2 = \mu_2 \mathbf{p}_0$ then for $\mu_2 > \mu_1 > 0$ we obtain:

$$\mathbf{p}_2 = \mu \mathbf{p}_1, \tag{4.2}$$

where $\mu = \mu_2/\mu_1 > 1$.

Since the problem is considered in the frame of kinematically linear theory, the total strain in general cases of $SEpP$ structures is a sum of individual partial strains. In particular, the distortion vector consists of clearance and plastic strains:

$$\boldsymbol{\varepsilon}_R = \boldsymbol{\varepsilon}_L + \boldsymbol{\varepsilon}_P. \tag{4.3}$$

Usually, during proportional loading of structures no local plastic unloading occurs. Such a behaviour corresponds to the path-independent (holonomic) model. Further considerations are restricted to this model.

If a $SEpP$ structure is subjected to proportional load $\mathbf{p}$, which induces clearance and plastic distortions, then the following inequality holds:

$$\mathbf{p}^T\mathbf{u} = \boldsymbol{\sigma}^T\boldsymbol{\varepsilon} = \boldsymbol{\sigma}^T(\boldsymbol{\varepsilon}_L + \boldsymbol{\varepsilon}_E + \boldsymbol{\varepsilon}_P) = \boldsymbol{\sigma}^T\boldsymbol{\varepsilon}_L + \boldsymbol{\sigma}^T\boldsymbol{\varepsilon}_E + \boldsymbol{\sigma}^T\boldsymbol{\varepsilon}_P > 0. \tag{4.4}$$

The inequality sign results from the following. The product of stress and elastic strains $\boldsymbol{\sigma}^T\boldsymbol{\varepsilon}_E$ is positive due to the definition of elasticity matrix. The clearance work $\boldsymbol{\sigma}^T\boldsymbol{\varepsilon}_L$ in slackened structures is always positive semi-definite (cf. [6]). The product of stresses and plastic strains $\boldsymbol{\sigma}^T\boldsymbol{\varepsilon}_P$ represents the positive semi-definite plastic dissipation in EpP sysytems. Relation (4.4) is also valid for the remaining kinds of structures (i.e. E, SE, EpP) because they are particular cases of the $SEpP$ structure.

The yield condition and contact condition are assumed to be convex. For the holonomic model, these assumptions can be expressed in the following mathematical form:

$$\begin{aligned}(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)^T\boldsymbol{\varepsilon}_{P1} &\geq 0,\\ (\boldsymbol{\varepsilon}_{L1} - \boldsymbol{\varepsilon}_{L2})^T\boldsymbol{\sigma}_1 &\geq 0.\end{aligned} \tag{4.5}$$

In (4.5) $\boldsymbol{\sigma}_1$, $\boldsymbol{\varepsilon}_{P1}$ and $\boldsymbol{\varepsilon}_{L1}$ denote true vectors of stress and strains, whereas $\boldsymbol{\sigma}_2$ and $\boldsymbol{\varepsilon}_{L2}$ are arbitrary statically admissible stress and kinematically admissible clearance strain vectors, respectively. Moreover, using inequalities (4.5) and assuming that $\boldsymbol{\sigma}_2$, $\boldsymbol{\varepsilon}_{P2}$ and $\boldsymbol{\varepsilon}_{L2}$ represent true associated stress and distortion states, we obtain

$$\begin{aligned}(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)^T(\boldsymbol{\varepsilon}_{P1} - \boldsymbol{\varepsilon}_{P2}) &\geq 0,\\ (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)^T(\boldsymbol{\varepsilon}_{L1} - \boldsymbol{\varepsilon}_{L2}) &\geq 0,\end{aligned} \tag{4.6}$$

hence

$$(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)^T[(\boldsymbol{\varepsilon}_{L1} + \boldsymbol{\varepsilon}_{P1} + \boldsymbol{\varepsilon}_D) - (\boldsymbol{\varepsilon}_{L2} + \boldsymbol{\varepsilon}_{P2} + \boldsymbol{\varepsilon}_D)] \geq 0, \tag{4.7}$$

where $\boldsymbol{\varepsilon}_D$ denotes a steady distortion vector. All the possible distortions which can occur in the class of time-independent structural systems considered herein can be presented as

$$\boldsymbol{\varepsilon}_{Li} + \boldsymbol{\varepsilon}_{Pi} + \boldsymbol{\varepsilon}_D = \boldsymbol{\varepsilon}_{Ri}; \qquad i = 1, 2. \tag{4.8}$$

Substituting (4.7) to inequality (3.2)′ yields

$$(\mathbf{p}_2 - \mathbf{p}_1)^T(\mathbf{u}_2 - \mathbf{u}_1) \geq 0. \tag{4.9}$$

Using (4.2) in Ineq. (4.9) we obtain

$$(\mu-1)\mathbf{p}_1^T(\mathbf{u}_2-\mathbf{u}_1) \geq 0,$$
$$(1-\mu^{-1})\mathbf{p}_2^T(\mathbf{u}_2-\mathbf{u}_1) \geq 0.$$

For proportional loading $(\mu-1)>0$ and $(1-\mu^{-1})>0$. Thus, we can state that

$$\begin{aligned}\mathbf{p}_1^T(\mathbf{u}_2-\mathbf{u}_1) &> 0,\\ \mathbf{p}_2^T(\mathbf{u}_2-\mathbf{u}_1) &> 0.\end{aligned} \tag{4.10}$$

Relations (4.10) will be used in further considerations.

4.2. Linear elastic systems

In elastic structures $\boldsymbol{\varepsilon}_i=\boldsymbol{\varepsilon}_{Ei}$ and $\varepsilon_{Ri}\equiv\mathbf{0}$ $(i=1,2)$. From $(3.5)_2$ we have

$$\mathbf{p}_1^T(\mathbf{u}_2-\mathbf{u}_1)\leq \Delta W_E \leq \mathbf{p}_2^T(\mathbf{u}_2-\mathbf{u}_1) \quad \text{for} \quad \mu>1.$$

According to $(4.10)_1$ $\mathbf{p}_1^T(\mathbf{u}_2-\mathbf{u}_1)>0$, hence $\Delta W_E=W_{E2}-W_{E1}>0$. It corresponds to the obvious conclusion that the elastic energy in linear elastic systems is an increasing function of the load multiplier.

It will be shown that the same conclusion is also valid for elastic systems with any initial, load-independent distortions. Consider an elastic structure that exhibits steady distortions $\boldsymbol{\varepsilon}_D$. Denote by subscripts 1 and 2 the elastic energies of the self-stresses and load $\mathbf{p}$, acting on the structure without distortions, respectively. Then

$$\begin{aligned}
&1:\ \mathbf{p}_1=\mathbf{0}, \quad \boldsymbol{\varepsilon}_{R1}=\boldsymbol{\varepsilon}_D, \quad \mathbf{C}\mathbf{u}_1=\boldsymbol{\varepsilon}_{E1}+\boldsymbol{\varepsilon}_D, \quad \boldsymbol{\sigma}_1=\mathbf{E}\boldsymbol{\varepsilon}_{E1}=\mathbf{Z}\boldsymbol{\varepsilon}_D;\\
&2:\ \mathbf{p}_2=\mathbf{p}, \quad \boldsymbol{\varepsilon}_{R2}=\mathbf{0}, \quad \mathbf{C}\mathbf{u}_2=\boldsymbol{\varepsilon}_{E2}, \quad \boldsymbol{\sigma}_2=\mathbf{E}\boldsymbol{\varepsilon}_{E2}.
\end{aligned}$$

The total elastic energy W_E including the distortion and load effects reads

$$\begin{aligned}W_E=\frac{1}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)^T(\boldsymbol{\varepsilon}_{E1}+\boldsymbol{\varepsilon}_{E2})=\frac{1}{2}\boldsymbol{\sigma}_1^T\boldsymbol{\varepsilon}_{E1}+\frac{1}{2}\boldsymbol{\sigma}_1^T\boldsymbol{\varepsilon}_{E2}\\
+\frac{1}{2}\boldsymbol{\sigma}_1^T\boldsymbol{\varepsilon}_{E2}+\frac{1}{2}\boldsymbol{\sigma}_2^T\boldsymbol{\varepsilon}_{E1}=W_{E1}+W_{E2}+\boldsymbol{\sigma}_1^T\boldsymbol{\varepsilon}_{E2}.\end{aligned}$$

The last term in the above expression vanishes due to the virtual work principle $(\mathbf{p}_1=\mathbf{0})$:

$$\boldsymbol{\sigma}_1^T\boldsymbol{\varepsilon}_{E2}=\boldsymbol{\sigma}_1^T\mathbf{C}\mathbf{u}_2=\mathbf{C}^T\boldsymbol{\sigma}_1\mathbf{u}_2=\mathbf{p}_1^T\mathbf{u}_2=0.$$

So, the elastic energy can be decomposed into the energy of steady distortions and the energy of external loads; the mutual, load-distortion energy is equal to zero. The same result has been obtained in [10]. However, this interesting observation is valid only for linear elastic systems. Since the external load energy is distortion-independent, the elastic energy is an increasing function of the load multiplier.

Finally, let us determine the explicit form of expression for the elastic energy of self-stresss:

$$
\begin{aligned}
(4.11) \qquad W_{E1} &= \frac{1}{2}\sigma_1^T \varepsilon_{E1} = \frac{1}{2}\varepsilon_D^T \mathbf{Z}\varepsilon_{E1} = \frac{1}{2}\varepsilon_D^T \mathbf{Z}(\mathbf{C}\mathbf{u}_1 - \varepsilon_D) = -\frac{1}{2}\varepsilon_D^T \mathbf{Z}\varepsilon_D \\
&= -\frac{1}{2}\varepsilon_D^T \mathbf{Z}(\mathbf{C}\mathbf{u}_1 - \varepsilon_{E1}) = \frac{1}{2}\varepsilon_D^T \mathbf{Z}\mathbf{E}^{-1}\sigma_1 = \frac{1}{2}\varepsilon_D^T(\mathbf{Z}\mathbf{E}^{-1}\mathbf{Z})\varepsilon_D.
\end{aligned}
$$

From (4.11) we conclude that $\mathbf{Z}\mathbf{E}^{-1}\mathbf{Z} = -\mathbf{Z}$. Indeed, using the definition of matrix $\mathbf{Z}$ and taking into account that $\mathbf{Z}\mathbf{C} \equiv \mathbf{0}$, we find

$$
(4.12) \qquad \mathbf{Z}\mathbf{E}^{-1}\mathbf{Z} = \mathbf{Z}\mathbf{E}^{-1}(\mathbf{E}\mathbf{C}\mathbf{K}^{-1}\mathbf{C}^T\mathbf{E} - \mathbf{E}) = \mathbf{Z}(\mathbf{C}\mathbf{K}^{-1}\mathbf{C}^T\mathbf{E} - \mathbf{I}) = -\mathbf{Z}.
$$

4.3. Elastic-perfectly plastic systems

For both levels of loads $\mathbf{p}_1$ and $\mathbf{p}_2$, the total strains consist of elastic and plastic (distortion) parts:

$$
\text{(a)} \qquad \varepsilon_i = \varepsilon_{Ei} + \varepsilon_{Ri}; \qquad \varepsilon_{Ri} = \varepsilon_{Pi}; \qquad i = 1, 2,
$$

so, from $(3.5)_1$ and (a) we obtain

$$
\Delta W_E \geq L_1 = (\mathbf{p}_2 - \mathbf{p}_1)^T \mathbf{u}_1 + (\sigma_1 - \sigma_2)^T \varepsilon_{R1} = (\mu - 1)\mathbf{p}_1^T \mathbf{u}_1 + (\sigma_1 - \sigma_2)^T \varepsilon_{P1}.
$$

Since $(\mu - 1) > 0$, and according to (4.4), $\mathbf{p}_1^T \mathbf{u}_1 > 0$, the first right-hand side term is positive. If the yield condition is convex, the second right-hand side term is non-negative (cf. (4.5)). Thus, $\Delta W_E > 0$ and the elastic energy is an increasing monotone function of load multiplier μ.

4.4. Slackened-elastic systems

In slackened systems the strain vector can be divided into elastic and clearance parts

$$
\text{(a)} \qquad \varepsilon_i = \varepsilon_{Ei} + \varepsilon_{Ri}; \qquad \varepsilon_{Ri} = \varepsilon_{Li}; \qquad i = 1, 2.
$$

The elastic energy is an increasing function of the load multiplier if L_2 is positive definite. Using inequality $(3.5)_2$ we obtain:

$$
\text{(b)} \qquad L_2 = \mathbf{p}_1^T(\mathbf{u}_2 - \mathbf{u}_1) + \sigma_1^T(\varepsilon_{R1} - \varepsilon_{R2}) = \mathbf{p}_1^T(\mathbf{u}_2 - \mathbf{u}_1) + \sigma_1^T(\varepsilon_{L1} - \varepsilon_{L2}) > 0.
$$

The positive definiteness of L_2 results from $(4.10)_2$ and from the convexity of the contact condition (cf. $(4.5)_2$). In view of (b) we can state that this conclusion holds also true in the case where steady distortions ε_D are additionally imposed on the slackened-elastic structure.

4.5. Slackened-elastic-perfectly plastic systems

Similarly to the previous case, the strain vector is the sum of elastic and distortion parts. However, distortion strains in slackened-elastic-plastic systems consist of clearance and plastic strains:

$$\text{(a)} \qquad \boldsymbol{\varepsilon}_i = \boldsymbol{\varepsilon}_{Ei} + \boldsymbol{\varepsilon}_{Ri}; \quad \boldsymbol{\varepsilon}_{Ri} = \boldsymbol{\varepsilon}_{Li} + \boldsymbol{\varepsilon}_{Pi}; \quad i = 1, 2.$$

Such systems demonstrate a lot of interesting effects and their behaviour is very complicated, particularly when plastic and clearance strains are simultaneously present. A complexity of this problem comes from the fact that clearance distortions, contrary to plastic ones, are always load-dependent. Therefore the signs of L_1, L_2, R_1 and R_2 in Ineqs. (3.5) cannot be evaluated. It is interesting to notice that even positive definiteness of right-hand sides of (3.5) does not have to be always guaranteed.

Let us consider, for example, the expression for R_2:

$$\text{(b)} \qquad R_2 = (\mathbf{u}_2 - \mathbf{u}_1)^T \mathbf{p}_2 + (\boldsymbol{\varepsilon}_{L1} - \boldsymbol{\varepsilon}_{L2})^T \boldsymbol{\sigma}_2 + (\boldsymbol{\varepsilon}_{P1} - \boldsymbol{\varepsilon}_{P2})^T \boldsymbol{\sigma}_2.$$

According to $(4.10)_2$, the first term in (b) is positive. On the other hand, the remaining terms consists of the non-positive definite part $(\boldsymbol{\varepsilon}_{L1} - \boldsymbol{\varepsilon}_{L2})^T \boldsymbol{\sigma}_2$ (cf. $(4.5)_2$) and the part due to plastic strains $(\boldsymbol{\varepsilon}_{P1} - \boldsymbol{\varepsilon}_{P2})^T \boldsymbol{\sigma}_2$, its sign being undeterminate, in general; however, for proportional loading the negative sign can be expected. Similar results can be obtained for L_1, L_2 and R_1. A numerical example of Sec. 5 will explain this problem.

5. Numerical Example

Consider a simple beam shown in Fig. 1.

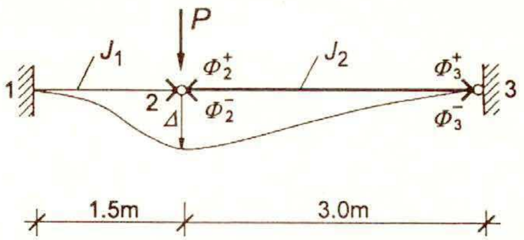

FIG. 1. Slackened beam with clearance hinges.

The beam is composed of two elements of ideal I-cross-sections. The moments of inertia and depths for both the elements are equal to $J_1 = 4500$ cm^4, $J_2 = 10000$ cm^4 and $h_1 = 30$ cm, $h_2 = 40$ cm, respectively. Two kinds of the material of the beam are assumed: the linear elastic of infinite strength, and the linear elastic-perfectly plastic with the yield stress $\sigma_Y = 300$ MPa. The corresponding full plastic bending moments of the cross-section for the beam-elements are $M_{Y1} = 90$ kNm and $M_{Y2} = 150$ kNm. The Young's modulus for both the materials is assumed to be the same: $E = 200$ GPa. In addition, at points 2 and 3 the so-called clearance hinges are introduced. In other words, the angle of free relative rotations of adjacent beam-elements $\phi_i (i = 2, 3)$ at these points can vary between the limits: $-\phi_i^- \le \phi_i \le \phi_i^+$. Angles ϕ_i play here the role of clearance strains. The cases where clearance hinges are introduced correspond to the systems which are slackened. If the clearance moduli (i.e. limit free rotations at clearance hinges) are equal to zero $\phi_i^- = \phi_i^+ = 0$, the beam becomes a common structure with bilateral constraints. Then the beam is fully fixed at both the supports (point 1 and point 3). So, we can consider the following four kinds of the system:

- elastic (E) $(\sigma_Y \to \infty, \phi_i^- = \phi_i^+ = 0)$,
- elastic-perfectly plastic (EpP) $(\sigma_Y = 300\ \text{MPa}, \phi_i^- = 0, \phi_i^+ = 0)$,
- slackened-elastic (SE) $(\sigma_Y \to \infty, \phi_i^- \ne \phi, \phi_i^+ \ne 0)$,
- slackened-elastic-perfectly plastic (SEpP) $(\sigma_Y = 300\ \text{MPa}, \phi_i^- \ne \phi, \phi_i^+ \ne 0)$.

Further considerations will be carried out for identical and symmetrically distributed rotation gaps, i.e. $\phi_2^- = \phi_2^+ = \phi_3^- = \phi_3^+ = \phi_0$. Variations of these gaps within the limits $< 0, 0.009$ rad $>$ allow us to analyse the elastic energy as a function of slackening intensity, including also the beam with bilateral constraints.

The beam is subjected to concentrated load P acting at point 2. The load increases proportionally up to $P_Y = 200$ kN (i.e. to the limit load for the elastic-perfectly plastic beam) and then the beam is proportionally unloaded.

Particular cases of the types specified above of the structure can be examined with respect to the elastic energy at given levels of the proportional loading. Additionally, the energy variations during unloading will be also presented.

The beam with rotation clearances belongs to a particular class of skeletal $SEpP$ structures where distortions are concentrated at the clearance, plastic or clearance-plastic hinges. The loading and unloading of the structure induce opening or closing of these hinges. As a consequence, the boundary conditions of elements (i.e. structure types) are changeable.

The current elastic energy W_E for particular kinds of the beam is calculated as a function of "deflection length" S_Δ or "load length" S_P. The current deflection of the beam Δ, deflection length S_Δ and load length S_P are defined as follows:

$$\Delta = \sum_{j=1}^{m} \Delta^{(j)}; \qquad S_\Delta = \sum_{j=1}^{m} \left|\Delta^{(j)}\right|; \qquad S_P = \sum_{j=1}^{m} \left|P^{(j)}\right|,$$

where $\Delta^{(j)}$ and $P^{(j)}$ denote the deflection rate of point 2 and the external load rate in the j-th step of the calculations, respectively. Symbol m denotes a current calculation step.

$P-\Delta$ diagrams for E, EpP, SE and $SEpP$ beams for $\phi_0 = 0.009$ rad are presented in Fig. 2a, while in Fig. 2b the elastic energy W_E versus the deflection

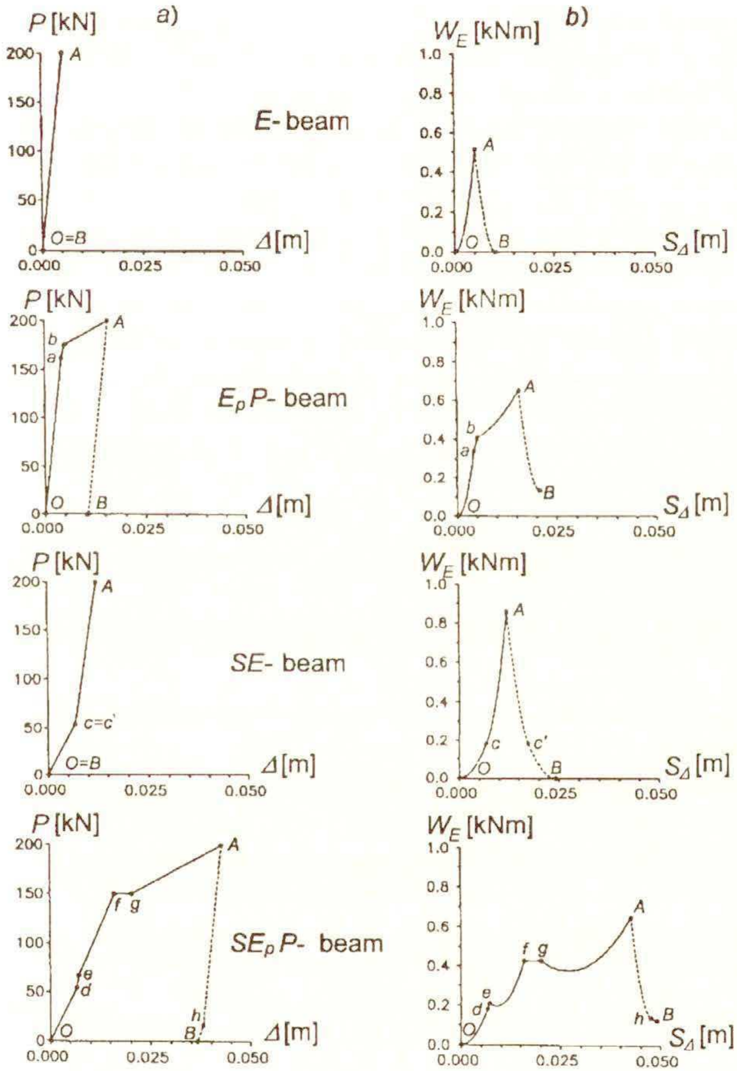

FIG. 2. Elastic energy for proportional loading of the beam; a) $P-\Delta$ diagrams, b) Elastic energy W_E versus deflection length S_Δ.

length S_Δ is plotted. Segments OA and segments AB correspond to proportional loading (solid lines) and unloading (dashed lines) of the beam, respectively. All the intermediate points indicate the structure type changes.

$P - \Delta$ relations for the E-beam and EpP–beam take a well-known form of concave functions. On the other hand, the presence of clearances induces locking effects which lead to convexity of $P(\Delta)$ functions. It is clearly seen for the SE–beam. The behaviour of $SEpP$–beam is much more complex; both the convexity (e.g. segment $O - d - e$) and concavity of $P(\Delta)$ function are noted. The $P(\Delta)$ convexity concerns also the unloading curve (segment $A - h - B$). Moreover, there exists the horizontal segment which corresponds to a "clearance-plastic mechanism" (cf. segment $f - g$). Obviously, the rates of elastic energy on this segment are equal to zero.

In the range of proportional loading, the elastic energy appears to be a monotone increasing function with respect to the beam deflection, except the case of the $SEpP$–beam (cf. Fig. 2b). It confirms the theoretical results of Sec. 4. Indeed, we can state that the elastic energy in the $SEpP$–beam can be a partially decreasing function of the load multiplier. Note that the energy of residual stresses does not have to coincide with that of the EpP–beam.

From Fig. 2 it follows that the elastic energy variations during the deformation processes must depend on the values of clearance moduli. In order to examine this problem we calculate W_E as a function of S_P during proportional loading for increasing values of rotation gaps, ϕ_0. Figure 3 shows $W_E(S_P)$ diagrams for particular kinds of the beam.

According to the results of Sec. 4, the elastic energy in the E–beam and SE–beam is an increasing function of the load multiplier (see Fig. 3a). From Fig. 3b it follows that for a sufficiently large values of ϕ_0, the elastic energy in the $SEpP$–beam can decrease while the load multiplier increases.

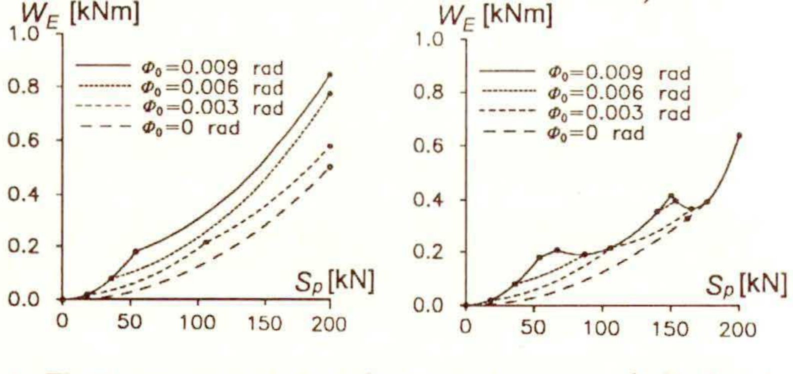

FIG. 3. Elastic energy variations for increasing gaps; a) Slackened-elastic beam, b) Slackened-elastic-perfectly plastic beam.

Now, the question arises: what is the physical and structural interpretation of the decreasing energy function?

Analysing the problem from the physical point of view we conclude that a part of the elastic energy can be converted into the plastic dissipation. Then the decrease in current elastic energy is observed. Obviously, such a phenomenon can occur only for structures whose material exhibits both the elastic and plastic deformations. To make the problem more clear, the current elastic energy W_E and the current total dissipation D in the $SEpP$–beam ($\phi_0 = 0.009$ rad) versus deflection length S_Δ are plotted in Fig. 4. It is seen that the elastic energy starts to drop down just as the plastic dissipation begins (cf. points e and g in Fig. 4).

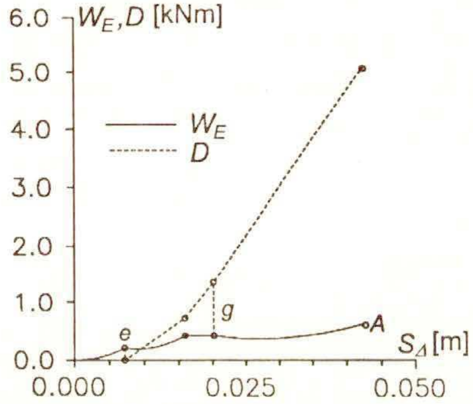

FIG. 4. Elastic energy W_E and total plastic dissipation D in $SEpP$–beam during proportional loading.

Next additional question is: "why can it occur only for the $SEpP$–beam?" An explanation of this problem can be found in Fig. 5 where changes of the structure type and the corresponding generalized stress (bending moment) distributions are presented. Figure 5a relates to $P = P_e = 65$ kN (point e in Fig. 4) and $P_{e\Delta} = P_e + \Delta P = 65 + 5 = 70$ kN. For $P = P_e$ the beam is fully fixed at the left-hand support and pin-ended at the right-hand support. The load increasing up to $P_{e\Delta}$ induces the structure type change; the beam becomes pin-ended at both the supports. Similar situation arises for $P = P_g = 150$ kN (point g in Fig. 4) and $P_{g\Delta} = P_g + \Delta P = 150 + 5 = 155$ kN. For $P = P_g$ at point 2 the new plastic hinge forms whereas at point 3 the clearance hinge closes and the beam becomes statically determinate. The structure-type changes give modifications of bending moment distributions. It can be easily checked that the elastic energy rates starting from $P = 65$ kN and $P = 150$ kN are negative. So, we can conclude that the elastic energy decrease is induced by deformation-dependent boundary condition changes. Such untypical changes can appear only for slackened-elastic-plastic structures where clearance and plastic strains simultaneously appear.

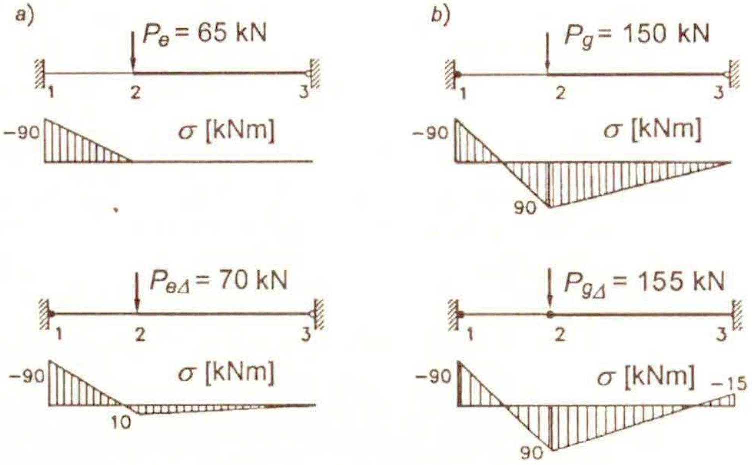

FIG. 5. Structure type and bending moment changes during proportional loading of $SEpP$–beam; a) load level e ($P = 65$ kN), b) load level g ($P = 150$ kN).

6. Final remarks

The present paper concerns the proportional loading of structures made of time-independent materials. It appears that this particular and simplest case of loading is not yet sufficiently recognized. It has been shown that the elastic energy can be a decreasing function of the load multiplier if unilateral constraints (i.e. gaps at structural connections) are introduced into an elastic-plastic structure. The results obtained in the paper seem to be of importance for the theory of structures and may have many theoretical implications. We have in mind, for instance, the damage mechanics where the elastic energy is usually assumed as an increasing function of the load multiplier. The problem appears to be much more significant due to the fact that damaged bodies contain internal gaps and therefore, this assumption seems to be not quite justified.

In spite the fact that the present work concerns discretized systems, the author believes that the results obtained herein can be generalized to continuous bodies made of time-independent materials.

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