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FIFTY YEARS OF THE ARCHIVES OF MECHANICS

The dependence of dynamic phase transitions on parameters

K. PIECHÓR (WARSZAWA)

WE CONSIDER phase changes described by a second order ordinary differential equation. The equation depends parametrically on the states of rest and the speed of the wave. We prove that, under some additional conditions, the solution is differentiable with respect to any of these parameters. As an application of the general theory we discuss the case when the data are close to the Maxwell line and obtain results generalising those of the previous authors.

1. Introduction

WE TREAT the phase boundary as a one-dimensional travelling wave connecting two different states of rest. The speed of the wave cannot be arbitrary but it is an unknown, determined totally by the value of just one of the states of rest. In other words, the question of existence of phase boundaries is a sort of nonlinear eigenvalue problem. For a very limited number of cases we know exactly the structure of the phase boundary and its speed [1, 2]. In the general case, it is only proved that once one state of rest is given, there is a unique value of speed and uniquely determined other state of rest such that the travelling wave connecting them exists and moves at this speed [3 – 8].

The aim of this paper is to formulate sufficient conditions ensuring differentiability of the phase boundary structure, the speed of the wave and the other state of rest as functions of one of the two states of rest.

The paper is organised as follows. In the next section we present the equation of the phase boundary deduced from the capillarity equations which we have derived from a model kinetic theory of van der Waals fluids [9]. In Sec. 3 we generalise this problem and prove a theorem on the differentiability of its solution with respect to a group of parameters treated as "independent". In the final Sec. 4, we apply this theory to our model equation of phase boundaries as well as to the case of isothermal phase transitions. We confine our interest to the case when the data are close to the so-called Maxwell line, in order to avoid complicated formulae. In the latter case our results not only agree with the previous authors' results but also generalise them. Moreover, we show that our model theory agrees qualitatively with the isothermal one.

2. The model equations of capillarity and the travelling wave problem

The model equations of capillarity we are going to consider consist of the following system of two partial differential equations [9]:

$$\frac{\partial}{\partial t}w - \frac{\partial}{\partial x}u = 0, \tag{2.1}$$

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}p(w,u) = \varepsilon\frac{\partial}{\partial x}\left(\mu\frac{\partial}{\partial x}u\right) + \alpha\varepsilon^2\frac{\partial}{\partial x}\left[\frac{5}{w^6}\left(\frac{\partial}{\partial x}w\right)^2 - \frac{2}{w^5}\frac{\partial^2}{\partial x^2}w\right]. \tag{2.2}$$

In Eqs. (2.1), (2.2), the variable $t > 0$ is the time, $x \in \mathrm{R}^1$ is the Lagrangian coordinate, u is the velocity, w is the specific volume, p is the pressure, and $\varepsilon\mu$ is the coefficient of viscosity.

The pressure formula reads

$$p = p(w,u) = \frac{1-u^2}{2(w-b)} - \frac{a}{w^2}, \tag{2.3}$$

where a and b are positive constants; a is the ratio of the mean value of the potential of the attractive intermolecular forces to the mean kinetic energy of molecules, and b characterises close packing. In the adopted dimensionless units b is equal to unity.

Next, $\varepsilon > 0$ characterises the order of magnitude of the viscosity effect, and $\mu = \mu(w,u)$ is given by [9]

$$\mu(w,u) = \frac{w^2(1-u^2) + 2b^2\rho^2(w)}{8w^3\rho(w)}, \qquad \rho(w) = \frac{w}{w-b}. \tag{2.4}$$

Finally, $\alpha\varepsilon^2$, with $\alpha = \text{const} > 0$, characterises the intensity of the capillarity effects which are represented by the space derivative of the term in the square brackets [ ].

We consider Eqs. (2.1), (2.2) in the domain $\mathcal{D}$ defined by [10]

$$\mathcal{D} = \left\{(w,u) : w > b, \qquad u^2 < 1 - \frac{a}{2b}, \qquad \frac{a}{2b} < 1\right\}. \tag{2.5}$$

For $(w,u) \in \mathcal{D}$, the mass density $1/w$ does not exceed the close-packing density $1/b$, and the pressure p and the viscosity μ are positive.

A travelling wave solution to (2.1), (2.2) is a solution of the form

$$(w,u)(x,t) = (w,u)(\xi), \qquad \xi = \frac{x - st}{\varepsilon} \in \mathrm{R}^1, \tag{2.6}$$

such that

$$\lim_{\xi\to-\infty}(w,u)(\xi) = (w_l,u_l), \tag{2.7}$$

$$\lim_{\xi\to+\infty}(w,u)(\xi) = (w_r,u_r), \tag{2.8}$$

$$\lim_{\xi\to\pm\infty}(w',u') = (0,0), \tag{2.9}$$

$$\lim_{\xi\to\pm\infty}(w'',u'') = (0,0), \tag{2.10}$$

where $s =$ const is the wave speed, and $()' = \dfrac{d}{d\xi}()$.

The following procedure is routine. We substitute (2.6) into Eqs. (2.1), (2.2), perform one integration with respect to ξ, and use the limit conditions (2.7)–(2.10). Having done that, we find that the left and right limit states are related by

$$\begin{aligned} sw_r + u_r &= sw_l + u_l, \\ -su_r + p(w_r,u_r) &= -su_l + p(w_l,u_l). \end{aligned} \tag{2.11}$$

These relations are called the Rankine-Hugoniot conditions and were in detail analysed in [10].

Next, we find the velocity u. It is given by

$$u = u_l - s(w - w_l), \tag{2.12}$$

where $w = w(\xi)$ is a solution of the following limit value problem:

$$\alpha\left[\frac{2}{w^5}w'' - \frac{5}{w^6}w'^2\right] + s\mu(w,s,w_l)w' + f(w,s,w_l) = 0, \tag{2.13}$$

where

$$\mu(w,s,w_l) = \mu(w,u_l - s\,(w-w_l)) > 0, \tag{2.14}$$

$$f(w,s,w_l) = p\,(w,u_l - s(w-w_l)) - p(w_l,u_l) + s^2(w-w_l), \tag{2.15}$$

subject to the conditions

$$\lim_{\xi\to-\infty} w(\xi) = w_l, \tag{2.16}$$

$$\lim_{\xi\to+\infty} w(\xi) = w_r, \tag{2.17}$$

$$\lim_{\xi\to\pm\infty} w'(\xi) = 0, \qquad \lim_{\xi\to\pm\infty} w''(\xi) = 0. \tag{2.18}$$

These conditions must be supplemented by Eqs.(2.11) which we write in the form

$$f(w_l, s, w_l) = 0, \qquad f(w_r, s, w_l) = 0. \tag{2.19}$$

In this paper we assume that

$$f'_w(w_l, s, w_l) < 0, \qquad f'_w(w_r, s, w_l) < 0. \tag{2.20}$$

Our problem contains a number of parameters like w_r, s, w_l, u_l, etc. These parameters are not independent, since some of them are related by the Rankine-Hugoniot conditions (2.11). However, our problem, under assumptions (2.19), (2.20), has a solution if and only if the parameters satisfy an additional relation, unknown in advance. The total number of relations, including the implicit one, is less than the total number of parameters. Therefore, we can split them into two groups: dependent and independent ones. Of course, this splitting is not dictated by the limit value problem itself, it is rather a result of our current interest. Also, it is not obligatory to consider the dependence of solutions on all parameters; simply, we can treat some of them as fixed.

Altogether, there is a great variety of specific problems we can be interested in. Therefore, in order to avoid repeating similar arguments, each time we ask a question concerning the character of dependence of the solution on certain parameters we choose, we formulate an "abstract" problem of dependence of the solution on the parameters and prove its solvability. In Sec. 4 we show how to reduce our specific problem to the "abstract" one.

Let us explain that we cannot answer the posed question basing on the well known theorem on continuous dependence of solutions of ordinary differential equations on the parameters, because it is not clear in advance whether the implicit, unknown relation between the parameters is a differentiable function or not.

3. The abstract problem

The problem we consider consists in determining a function and a set of functions $y(\xi, \lambda) : \mathrm{R}^1 \times \Lambda \to Y \subset \mathrm{R}^1$, and a set of functions $\kappa(\lambda) = (\kappa_1(\lambda), \kappa_2(\lambda), ..., \kappa_k(\lambda)) : \Lambda \to K$, where Λ is an open subset of R^l, K is an open subset of R^k , and the range Y of $y(\xi, \lambda)$ contains the closed interval [0,1]. The functions $y(\xi, \lambda)$ and $\kappa(\lambda)$ are such that:

i) $y(\xi, \lambda)$ satisfies the differential equation

$$y'' = g(y, y', \kappa, \lambda), \tag{3.1}$$

and the limit conditions: for any $\lambda \in \Lambda$

$$\lim_{\xi \to -\infty} y(\xi, \lambda) = 0, \qquad \lim_{\xi \to +\infty} y(\xi, \lambda) = 1, \tag{3.2}$$

$$\lim_{\xi \to -\infty} (y'(\xi, \lambda), y''(\xi, \lambda)) = (0, 0), \tag{3.3}$$

$$\lim_{\xi \to +\infty} (y'(\xi, \lambda), y''(\xi, \lambda)) = (0, 0), \tag{3.4}$$

where the dash denotes differentiation with respect to ξ.

ii) The functions $\kappa_i(\lambda)$, $i = 1, 2, ..., k-1$, satisfy a system of $k-1$ algebraic equations of the form

$$G_i(\kappa, \lambda) = 0, \qquad i = 1, 2, ..., k-1. \tag{3.5}$$

We take the following assumptions concerning the functions g and $G(\kappa, \lambda) = (G_1(\kappa, \lambda), G_2(\kappa, \lambda), ..., G_{k-1}(\kappa, \lambda))$:

H1. $g(y, z, \kappa, \lambda) \in C^{m+\tau}(Y \times \mathrm{R} \times K \times \Lambda)$ for some integer $m > 0, \tau > 0$.

H2. For any $(\kappa, \lambda) \in K \times \Lambda$

$$g(0, 0, \kappa, \lambda) = 0, \tag{3.6}$$

$$g(1, 0, \kappa, \lambda) = 0, \tag{3.7}$$

$$g'_y(0, 0, \kappa, \lambda) > 0, \tag{3.8}$$

$$g'_y(1, 0, \kappa, \lambda) > 0. \tag{3.9}$$

H3. $G_i(\kappa, \lambda) \in C^{m+\tau}(K \times \Lambda), \qquad i = 1, 2, ..., k-1.$

COMMENTS

i) Conditions (3.6), (3.7) make Eq. (3.1) and the limit values (3.2)–(3.4) compatible.

ii) Conditions (3.8), (3.9) are crucial for our considerations. They mean that the rest points (0,0) and (1,0) in the (y, y') – plane are saddle points.

Equation (3.1) is autonomous, i.e. if $y(\xi)$ is its solution so is $y(\xi + c)$, for any constant c. To get rid of this ambiguity we impose an additional condition:

$$y(0, \lambda) = \frac{1}{2}\left[y(-\infty, \lambda) + y(+\infty, \lambda)\right] = \frac{1}{2}. \tag{3.10}$$

Our aim is to prove that, roughly speaking, if $y_0(\xi), \kappa_0 \in \mathrm{R}^k, \lambda_0 \in \mathrm{R}^l$ is a solution to (3.1)–(3.5) then, under some additional conditions to be specified, the problem has also a solution in a vicinity of λ_0. The Implicit Function Theorem seems to be the proper tool to perform this task, but some difficulty arises from the fact that the number of unknowns is greater than that of the equations. Elimination of this difficulty is possible owing to the fact that we are looking for special solutions, namely those which satisfy (3.2), (3.3) for any λ. It means that we have to be cautious and choose suitable functional spaces.

Since our course of action follows the Implicit Function Theorem we start, for the reader's convenience, from its presentation (cf. [12])

IMPLICIT FUNCTION THEOREM [12]. *Let*

(i) *X, Y, Z be normed affine spaces and $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ the corresponding vector spaces;*

(ii) *$\mathcal{D}(\mathfrak{Y}, \mathfrak{Z})$ be the set of linear continuous mappings of the space $\mathfrak{Y}$ onto $\mathfrak{Z}$;*

(iii) *W be an open subset of $X \times Y$, and $(x_0, y_0) \in W, x_0 \in X, y_0 \in Y$;*

(iv) *$F : W \to Z$ be a continuous mapping of W onto Z and $F(x_0, y_0) = z_0$, $z_0 \in Z$.*

If

i) *for every fixed $x \in X$ and $(x, y) \in W$, the mapping F has the Fréchet derivative $F_y \in \mathcal{L}(\mathfrak{Y}, \mathfrak{Z})$;*

ii) *$F_y : W \to \mathcal{L}(\mathfrak{Y}, \mathfrak{Z})$ is a linear continuous mapping of W onto $\mathcal{L}(\mathfrak{Y}, \mathfrak{Z})$;*

iii) *the linear mapping $F_y(x_0, y_0) : \mathfrak{Y} \to \mathfrak{Z}$ has continuous inverse linear mapping.*

Then there are subsets $U \subset X$, $V \subset Y$ open in X, Y, respectively, $x_0 \in U$, $y_0 \in V$, such that for every $x \in U$ there is a unique element $y \in V$, denoted by $y = f(x)$, satisfying $f(x) \in V$, $F(x, f(x)) = z_0$, $f(x_0) = y_0$; $f(x)$ is a continuous mapping of U onto V.

If additionally, the Fréchet derivative $F_x(x_0, y_0)$ exists and is a linear continuous mapping of $\mathfrak{X}$ onto $\mathfrak{Z}$, then the mapping f is differentiable at the point x_0 and its Fréchet derivative is given by the formula

$$f'(x_0) = -F_y^{-1}(x_0, y_0) \circ F_x(x_0, y_0),$$

or implicitly

$$F_x(x_0, y_0) + F_y(x_0, y_0) \circ f'(x_0) = 0.$$

■

Now we define the spaces suitable for our problem.

DEFINITION 1. *The space $\mathfrak{X}$ is defined as the Euclidean space R^l with elements denoted by $\lambda \in \Lambda \subset R^l$, the affine space $X = A(R^l)$, where $A(R^n)$ denotes the affine space associated with R^n.* ■

The definitions of the other spaces are more complicated.

DEFINITION 2. *The set of functions $y(\xi) \in C^i(R^1)$, for $i = 0, 1, 2$, vanishing exponentially together with their first i derivatives as $|\xi| \to \infty$ we denote by $\mathfrak{B}_i$; the norms are taken in the form*

$$||y||_i = \sup_{\xi \in \mathrm{R}'} \left(|y(\xi)| + ... + |y^{(i)}(\xi)| \right).$$

The subspace of $\mathfrak{B}_i$ consisting of functions such that

$$y(0) = \frac{1}{2} [y(-\infty) + y(+\infty)] = 0$$

is denoted by $\mathfrak{B}_i^0$. ■

Of course, $\mathfrak{B}_i$ and $\mathfrak{B}_i^0$ are Banach spaces.

DEFINITION 3. *The affine space* B_2^0 *associated with the normed vector space* $\mathfrak{B}_2^0$ *is defined as the set of functions* $y(\xi) \in C^2(R^1)$ *satisfying exponentially* (3.2) *and such that* $\lim\limits_{|\xi|\to\infty} y^{(i)}(\xi) = 0$, $i = 1, 2$, *also exponentially.* ■

DEFINITION 4. *The normed vector space* $\mathfrak{Y}$ *is defined by the equality* $\mathfrak{Y} = \mathfrak{B}_2^0 \times R^k$ *with the usual product norm; the affine space* Y *associated with* $\mathfrak{Y}$ *is defined by* $Y = B_2^0 \times A(R^k)$. ■

DEFINITION 5. *The normed vector space* $\mathfrak{Z}$ *is defined by the equality* $\mathfrak{Z} = \mathfrak{B}_0 \times R^{k-1}$ *with the usual product norm; the affine space* Z *associated with* $\mathfrak{Z}$ *is defined by* $Z = B_0 \times A(R^{k-1})$. ■

As to the mapping F mentioned in the Implicit Function Theorem, we take the pair $F = (A(y,\kappa,\lambda), G(\kappa,\lambda)) : X \times Y \to Z$, where $A(y)$ is defined by

$$A(y,\kappa,\lambda) = y'' - g(y, y', \kappa, \lambda).$$

Let $(y_0(\xi), \kappa_0, \lambda_0)$ be a solution to (3.1)–(3.5) with $y_0(\xi) \in \mathfrak{B}_2^0$. The Fréchet derivative of F with respect to (y,κ) evaluated at this solution is equal to

$$D_{(y,\kappa)}F(y_0,\kappa_0,\lambda_0)(h,\Delta\kappa) = \left(L[y_0,\kappa_0,\lambda_0](h,\Delta\kappa), \nabla_k G(\kappa_0,\lambda_0)\cdot\Delta\kappa\right),$$

where $h \in \mathfrak{B}_2^0$, $\Delta\kappa \in \mathrm{R}^k$, and the operator $L[y_0,\kappa_0,\lambda_0]$ is the Fréchet derivative of $A(y,\kappa,\lambda)$. Explicitly,

$$\begin{aligned} L[y_0,\kappa_0,\lambda_0](h,\Delta\kappa) &= h'' - g'_z(y_0,y'_0,\kappa_0,\lambda_0)h' \\ &\quad - g'_y(y_0,y'_0,\kappa_0,\lambda_0)h - \nabla_\kappa g(y_0,y'_0,\kappa_0,\lambda_0)\cdot\Delta\kappa : \mathfrak{Y} \to \mathfrak{Z}. \end{aligned}$$

Let $L_{\mathrm{hom}}[y_0,\kappa_0,\lambda_0]h$ denote the "homogeneous part" of $L[y_0,\kappa_0,\lambda_0]$, i.e.

$$\begin{aligned} L_{\mathrm{hom}}[y_0,\kappa_0,\lambda_0]h &= h'' - g'_z(y_0,y'_0,\kappa_0,\lambda_0)h' \\ &\quad - g'_y(y_0,y'_0,\kappa_0,\lambda_0)h : \mathfrak{B}_2^0 \to \mathfrak{B}_0. \end{aligned} \tag{3.11}$$

The adjoint operator $L^*_{\mathrm{hom}}[y_0,\kappa_0,\lambda_0] : \mathfrak{B}_2^0 \to \mathfrak{B}_0$ is defined by:

$$\int\limits_{-\infty}^{+\infty} g(\xi)(L_{\mathrm{hom}}[y_0,\kappa_0,\lambda_0]h)(\xi)\,d\xi = \int\limits_{-\infty}^{+\infty} (L^*_{\mathrm{hom}}[y_0,\kappa_0,\lambda_0]g)(\xi)h(\xi)\,d\xi$$

for any two functions g and h from $\mathfrak{B}_2^0$ or, explicitly,

$$\begin{aligned} L^*_{\mathrm{hom}}[y_0,\kappa_0,\lambda_0]h &= h'' + g'_z(y_0,y'_0,\kappa_0,\lambda_0)h' \\ &\quad - \left[g'_y(y_0,y'_0,\kappa_0,\lambda_0) - \frac{d}{d\xi}g'_z(y_0,y'_0,\kappa_0,\lambda_0)\right]h. \end{aligned} \tag{3.12}$$

We have

PROPOSITION 1. [11] The equation

$$L_{\text{hom}}[y_0, \kappa_0, \lambda_0]h = 0 \tag{3.13}$$

has two linearly independent solutions of the class C^2:

$$h_1(\xi) = y_0'(\xi) \in \mathcal{B}_2^0 \qquad \text{and} \qquad h_2(\xi) = \vartheta(\xi), \tag{3.14}$$

where

$$\vartheta(\xi) = y_0'(\xi) \int\limits_0^\xi \frac{d\zeta}{y_0'^2(\zeta) q(\zeta)}, \tag{3.15}$$

with

$$q(\xi) = q(\xi, \kappa, \lambda) = \exp\left[- \int\limits_0^\xi g_z'(y_0(\zeta), y_0'(\zeta), \kappa, \lambda) d(\zeta) \right]. \tag{3.16}$$

PROPOSITION 2. In the class C^2, the equation $L^*_{\text{hom}}[y_0]h = 0$ has two linearly independent solutions which are

$$h_i^*(\xi) = h_i(\xi) q(\xi), \qquad i = 1, 2, \tag{3.17}$$

where $h_i(\xi)$ are given by (3.14), with $h_l^* \in \mathcal{B}_0^0$.

Proof. This result can be verified by a direct check.

PROPOSITION 3. The range of $L_{\text{hom}}[y_0, \kappa_0, \lambda_0]$ as an operator from $\mathcal{B}_2^0$ into $\mathcal{B}_0$ is

$$\mathcal{B}_0^\perp = \left\{ h \in \mathcal{B}_0 : \int\limits_{-\infty}^{+\infty} y_0'(\zeta) q(\zeta) h(\zeta) d\zeta = 0 \right\}.$$

Proof. The result follows immediately from the definition of the adjoint operator $L^*_{\text{hom}}[y_0, \kappa_0, \lambda_0]$ and Proposition 2. The proof is complete.

PROPOSITION 4. The equation

$$L_{\text{hom}}[y_0, \kappa_0, \lambda_0]h = f \tag{3.18}$$

has a solution in $\mathcal{B}_2^0$ if and only if $f \in \mathcal{B}_0^\perp$. The solution is unique and given by

$$h(\xi) = \int\limits_{-\infty}^{+\infty} K(\xi, \zeta) f(\zeta) d\zeta, \tag{3.19}$$

where

$$K(\xi, \zeta) = q(\zeta) \Big\{ [H(-\zeta) - H(\xi - \zeta)] \vartheta(\zeta) y_0'(\xi) + \frac{1}{2} [H(\xi - \zeta) - H(\zeta - \xi)] \vartheta(\xi) y_0'(\zeta) \Big\},$$

$H(x)$ is the Heaviside step function

$$H(x) = \begin{cases} 1 & \text{for} \quad x > 0, \\ 0 & \text{for} \quad x < 0. \end{cases}$$

P r o o f. The first part of the statement follows from Proposition 3, whereas the second one is the result of the theory of linear differential equations [11].

THEOREM 1. *Let the functions* $g(y, z, \kappa, \lambda)$ *and* $G(\kappa, \lambda)$ *satisfy Hypotheses* H1–H3, *and let* $y_0(\xi) \in \mathfrak{B}_2, \kappa_0 \in \mathrm{R}^k, \lambda_0 \in \mathrm{R}^l$ *be a solution to* (3.1)–(3.5). *If the determinant of the matrix*

$$D_\kappa(\kappa, \lambda) = \begin{pmatrix} \nabla_\kappa G_1(\kappa, \lambda) \\ \nabla_\kappa G_2(\kappa, \lambda) \\ \cdot \\ \cdot \\ \cdot \\ \nabla_\kappa G_{k-1}(\kappa, \lambda) \\ Q_\kappa(\kappa, , \lambda) \end{pmatrix}, \tag{3.20}$$

where

$$Q_\kappa(\kappa, \lambda) = \int\limits_{-\infty}^{+\infty} y'(\zeta, \lambda) q(\zeta, \kappa, \lambda) \nabla_\kappa g(y(\zeta, \lambda), y'(\zeta, \lambda), \kappa, \lambda) d\zeta,$$

evaluated at $y(\xi, \lambda) = y_0(\xi), \kappa = \kappa_0, \lambda = \lambda_0$, *is different from zero, then*

1. *Problem* (3.1)–(3.5) *has a unique solution* $y = y(\xi, \lambda), \kappa = \kappa(\lambda)$ *for* $\xi \in \mathrm{R}^l$ *and* λ *contained in a vicinity of* λ_0, *such that for any fixed value of* λ $y(\xi, \lambda) \in \mathfrak{B}_2^0$.

2. *These functions satisfy the equalities*

$$y(\xi, \lambda_0) = y_0(\xi), \qquad \kappa(\lambda_0) = \kappa_0.$$

3. *These functions are continuously differentiable* m *times with respect to* λ, *and the gradients* $\nabla_\lambda y(\xi, \lambda), \nabla_\lambda \kappa(\lambda)$ *are given by* (3.22) *and* (3.23), (3.24), *respectively.*

O u t l i n e   o f   t h e   p r o o f

According to the Inverse Function Theorem it is sufficient to prove that the Fréchet derivative $D_{(y,\kappa)}F(y_0, \kappa_0, \lambda_0)$ has an inverse. Indeed, let us take $(f, \varphi) \in \mathfrak{B}_0 \times \mathrm{R}^{k-1}$. We are looking for $(h, \Delta\kappa) \in \mathfrak{B}_2^0 \times \mathrm{R}^k$ such that $D_{(y,\kappa)}F(y_0, \kappa_0, \lambda_0)(h, \Delta\kappa) = (f, \varphi)$. Explicitly, this equation is equivalent to the following system of linear equations:

$$\begin{aligned} L[y_0, \kappa_0, \lambda_0](h, \Delta\kappa) &= f, \\ \nabla_\kappa G(\kappa_0, \lambda_0) \cdot \Delta\kappa &= \varphi. \end{aligned} \tag{3.21}$$

The first equation is equivalent to

$$L_{\mathrm{hom}}[y_0, \kappa_0, \lambda_0]h = \nabla_\kappa g(y_0, \kappa_0, \lambda_0) \cdot \Delta\kappa + f.$$

According to Proposition 4, this equation has a unique solution in $\mathcal{B}_2^0$ if and only if

$$\int_{-\infty}^{\infty} y_0'(\zeta) q(\zeta, \kappa_0, \lambda_0) \nabla_\kappa g(y_0(\zeta), y_0'(\zeta), \kappa_0, \lambda_0) d\zeta \cdot \Delta\kappa = -\int_{-\infty}^{\infty} y_0'(\zeta) q(\zeta, \kappa_0, \lambda_0) f(\zeta) d\zeta.$$

This equation together with $(3.21)_2$ constitute a system of k linear algebraic equations for k unknowns $\Delta\kappa$. It has a unique solution if and only if the determinant of the matrix (3.20) is different from zero.

From the Implicit Function Theorem we obtain the following expressions for the derivatives $\nabla_\lambda y(\xi, \lambda)$ and $\nabla_\lambda \kappa(\lambda)$:

$$\nabla_\lambda y(\xi, \lambda) = \int_{-\infty}^{+\infty} K(\xi, \zeta) \nabla_\kappa g(y(\zeta, \lambda), y'(\zeta, \lambda), \kappa(\lambda), \lambda) d\zeta \cdot \nabla_\lambda \kappa(\lambda) + \int_{-\infty}^{+\infty} K(\xi, \zeta) \nabla_\lambda g(y(\zeta, \lambda), y'(\zeta, \lambda), \kappa(\lambda), \lambda) d\zeta, \tag{3.22}$$

and

$$\nabla_\kappa G(\kappa(\lambda), \lambda) \cdot \nabla_\lambda \kappa(\lambda) = -\nabla_\lambda G(\kappa(\lambda), \lambda), \tag{3.23}$$

$$Q_\kappa(\kappa(\lambda), \lambda) \cdot \nabla_\lambda \kappa(\lambda) = -Q_\lambda(\kappa(\lambda), \lambda). \tag{3.24}$$

The proof is complete. ■

4. Applications to phase change problems

We consider the following limit value problem:
find a function $w = w(\xi), \xi \in \mathrm{R}^1$, *satisfying the differential equation*

$$A(w)w'' + \frac{1}{2}A_w'(w)w'^2 + s\mu(w, s, w_l)w' + f(w, s, w_l) = 0 \tag{4.1}$$

and the conditions (2.16)–(2.18).

Here, $A(w)$ is assumed to be a strictly positive and continuously differentiable function defined for all $w > b$, and $\mu(w, s, w_l)$, $f(w, s, w_l)$ are defined by (2.14), (2.15), respectively. Also s, w_r, w_l, etc. are the same as in Sec. 2.

We introduce the transformation

$$w \to y(w, w_r, w_l) = \frac{\int\limits_{w_l}^{w} \sqrt{A(\zeta)}d\zeta}{\int\limits_{w_l}^{w_r} \sqrt{A(\zeta)}d\zeta}. \tag{4.2}$$

Since $A(\zeta) > 0$, then $y'_w(w, w_r, w_l) > 0$ for $w_l \le w \le w_r$, or $y'_w(w, w_r, w_l) < 0$ for $w_r \le w \le w_l$. Hence, this transformation has the inverse $y \to W(y, w_r, w_l)$ such that

$$W(0, w_r, w_l) = w_l, \qquad W(l, w_r, w_l) = w_r. \tag{4.3}$$

By applying transformation (4.2) to Eq. (4.1) we obtain for y an equation of the type (3.1) with

$$g(y, y', s, w_r, w_l) = -\frac{s\mu(W(y, w_r, w_l), s, w_l)}{A(W((y, w_r, w_l))}y' - \frac{f(W(y, w_r, w_l), w, w_l)}{\sqrt{A(W(y, w_r, w_l))}\int\limits_{w_l}^{w_r} \sqrt{A(\zeta)}d\zeta}. \tag{4.4}$$

We check easily that $g(y, y', s, w_r, w_l)$ as defined by (4.4) satisfies Hypotheses H1, H2 formulated in the previous section. Let us also notice that (4.2) transforms the limit conditions (2.16)–(2.18) into (3.2)–(3.4), and (3.10) is a counterpart of

$$w(\xi = 0) = \frac{1}{2}(w_r + w_l). \tag{4.5}$$

We take w_l as the independent parameter λ of Sec. 3, and as the dependent parameters κ we take (s, w_r); the function $G(\kappa, \lambda)$ is assumed in the form:

$$G(s, w_r, w_l) = p(w_r, u_l - s(w_r - w_l)) - p(w_l, u_l) + s^2(w_r - w_l). \tag{4.6}$$

Then the equation $G(s, w_r, w_l) = 0$ expresses the Rankine-Hugoniot condition (2.19). The other parameters such as u_l, a, b are assumed to be fixed.

We can apply now the theory developed in the previous section to the present case of g given by (4.4), G defined by (4.6), and $k = 2$, $l = 1$, assuming of course that we know a solution $w_0(\xi)$, s_0, w_r^0, w_l^0 of (4.1) and (2.16)–(2.19), or equivalently, $y_0(\xi)$, s_0, w_r^0, w_l^0 of (3.1)–(3.5). Having done that we have to retransform the condition $D_\kappa \neq 0$ back to $w = W(y, w_r, w_l)$. However, we resign of doing that because we would obtain very complicated formulae. That is why we limit ourselves to the simpler but physically the most important case when the parameters s, w_r, w_l are near the Maxwell line. This is a particular phase

The first equation is equivalent to

$$L_{\mathrm{hom}}[y_0, \kappa_0, \lambda_0]h = \nabla_\kappa g(y_0, \kappa_0, \lambda_0) \cdot \Delta\kappa + f.$$

According to Proposition 4, this equation has a unique solution in $\mathfrak{B}_2^0$ if and only if

$$\int_{-\infty}^{\infty} y_0'(\zeta) q(\zeta, \kappa_0, \lambda_0) \nabla_\kappa g(y_0(\zeta), y_0'(\zeta), \kappa_0, \lambda_0) d\zeta \cdot \Delta\kappa$$

$$= - \int_{-\infty}^{\infty} y_0'(\zeta) q(\zeta, \kappa_0, \lambda_0) f(\zeta) d\zeta.$$

This equation together with $(3.21)_2$ constitute a system of k linear algebraic equations for k unknowns $\Delta\kappa$. It has a unique solution if and only if the determinant of the matrix (3.20) is different from zero.

From the Implicit Function Theorem we obtain the following expressions for the derivatives $\nabla_\lambda y(\xi, \lambda)$ and $\nabla_\lambda \kappa(\lambda)$:

$$\nabla_\lambda y(\xi, \lambda) = \int_{-\infty}^{+\infty} K(\xi, \zeta) \nabla_\kappa g(y(\zeta, \lambda), y'(\zeta, \lambda), \kappa(\lambda), \lambda) d\zeta \cdot \nabla_\lambda \kappa(\lambda)$$
$$+ \int_{-\infty}^{+\infty} K(\xi, \zeta) \nabla_\lambda g(y(\zeta, \lambda), y'(\zeta, \lambda), \kappa(\lambda), \lambda) d\zeta, \tag{3.22}$$

and

$$\nabla_\kappa G(\kappa(\lambda), \lambda) \cdot \nabla_\lambda \kappa(\lambda) = -\nabla_\lambda G(\kappa(\lambda), \lambda), \tag{3.23}$$

$$Q_\kappa(\kappa(\lambda), \lambda) \cdot \nabla_\lambda \kappa(\lambda) = -Q_\lambda(\kappa(\lambda), \lambda). \tag{3.24}$$

The proof is complete. ■

4. Applications to phase change problems

We consider the following limit value problem:

find a function $w = w(\xi), \xi \in \mathrm{R}^1$, *satisfying the differential equation*

$$A(w)w'' + \frac{1}{2} A_w'(w) w'^2 + s\mu(w, s, w_l) w' + f(w, s, w_l) = 0 \tag{4.1}$$

and the conditions (2.16)–(2.18).

Here, $A(w)$ is assumed to be a strictly positive and continuously differentiable function defined for all $w > b$, and $\mu(w, s, w_l)$, $f(w, s, w_l)$ are defined by (2.14), (2.15), respectively. Also s, w_r, w_l, etc. are the same as in Sec. 2.

We introduce the transformation

$$w \to y(w, w_r, w_l) = \frac{\int\limits_{w_l}^{w} \sqrt{A(\zeta)}d\zeta}{\int\limits_{w_l}^{w_r} \sqrt{A(\zeta)}d\zeta}. \tag{4.2}$$

Since $A(\zeta) > 0$, then $y'_w(w, w_r, w_l) > 0$ for $w_l \leq w \leq w_r$, or $y'_w(w, w_r, w_l) < 0$ for $w_r \leq w \leq w_l$. Hence, this transformation has the inverse $y \to W(y, w_r, w_l)$ such that

$$W(0, w_r, w_l) = w_l, \qquad W(l, w_r, w_l) = w_r. \tag{4.3}$$

By applying transformation (4.2) to Eq. (4.1) we obtain for y an equation of the type (3.1) with

$$g(y, y', s, w_r, w_l) = -\frac{s\mu(W(y, w_r, w_l), s, w_l)}{A(W((y, w_r, w_l))}y' - \frac{f(W(y, w_r, w_l), w, w_l)}{\sqrt{A(W(y, w_r, w_l))}\int\limits_{w_l}^{w_r} \sqrt{A(\zeta)}d\zeta}. \tag{4.4}$$

We check easily that $g(y, y', s, w_r, w_l)$ as defined by (4.4) satisfies Hypotheses H1, H2 formulated in the previous section. Let us also notice that (4.2) transforms the limit conditions (2.16)–(2.18) into (3.2)–(3.4), and (3.10) is a counterpart of

$$w(\xi = 0) = \frac{1}{2}(w_r + w_l). \tag{4.5}$$

We take w_l as the independent parameter λ of Sec. 3, and as the dependent parameters κ we take (s, w_r); the function $G(\kappa, \lambda)$ is assumed in the form:

$$G(s, w_r, w_l) = p(w_r, u_l - s(w_r - w_l)) - p(w_l, u_l) + s^2(w_r - w_l). \tag{4.6}$$

Then the equation $G(s, w_r, w_l) = 0$ expresses the Rankine-Hugoniot condition (2.19). The other parameters such as u_l, a, b are assumed to be fixed.

We can apply now the theory developed in the previous section to the present case of g given by (4.4), G defined by (4.6), and $k = 2$, $l = 1$, assuming of course that we know a solution $w_0(\xi)$, s_0, w_r^0, w_l^0 of (4.1) and (2.16)–(2.19), or equivalently, $y_0(\xi)$, s_0, w_r^0, w_l^0 of (3.1)–(3.5). Having done that we have to retransform the condition $D_\kappa \neq 0$ back to $w = W(y, w_r, w_l)$. However, we resign of doing that because we would obtain very complicated formulae. That is why we limit ourselves to the simpler but physically the most important case when the parameters s, w_r, w_l are near the Maxwell line. This is a particular phase

It is reasonable due to the physical reasons to introduce the characteristic speeds $c_\pm(w,u)$ being an extension of the notion of the sound speed to general hyperbolic systems. In our case they are defined as the real solutions (if they do exist) of the quadratic equation [10]

$$c^2 - cp'_u(w,u) + p'_w(w,u) = 0.$$

We have

$$p'_w(w,u) = c_-(w,u)c_+(w,u). \tag{4.20}$$

Using (4.17) and (4.20) in (4.15), (4.16) we obtain

i) if $w_l = w_m$:

$$\left[\int\limits_{w_m}^{w_M} \mu(\zeta,u_l)\sqrt{-\frac{2}{A(\zeta)}\int\limits_{w_m}^{\zeta} f(\xi,0,w_m)d\xi d\zeta} - \int\limits_{w_m}^{w_M}(\zeta - w_m)p'_u(\zeta,u_l)d\zeta\right]\left.\frac{ds}{dw_l}\right|_{w_l=w_m} = (w_M - w_m)c_-(w_m,u_l)c_+(w_m,u_l), \tag{4.21$_1$}$$

$$\left.\frac{dw_r}{dw_l}\right|_{w_l=w_m} = \frac{c_-(w_m,u_l)c_+(w_m,u_l)}{c_-(w_M,u_l)c_+(w_M,u_l)}; \tag{4.22$_1$}$$

and

ii) if $w_l = w_M$,

$$\left[\int\limits_{w_m}^{w_M} \mu(\zeta,u_l)\sqrt{-\frac{2}{A(\zeta)}\int\limits_{w_m}^{\zeta} f(\xi,0,w_m)d\xi d\zeta} - \int\limits_{w_m}^{w_M}(w_M - \zeta)p'_u(\zeta,u_l)d\zeta\right]\left.\frac{ds}{dw_l}\right|_{w_l=w_M} = -(w_M - w_m)c_-(w_M,u_l)c_+(w_M,u_l), \tag{4.21$_2$}$$

$$\left.\frac{dw_r}{dw_l}\right|_{w_l=w_M} = \frac{c_-(w_M,u_l)c_+(w_M,u_l)}{c_-(w_m,u_l)c_+(w_m,u_l)} + \left(\frac{1}{c_-(w_M,u_l)} + \frac{1}{c_+(w_M,u_l)}\right)(w_M - w_m)\left.\frac{ds}{dw_l}\right|_{w_l=w_m}. \tag{4.22$_2$}$$

Let us notice that, in general, the coefficient of ds/dw_l in (4.21) can vanish for some value u_l^* of u_l. Unfortunately it is difficult to determine all such critical values of this parameter due to the complexity of the equation resulting from equating this coefficient to zero. That is why we limit ourselves to two particular, but important, cases for which we can explain this problem.

EXAMPLE 1. In many papers ([1 – 4, 6]), so-called *isothermal phase transitions* were discussed. In this case

$$p'_u(w,u) \equiv 0. \tag{4.23}$$

Due to that the problem of the critical values of u_l does not exist and we obtain from (4.21)

$$\left.\frac{ds}{dw_l}\right|_{w_l=w_m} = -\frac{(w_M - w_m)c^2(w_m)}{\int\limits_{w_m}^{w_M} \mu(\zeta,u_l)\sqrt{-\frac{2}{A(\zeta)}\int\limits_{w_m}^{\zeta} f(\xi,0,w_m)d\xi d\zeta}}, \tag{4.24$_1$}$$

$$\left.\frac{dw_r}{dw_l}\right|_{w_l=w_m} = \frac{c^2(w_m)}{c^2(w_M)}, \tag{4.25$_1$}$$

or

$$\left.\frac{ds}{dw_l}\right|_{w_l=w_M} = \frac{(w_M - w_m)c^2(w_M)}{\int\limits_{w_m}^{w_M} \mu(\zeta,u_l)\sqrt{-\frac{2}{A(\zeta)}\int\limits_{w_M}^{\zeta} f(\xi,0,w_m)d\xi d\xi}}, \tag{4.24$_2$}$$

$$\left.\frac{dw_r}{dw_l}\right|_{w_l=w_M} = \frac{c^2(w_M)}{c^2(w_m)}. \tag{4.25$_2$}$$

Here, we made use of the fact that in the isothermal case

$$c_+(w,u) = c(w) = -c_-(w,u),$$

where

$$c(w) = \sqrt{-p'_w(w)}.$$

Formulae (4.24), (4.25) generalise the corresponding expressions obtained by TRUSKINOVSKY [2], who assumed additionally that $A(w) = \text{const}$ and $\mu(w,u) = \text{const}$.

EXAMPLE 2. *The model equations of hydrodynamics* [10, 11]. In this case $p(w,u)$ is given by Eq. (2.3). We have

$$-\int_{w_m}^{w_M} (\zeta - w_m) p'_u(\zeta, u_l) d\zeta = u_l \int_{w_m}^{w_M} \frac{\zeta - w_m}{\zeta - b} d\zeta,$$
$$-\int_{w_m}^{w_M} (w_M - \zeta) p'_u(\zeta, u_l) d\zeta = u_l \int_{w_m}^{w_M} \frac{w_M - \zeta}{\zeta - b} d\zeta.$$

We see that in both cases the coefficient of ds/dw_l is positive for $u_l \geq 0$. Hence, it remains positive for negative (but sufficiently close to zero) values of this parameter. Unfortunately, we are unable to say whether the discussed coefficient can vanish for some negative u_l. Consequently, we can claim only that, *at least for small values of* $|u_l|$ *and for* w_l *close to* $w_m = w_m(u_l)$ *or* $w_M(u_l)$ (we remind that the solutions of (4.7), (4.8) depend on u_l), *the solution to the problem* (2.13)–(2.19) *exists, is unique and is differentiable with respect to* w_l.

Of course, we can take the right state of equilibrium w_r or the speed s as the independent parameter and (s, w_l) or, respectively, (w_l, w_r) as the dependent ones and obtain a similar theorem. But from our theory we can deduce more. Namely, we have $\left.\frac{ds}{dw_l}\right|_{w_l=w_m} > 0$ and $\left.\frac{ds}{dw_l}\right|_{w_l=w_M} < 0$, at least in the considered examples. Also we can use the Taylor formula, as we have proved the existence of all the necessary derivatives, to obtain

$$s(w_l) = \begin{cases} (w_l - w_m) \left.\dfrac{ds}{dw_l}\right|_{w_l=w_m} + O\left((w_l - w_m)^2\right), \\[2ex] (w_l - w_M) \left.\dfrac{ds}{dw_l}\right|_{w_l=w_M} + O\left((w_l - w_M)^2\right). \end{cases} \tag{4.26}$$

This is the so-called "normal growth" approximation [2] introduced intuitively on physical grounds.

In this way we obtain the following conclusions:

The speed of the phase boundary is positive if either $w_l < w_m$ *and* w_l *is close to* w_m *(condensation), or* $w_l > w_M$ *and* w_l *is close to* w_M *(evaporation).*

The speed of the phase boundary is negative if either $w_l > w_m$ *and* w_l *is close to* w_m *(evaporation), or* $w_l < w_M$ *and* w_l *is close to* w_M *(condensation).*

The above results constitute an extension of a theorem proved by SHEARER [6].

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POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH
e-mail: kpiechor@ippt.gov.pl

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