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Receding contact between an orthotropic layer and an orthotropic half-space

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THE PROBLEM of smooth receding contact between an orthotropic layer and an orthotropic half-space has been considered in this papaer. The paper includes a generalization of the results of the paper by L. M. KEER, J. DUNDURS, K. C. TSAI [4] concerning receding contact between an isotropic layer and an isotropic half-space. It is observed that the task of finding the extent of contact in the loaded configuration can be reduced to an eigenvalue problem of a homogenous Fredholm integral equation. The kernel of the Fredholm integral equation is found to be dependent on the elastic constants of the layer but is independent of the elastic constants of the half-space below, which is in contrast to the study for the isotropic materials, where the kernel is independent of the elastic constants. Finally some numerical results have been presented in graphs in order to compare the results of interest for isotropic and orthotropic materials.

1. Introduction

WHEN TWO ELASTIC media are in contact, the determination of the state of stress in the media has been the subject of study in literature for many years, and the problems are usually termed as contact problems. Usually the contact problems are of two types:

(a) the bodies in contact are bonded together and consequently, the contact regions are known *a priori*, and the main task is to determine the stress distribution in the media;

(b) the bodies are in contact without bond so that the region of contact is not known. In such class of problems determination of the contact region (which depends upon geometric features of the bodies as well as upon the load distribution on the system) becomes an additional task barring the stress distribution in the system. In 1970, (DUNDURS and STIPPES [2]) observed the role of elastic constants in unbonded frictionless contact of two elastically isotropic bodies. They classified the contact between two bodies into three categories, viz. (i) receding contact, (ii) stationary contact, (iii) advancing contact. Clearly, in a contact problem of two unbonded media, the initial region of contact does not always remain the same when deformation occurs in the body. If R' represents the contact region which was initially R, then the contact is receding if $R' \subset R$, stationary if $R' = R$ and advancing if $R' \supset R$. It has been observed that in linear elastostatics, the problems of receding contact have several simplifying features compared to the advancing contacts. The most important fact is that the extent of receding contact is independent of the level of loading, which implies, in turn, that if uplift is to occur, the change from initial contact to the contact in the loaded configuration takes place discontinuously. Among the studies of receding contact between two bodies recorded in literature, mention may be made of the works of WEITSMAN [8], PU and HUSSAIN [7], WILSON and GOREE [9], KEER, DUNDURS and TSAI [4] and more recently of LI and DEMPSEY [6]. In all the above papers the discussion is limited to isotropic media only.

The aim of the present paper is to study the properties of smooth receding contact when an orthotropic layer is resting unbonded on an orthotropic half-space and to provide information regarding the extent of the contact region. As in [4], it is observed that the task of finding the extent of contact in the loaded configuration can be reduced to an eigenvalue problem of a homogeneous Fredholm integral equation which may be solved numerically to any desired degree of accuracy, when loading intervals are not very large in comparison to the thickness of the layer.

One notable feature in the present study is that the kernel of the Fredholm integral equations is dependent on the elastic constants of the layer but is independent of the elastic constants of the half-space below, whereas in the isotropic media the kernel is totally independent of the elastic constants. Finally, all the results for the isotropic media can be obtained from the corresponding results of our discussion by assigning limiting values to the elastic parameters.

2. The basic equations

For an orthotropic elastic medium, we choose the Cartesian coordinate axes to coincide with the principal material axes and define the plane strain state by

$$u_x = u(x,z), \quad u_y = 0, \quad u_z = w(x,z)\,, \tag{2.1}$$

where u_x, u_y, u_z are the displacements along x, y and z directions, respectively. The stress-displacement relations are

$$\begin{aligned}
\sigma_{xx} &= A_{11}\frac{\partial_x}{\partial x} + A_{13}\frac{\partial u_z}{\partial z},\\
\sigma_{zz} &= A_{13}\frac{\partial u_x}{\partial x} + A_{33}\frac{\partial u_z}{\partial z},\\
\sigma_{xz} &= A_{55}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right),\\
\sigma_{yz} &= 0 = \sigma_{xy},
\end{aligned} \tag{2.2}$$

where A_{ij} are anisotropic constants of the orthotropic material. In terms of the displacement components the equations of equilibrium are

$$(2.3)\qquad \begin{aligned} A_{11}\frac{\partial^2 u_x}{\partial x^2} + A_{55}\frac{\partial^2 u_x}{\partial z^2} + (A_{13}+A_{55})\frac{\partial^2 u_z}{\partial x \partial z} = 0,\\ A_{55}\frac{\partial^2 u_z}{\partial x^2} + A_{33}\frac{\partial^2 u_z}{\partial z^2} + (A_{55}+A_{13})\frac{\partial^2 u_x}{\partial x \partial z} = 0. \end{aligned}$$

At this stage we decide to represent the solution u_x, u_z of Eqs. (2.3) in terms of potential functions. Obviously, the usual potential representation of displacement components in isotropic medium will not be suitable here. However, we can employ the method discussed in LEKHNITSKII [5] to represent u_x, u_z in terms of potential functions as follows:

Let us seek a solution of Eqs. (2.3) in the form

$$(2.4)\qquad u_x = \frac{\partial \phi_0}{\partial x}, \qquad u_z = \lambda \frac{\partial \phi_0}{\partial z},$$

where λ is a constant and $\phi_0 \equiv \phi_0(x,z)$. Equations (2.3) are satisfied if

$$(2.5)\qquad \frac{A_{55}+\lambda(A_{13}+A_{55})}{A_{11}} = \frac{\lambda A_{33}}{\lambda A_{55}+A_{13}+A_{55}} = \delta^2,$$

where δ^2 is another constant.

From the relation (2.5) we get two quadratic equations, one in λ and the other in δ^2. If λ_1, λ_2 and δ_1^2, δ_2^2 are the roots of the quadratic equations

$$(2.6)\qquad \lambda^2 A_{55}(A_{13}+A_{55}) + \lambda\left[A_{13}+A_{55})^2 + A_{55}^2 - A_{11}A_{33}\right] + A_{55}(A_{13}+A_{55}) = 0,$$

$$(2.7)\qquad \delta^4 A_{11}A_{55} + \delta^2\left[(A_{13}+A_{55})^2 - A_{55}^2 - A_{11}A_{33}\right] + A_{33}A_{55} = 0,$$

respectively, then we have two potential functions $\phi(x,z)$ and $\psi(x,z)$ satisfying the differential equations

$$(2.8)\qquad \begin{aligned} \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z_1^2} &= 0,\\ \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z_2^2} &= 0, \end{aligned}$$

where

$$(2.9)\qquad z_1 = z/\delta_1, \qquad z_2 = z/\delta_2.$$

Hence the displacements and stresses can be expressed in terms of ϕ and ψ as

$$u_x = \frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial x},$$

(2.10)

$$u_z = \frac{\lambda_1}{\delta_1}\frac{\partial\phi}{\partial z_1} + \frac{\lambda_2}{\delta_2}\frac{\partial\psi}{\partial z_2},$$

$$\frac{\sigma_{xx}}{A_{55}} = \frac{1+\lambda_1}{\delta_1^2}\frac{\partial^2\phi}{\partial x^2} + \frac{1+\lambda_2}{\delta_2^2}\frac{\partial^2\psi}{\partial x^2},$$

(2.11)

$$\frac{\sigma_{zz}}{A_{55}} = (1+\lambda_1)\frac{\partial^2\phi}{\partial z_z^2} + (1+\lambda_2)\frac{\partial^2\psi}{\partial z_2^2},$$

$$\frac{\sigma_{xz}}{A_{55}} = \frac{1+\lambda_1}{\delta_1}\frac{\partial^2\phi}{\partial x\partial z_1} + \frac{1+\lambda_2}{\delta_2}\frac{\partial^2\psi}{\partial x\partial z_2}.$$

3. Formulation and solution of the problem

We consider the plane elastostatic problem in which an unbonded layer of thickness h is in smooth contact with a semi-infinite base. The coordinate system used in relation to the two bodies is shown in Fig. 1. The layer is pressed against the base by normal tractions distributed over the segment $-a \le x \le a$ on the top surface of the layer, while the rest of the surface is free from tractions. The layer is in contact with the base below over the segment $-c \le x \le c$ in the deformed state while separation takes place outside this interval.

For simplicity, we consider only the case where the applied normal tractions are symmetric with respect to the centre of the loaded segment. Integral solutions of Eq. (2.8) which will be appropriate for the present problem can be written in the following forms:

$$\phi^{(1)} = \int_0^\infty \left[A(\xi)\text{sh}\left\{\xi(z+h)/\delta_1^{(1)}\right\} + B(\xi)\text{ch}\left\{\xi(z+h)/\delta_1^{(1)}\right\}\right]\cos\xi x\, d\xi\,, \tag{3.1}$$

$$\psi^{(1)} = \int_0^\infty \xi^{-1}\left[C(\xi)\text{sh}\left\{\xi(z+h)/\delta_2^{(1)}\right\} + D(\xi)\text{ch}\left\{\xi(z+h)/\delta_2^{(1)}\right\}\right] \cdot\cos\xi x\, d\xi, \qquad -h \le z \le 0\,, \tag{3.2}$$

and

$$\phi^{(2)} = \int_0^\infty E(\xi)e^{-\xi z/\delta_1^{(2)}}\cos\xi x\, d\xi\,, \tag{3.3}$$

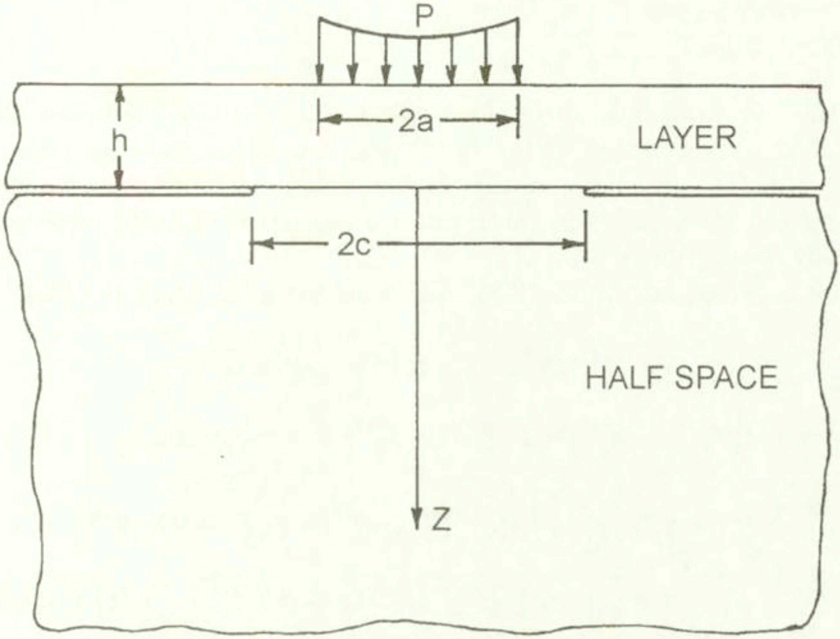

FIG. 1. Layer pressed against a half space.

$$\psi^{(2)} = \int_0^\infty F(\xi)\xi^{-1} e^{-\xi z/\delta_2^{(2)}} \cos \xi x \, d\xi, \qquad z \geq 0, \tag{3.4}$$

where $A(\xi)$, $B(\xi)$, $C(\xi)$, $D(\xi)$, $E(\xi)$ and $F(\xi)$ are arbitrary functions, to be determined from the boundary conditions.

The boundary conditions for the problem are

$$\sigma_{xz}^{(1)}(x, -h) = 0, \qquad 0 \leq |x| < \infty ; \tag{3.5}$$

$$\sigma_{zz}^{(1)}(x, -h) = \begin{cases} -p(x) = -\dfrac{P}{\pi} f(x), & 0 \leq |x| \leq a , \\ 0, & a < |x| < \infty; \end{cases} \tag{3.6}$$

$$\sigma_{xz}^{(1)}(x, 0) = 0, \qquad 0 \leq |x| < \infty; \tag{3.7}$$

$$\sigma_{xz}^{(2)}(x, 0) = 0, \qquad 0 \leq |x| < \infty ; \tag{3.8}$$

$$\sigma_{zz}^{(1)}(x, 0) = \sigma_{zz}^{(2)}(x, 0) \leq 0, \qquad 0 \leq |x| < \infty; \tag{3.9}$$

$$\sigma_{zz}^{(2)}(x, 0) = 0, \qquad c < |x| < \infty; \tag{3.10}$$

$$\frac{\partial u_z^{(1)}(x,0)}{\partial x} = \frac{\partial u_z^{(2)}(x,0)}{\partial x}, \quad 0 \le |x| < c, \tag{3.11}$$

where the uknown length of contact between the layer and the base is $2c$. The superscripts 1 and 2 in Eqs. (3.1) – (3.11) refer to the layer and base, respectively. As the normal displacements in the contact region are continuous, we write the boundary condition (3.11) in the form of a derivative to avoid involvement of rigid body displacements.

From boundary conditions (3.5) – (3.9) and using (2.11), (3.1) – (3.4), we get

$$\xi A(\xi)\,(1+\lambda_1^{(1)})/\delta_1^{(1)} = -C(\xi)\,(1+\lambda_2^{(1)})/\delta_2^{(1)}, \tag{3.12}$$

$$\xi B(\xi)\,(1+\lambda_1^{(1)}) + D(\xi)\,(1+\lambda_2^{(1)}) = -\frac{1}{\xi A_{55}^{(1)}}\hat{p}(\xi), \tag{3.13}$$

$$\begin{aligned}\left\{\xi(1+\lambda_1^{(1)})/\delta_1^{(1)}\right\}\left[A(\xi)\,\mathrm{ch}(\xi h/\delta_1^{(1)}) + B(\xi)\,\mathrm{sh}(\xi h/\delta_1^{(1)})\right]\\ = -\left\{(1+\lambda_2^{(1)})/\delta_2^{(1)}\right\}\left[C(\xi)\,\mathrm{ch}(\xi h/\delta_2^{(1)}) + D(\xi)\,\mathrm{sh}(\xi h/\delta_2^{(1)})\right],\end{aligned} \tag{3.14}$$

$$\xi E(\xi)(1+\lambda_1^{(2)})/\delta_1^{(2)} = -F(\xi)\,(1+\lambda_2^{(2)})/\delta_2^{(2)}, \tag{3.15}$$

$$\begin{aligned}\xi\Big[(1+\lambda_1^{(1)})\left\{A(\xi)\,\mathrm{sh}(\xi h/\delta_1^{(1)}) + B(\xi)\,\mathrm{ch}(\xi h/\delta_1^{(1)})\right\}\\ -\zeta(1+\lambda_1^{(2)})E(\xi)\Big] = -(1+\lambda_2^{(1)})\left\{C(\xi)\,\mathrm{sh}(\xi h/\delta_2^{(1)})\right.\\ \left. +D(\xi)\,\mathrm{ch}(\xi h/\delta_2^{(1)})\right\} + \zeta(1+\lambda_2^{(2)})F(\xi),\end{aligned} \tag{3.16}$$

where

$$\hat{p}(\xi) = (2/\pi)\int_0^a p(x)\,\cos\xi x\,dx, \tag{3.17}$$

$$\zeta = A_{55}^{(2)}/A_{55}^{(1)}. \tag{3.18}$$

Equation (3.17) is the Fourier cosine transform of the normal tractions acting on the top surface of the layer. Using Eqs. (3.12), (3.13), (3,15) we get from (3.14) and (3.16)

$$\begin{aligned}A(\xi) = \left\{\delta_1^{(1)}/L_1(\xi)\right\}\Big[\left\{\hat{p}(\xi)\,\mathrm{sh}(\xi h/\delta_2^{(1)})\right\}\\ /\left\{\xi^2 A_{55}^{(1)}\delta_2^{(1)}(1+\lambda_1^{(1)})\right\} + B(\xi)L_2(\xi)\Big],\end{aligned} \tag{3.19}$$

$$B(\xi) = \frac{1}{(1+\lambda_1^{(1)})L_B(\xi)}\left[\zeta\eta_1(1+\lambda_1^{(2)})E(\xi)L_1(\xi) + \frac{\hat{p}(\xi)L_4(\xi)}{\xi^2 A_{55}^{(1)}}\right], \tag{3.20}$$

where

$$L_1(\xi) = \mathrm{ch}(\xi h/\delta_1^{(1)}) - \mathrm{ch}(\xi h/\delta_2^{(1)}), \tag{3.21}$$

$$L_2(\xi) = (1/\delta_2^{(1)})\mathrm{sh}(\xi h/\delta_2^{(1)}) - (1/\delta_1^{(1)})\mathrm{sh}(\xi h/\delta_1^{(1)}), \tag{3.22}$$

$$L_3(\xi) = 2 - 2\mathrm{ch}(\xi h/\delta_1^{(1)})\mathrm{ch}(\xi h/\delta_2^{(1)}) + (\delta_1^{(1)}/\delta_2^{(1)} + \delta_2^{(1)}/\delta_1^{(1)})\mathrm{sh}(\xi h/\delta_1^{(1)})\mathrm{sh}(\xi h//\sigma_2^{(1)}), \tag{3.23}$$

$$L_4(\xi) = \mathrm{ch}(\xi h/\delta_1^{(1)})\mathrm{ch}(\xi h/\delta_2^{(1)}) - (\delta_1^{(1)}/\delta_2^{(1)})\mathrm{sh}(\xi h/\delta_1^{(1)})\mathrm{sh}(\xi h/\delta_2^{(1)}) - 1, \tag{3.24}$$

$$\eta_1 = 1 - (\delta_2^{(2)}/\delta_1^{(2)}). \tag{3.25}$$

From boundary condition (3.10) and Eq. (3.15), we obtain

$$\int_0^\infty \xi^2 E(\xi) \cos\xi x \, d\xi = 0, \qquad c < |x| < \infty \, . \tag{3.26}$$

Using the result [3]

$$\int_0^\infty J_0(\xi t) \ \cos \ \xi x \ d\xi = \begin{cases} 0, & 0 < t < x, \\ 1/\sqrt{t^2 - x^2}, & 0 < x < t \end{cases} \tag{3.27}$$

Eq. (3.26) is identically satisfied if we take

$$E(\xi) = -\xi^{-2} \int_0^c M(t) J_0(\xi t) \, dt, \tag{3.28}$$

where $M(t)$ is an unknown function which will be determined so as to satisfy the last boundary condition (3.11). Using Eqs. (3.12), (3.13), (3.15), (3.19), (3.20) and (3.28), the boundary condition (3.11) reduces, after some calculations, to

$$\int_0^c M(t) \int_0^\infty T_1(\xi) J_0(\xi t) \ \sin\xi x \ d\xi \ dt = \eta_2 \int_0^\infty T_2(\xi) \ \hat{p}(\xi) \ \sin \ \xi x \ d\xi \ , \tag{3.29}$$

where

$$\eta_2 = \frac{\left(\lambda_1^{(1)}/\delta_1^{(1)}\right) - \left[\left\{\lambda_2^{(1)}(1+\lambda_1^{(1)})\right\} / \left\{\delta_1^{(1)}(1+\lambda_2^{(1)})\right\}\right]}{(\lambda_1^{(2)}/\delta_1^{(2)}) - \left[\left\{\lambda_2^{(2)}(1+\lambda_1^{(2)})\right\} / \left\{\delta_1^{(2)}(1+\lambda_2^{(2)})\right\}\right]}, \tag{3.30}$$

$$T_1(\xi) = 1 + \frac{\zeta\eta_1\eta_2(1+\lambda_1^{(2)})}{(1+\lambda_1^{(1)})L_3(\xi)}\left[\left(\delta_1^{(1)}/\delta_2^{(1)}\right)\mathrm{sh}\left(\xi h/\delta_2^{(1)}\right)\right.$$
$$\left.\mathrm{ch}\left(\xi h/\delta_1^{(1)}\right) - \mathrm{sh}(\xi h/\delta_1^{(1)})\mathrm{ch}(\xi h/\delta_2^{(1)})\right], \tag{3.31}$$

$$T_2(\xi) = \left\{\delta_1^{(1)}L_2(\xi)\right\}/\left\{A_{55}^{(1)}\left(1+\lambda_1^{(1)}\right)L_3(\xi)\right\}. \tag{3.32}$$

We observe that the integrals in Eq. (3.29) have singularities at $\xi = 0$ which poses the convergence problem for the integrals. This difficulty can be eliminated by taking advantage of the fact that the load applied over the interval $|x| \le c$ on the top surface of the layer is balanced by the contact pressure acting over $|x| < c$ on the bottom surface. So from Eq. (2.11) and using (3.28), we get

$$\frac{\sigma_{zz}^{(2)}(x,\,0)}{A_{55}^{(2)}} = -\eta_1\left(1+\lambda_1^{(2)}\right)\int_0^c M(t)\int_0^\infty J_0(\xi t)\ \cos\ \xi x\ d\xi\ dt\,. \tag{3.33}$$

Using (3.27), Eq. (3.33) reduced to

$$\frac{\sigma_{zz}^{(2)}(x,\,0)}{A_{55}^{(2)}} = -\eta_1\left(1+\lambda_1^{(2)}\right)\int_x^c \frac{M(t)}{\sqrt{t^2-x^2}}\,dt, \quad 0 \le x < c\,. \tag{3.34}$$

From Eq. (3.34) it is clear that if $M(t)$ is finite in $[0, c]$, the contact pressure vanishes as $x \to c$.

The condition of equilibrium is given by

$$\int_0^c \sigma_{zz}^{(2)}(x,\,0)\mathrm{d}x = -\int_0^a p(x)\ \mathrm{d}x = -\frac{1}{2}P\,. \tag{3.35}$$

Substituting Eq. (3.34) into (3.35) we get

$$\eta_1 A_{55}^{(2)}\left(1+\lambda_1^{(2)}\right)\int_0^c M(t)dt = \frac{2}{\pi}\int_0^a p(x)\ dx$$
$$= \hat{p}(0) = \frac{P}{\pi}\hat{f}(0). \tag{3.36}$$

Rewrite Eq. (3.29) in the form

$$(1+\eta_3\eta_4)\int_0^c M(t)\int_0^\infty J_0(\xi t)\ \sin\ \xi x\ d\xi\ dt\ d\xi + \frac{\zeta\eta_1\eta_2\left(1+\lambda_1^{(2)}\right)}{\left(1+\lambda_1^{(1)}\right)}$$
$$\cdot\int_0^c M(t)\int_0^\infty [\{L_5(\xi)/L_3(\xi)\} - \eta_3]\,J_0(\xi t)\times\sin\ \xi x\ d\xi\ dt$$
$$-\eta_2\int_0^\infty T_2(\xi)\hat{p}(\xi)\ \sin\xi x\ dx\ d\xi = 0, \tag{3.37}$$

where

$$\eta_3 = \left\{1 - \left(\delta_2^{(1)}/\delta_1^{(1)}\right)\right\}^{-1}, \tag{3.38}$$

$$\eta_4 = \zeta\eta_1\eta_2\left(1+\lambda_1^{(2)}\right)/\left(1+\lambda_1^{(1)}\right), \tag{3.39}$$

$$L_5(\xi) = \left(\delta_1^{(1)}/\delta_2^{(1)}\right)\,\mathrm{sh}\left(\xi h/\delta_2^{(1)}\right)\,\mathrm{ch}\left(\xi h/\delta_1^{(1)}\right) - \mathrm{sh}\left(\xi h/\delta_1^{(1)}\right)\,\mathrm{ch}\left(\xi h/\delta_2^{(1)}\right). \tag{3.40}$$

Using Eq. (3.36) and the result [3]

$$\int_0^\infty J_0(\xi t)\,\sin\xi x\,d\xi = \begin{cases} \dfrac{1}{\sqrt{x^2-t^2}}, & 0<t<x\,, \\ 0, & 0<x<t \end{cases} \tag{3.41}$$

the integral equation (3.37) reduces to

$$(1+\eta_3\eta_4)\int_0^x \frac{M(t)\,dt}{\sqrt{x^2-t^2}} + \frac{\zeta\eta_1\eta_2\left(1+\lambda_1^{(2)}\right)}{\left(1+\lambda_1^{(1)}\right)}\int_0^c M(t)\int_0^\infty \frac{1}{L_3(\xi)} \cdot\left[\left\{L_5(\xi)-\eta_3L_3(\xi)\right\}J_0(\xi t) - \delta_1^{(1)}L_2(\xi)\hat{f}(\xi)\right]\sin\,\xi x\,d\xi\,dt = 0\,. \tag{3.42}$$

Although the second integral in Eq. (3.42) is an unknown function of x, considering Eq. (3.42) as an Abel's integral equation and solving for $M(t)$, we get a homogeneous Fredholm integral equation of the second kind of the form

$$\int_0^c U(s,t)\,M(t)\,dt = \frac{2}{1+\alpha}M(s)\,, \tag{3.43}$$

where

$$U(s,t) = \frac{S}{\eta_3}\int_0^\infty \frac{1}{L_3(\xi)}\left[\delta_1^{(1)}L_2(\xi)\,\hat{f}(\xi) - \left\{L_5(\xi)-\eta_3L_3(\xi)\right\}J_0(\xi t)\right]\xi J_0(\xi s)\,d\xi, \tag{3.44}$$

$$\alpha = (Q_2-Q_1)/(Q_2+Q_1)\,, \tag{3.45}$$

$$Q_1 = A_{11}^{(2)} A_{55}^{(1)} \left(A_{13}^{(1)} + A_{55}^{(1)} \right) \left(1 + \lambda_1^{(1)} \right) \left(1 + \lambda_2^{(1)} \right) \left(\delta_1^{(2)} + \delta_2^{(2)} \right), \tag{3.46}$$

$$Q_2 = A_{11}^{(1)} A_{55}^{(2)} \left(A_{13}^{(2)} + A_{55}^{(2)} \right) \left(1 + \lambda_1^{(2)} \right) \left(1 + \lambda_2^{(2)} \right) \left(\delta_1^{(1)} + \delta_2^{(1)} \right). \tag{3.47}$$

It is evident from Eqs. (3.45), (3.46), (3.47) that the parameter α depends on the elastic constants of both the media. For a given combination of materials, α is known and hence Eq. (3.43) determines the extent of contact between the bodies. But it would be more convenient from the mathematical point of view to consider Eq. (3.43) as an eigenvalue problem for the eigenfunction $M(t)$. This may be done by considering c as specified and determining the eigenvalue $2/(1+\alpha)$ of the homogeneous integral equation (3.43) for which the extent of contact c is realized. Eigenvalues are determined by considering physically possible values of α, and then the corresponding eigenfunction $M(t)$ determines the pressure in the contact zone from Eq. (3.34).

From Eq. (3.43) it is observed that in contrast to isotropic materials, the contact region and contact pressure are not functions of a single parameter depending on the elastic constants of the two media, and even if the layer and the half-space are of the same orthotropic material, the contact region and the contact pressure are not completely independent of the elastic parameters, which was predicted by the general results for isotropic medium in [2], and was verified later in [4].

4. Numerical results

To determine the eigenfunction $M(t)$ corresponding to the eigenvalue $2/(1+\alpha)$ of Eq. (3.43), we discretise the integral to get a system of linear homogeneous equations in $M(t_1)$. For a specified value of c, the determination of $M(t_1)$ demands nonvanishing of the coefficient determinant. This in turn requires choosing appropriate values of α such that the coefficient determinant should be nonzero and the value of α should be acceptable on physical grounds. It may be observed that on physical grounds, α should satisfy the inequality $-1 \le \alpha \le 1$. Moreover, it is necessary that the corresponding eigenvector $M(t_1)$ should yield purely compressive tractions in the zone of contact. In all cases the components of the eigenvector associated with an admissible value of α are of the same sign. In this problem, the results have been computed for the concentrated force for which $\hat{f}(\xi) = 1$. During the numerical computations for orthotropic materials we have used the values 2.862 and 0.047 for $\delta_1^{(1)^2}$ and $\delta_2^{(1)^2}$, respectively, for boron-epoxy composite material as given in [1]. The curves of c/h versus α for orthotropic and isotropic materials are shown in Fig. 2. The contact tractions may be calculated from Eq. (3.43) and are shown in Fig. 3 and Fig. 4 for orthotropic and isotropic cases, respectively.

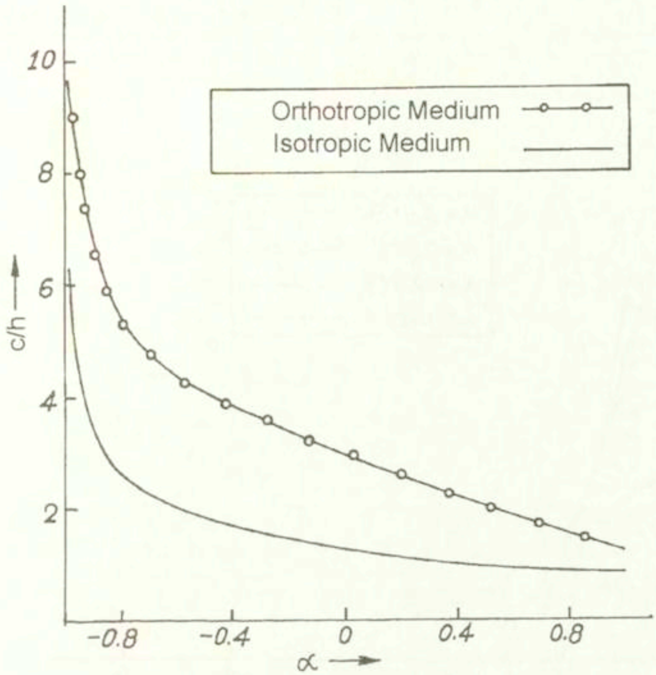

FIG. 2. Extent of contact for loading by a concentrated force in plane problem.

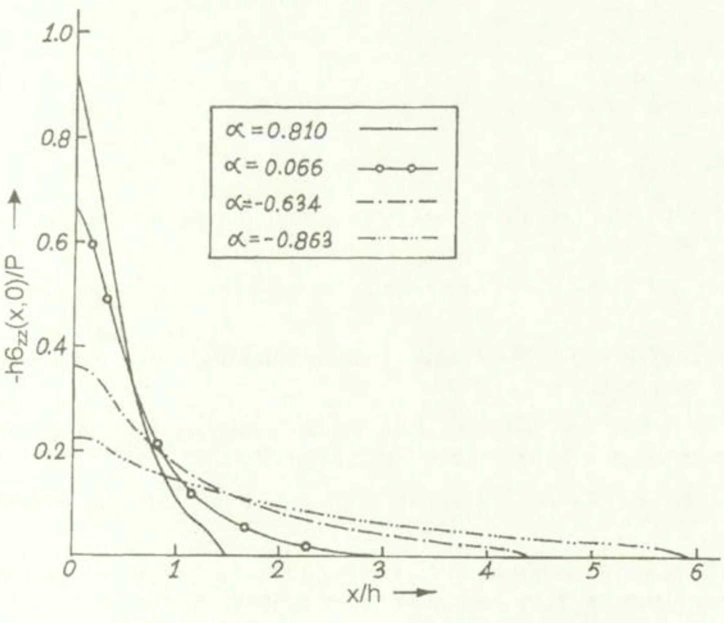

FIG. 3. Contact pressure for loading by a concentrated force in plane problem for orthotropic medium.

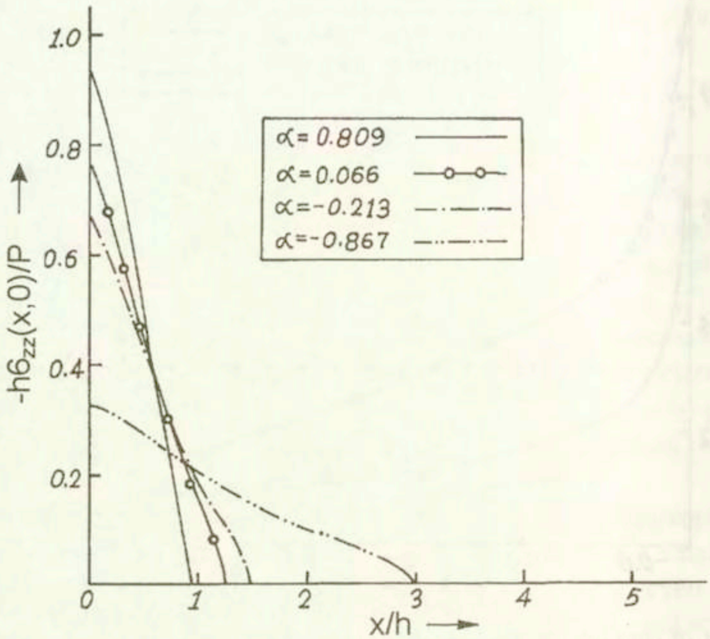

FIG. 4. Contact pressure for loading by a concentrated force in plane problem for isotropic medium.

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