Arch. Mech., 50, 4, pp. 731–741, Warszawa 1998
FIFTY YEARS OF THE ARCHIVES OF MECHANICS

On symmetric tensor-valued isotropic functions of a symmetric tensor and a skewsymmetric tensor and related transversely isotropic functions

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A NEW GENERATING set consisting of seven polynomial tensor generators is presented for symmetric second order tensor-valued isotropic functions of a symmetric second order tensor and a skewsymmetric second order tensor. It is smaller than the existing corresponding generating set consisting of eight tensor generators and shown to be minimal in all possible generating sets consisting of homogeneous polynomial tensor generators. This result indicates that the well-known results for isotropic functions may be sharpened. In addition, from the presented result a minimal generating set consisting of six tensor generators is derived for the symmetric second order tensor-valued transversely isotropic functions of a symmetric second order tensor relative to the transverse isotropy group $C_{\infty h}$ consisting of all rotations about a fixed axis.

1. Introduction

IN CONTINUUM MECHANICS, symmetric second order tensor-valued isotropic functions of vectors and second order tensors serve to formulate constitutive relations of isotropic materials, such as stress-deformation relations etc. The isotropy places restrictions on the tensor function forms of constitutive relations. It is important to determine general reduced forms, i.e., representations of the latter under such restrictions. Now general results for isotropic functions of vectors and second order tensors, in the sense of polynomial or nonpolynomial representation, are available and well known (see, e.g., [1 – 7]), and in [1] and [8 – 9] these results are given for use in convenient tabular forms.

To arrive at concise and efficient formulation of complicated material behaviours, representations for constitutive equations of materials should be made as compact as possible. Although the above-mentioned representations for isotropic functions have been proved to be irreducible in the sense of polynomial or nonpolynomial representation, until now it has not been known whether or not each of them is minimal in a suitable sense. Further development requires examining and sharpening the existing results and finally, arriving at minimal representations in a suitable sense. In this paper, we are mainly concerned with symmetric second order tensor-valued isotropic functions of a second order symmetric tensor and a skewsymmetric second order tensor. For such tensor functions, earlier TELEGA [10] made an attempt to find a generating set that is smaller than the existing one (see the end of this section for detail). Although later this early attempt was shown to be unsuccessful in [5], in this paper we shall show that the expected goal may indeed be arrived at. Specifically, we shall provide a new generating set consisting of seven polynomial tensor generators for the isotropic functions at issue, which is smaller than the existing generating set consisting of eight generators, and furthermore we shall prove that the presented set is a minimal one in all possible generating sets consisting of homogeneous polynomial tensor generators.

It shoud be pointed out that the tensor functions at issue include symmetric second order tensor-valued isotropic functions of an asymmetric second order tensor, since each asymmetric second order tensor has a unique additive decomposition into a symmetric tensor and a skewsymmetric tensor. Moreover, the tensor functions at issue also include symmetric second order tensor-valued transversely isotropic functions of a symmetric second order tensor relative to the transverse isotropy group $C_{\infty h}$, as will be shown in Sec. 3. Accordingly, minimal generating sets for the two types of tensor functions just mentioned can be derived from the presented minimal generating set for the tensor functions at issue. In so doing, by virtue of (1.2) given later, the derivation is direct for the former type of tensor functions. However, the derivation is not so direct for the latter type of tensor functions and will be discussed at the end of Sec. 3.

To facilitate the subsequent account, in what follows we outline some related facts.

A symmetric second order tensor-valued function $\psi(\mathbf{A},\mathbf{W})$ of a symmetric second order tensor $\mathbf{A}$ and a skewsymmetric second order tensor $\mathbf{W}$ is *isotropic* if

$$\psi(\mathbf{Q}\mathbf{A}\mathbf{Q}^{\mathbf{T}},\mathbf{Q}\mathbf{W}\mathbf{Q}) = \mathbf{Q}\psi(\mathbf{A},\mathbf{W})\mathbf{Q}^{\mathbf{T}}, \tag{1.1}$$

for each orthogonal tensor $\mathbf{Q}$. Henceforth, tensor means second order tensor if no confusion arises. Throughout, the superscript T is used to signify the transpose of a tensor.

As mentioned before, each symmetric tensor-valued isotropic function of an asymmetric tensor $\mathbf{B}$ is equivalent to a symmetric tensor-valued isotropic function of a symmetric tensor $\mathbf{A}$ and a skewsymmetric tensor $\mathbf{W}$. Indeed, this can be done just by the replacement

$$\begin{aligned} \mathbf{A} &= \frac{1}{2}(\mathbf{B} + \mathbf{B}^{\mathrm{T}}), \\ \mathbf{W} &= \frac{1}{2}(\mathbf{B} - \mathbf{B}^{\mathrm{T}}). \end{aligned} \tag{1.2}$$

A finite number of symmetric tensor-valued isotropic functions, $\psi_1(\mathbf{A},\mathbf{W})$, $\cdots$, $\psi_r(\mathbf{A},\mathbf{W})$, of a symmetric tensor $\mathbf{A}$ and a skewsymmetric tensor $\mathbf{W}$ form a *generating set* for the symmetric tensor-valued isotropic functions of $\mathbf{A}$ and $\mathbf{W}$

if every symmetric tensor-valued isotropic function $\psi(\mathbf{A}, \mathbf{W})$ is expressible as a linear combination of these tensor functions with invariant coefficients, i.e.

$$\psi(\mathbf{A}, \mathbf{W}) = \mathbf{c}_1\psi_1(\mathbf{A}, \mathbf{W}) + \cdots + \mathbf{c}_r\psi_r(\mathbf{A}, \mathbf{W}), \tag{1.3}$$

where each coefficient $c_i = \bar{c}_i(\mathbf{A}, \mathbf{W})$ is an *isotropic invariant* of the tensors $\mathbf{A}$ and $\mathbf{W}$, and each $\psi_i(\mathbf{A},\mathbf{W})$ is called a *tensor generator*. A generating set is said to be *irreducible* if it contains no redundant tensor generators, i.e. none of its proper subsets is again a generating set.

The following generating set for the isotropic tensor functions at issue is well-known [2 – 3]:

$$\{\mathbf{I}, \mathbf{W}^2, \mathbf{A}, \mathbf{A}^2, \mathbf{AW} - \mathbf{WA}, \mathbf{A}^2\mathbf{W} - \mathbf{WA}^2, \mathbf{WAW}, \mathbf{W}^2\mathbf{AW} - \mathbf{WAW}^2\}\,. \tag{1.4}$$

Here and hereafter, $\mathbf{I}$ is used to denote the identity tensor. Earlier, it was thought [10] that the generator $\mathbf{WAW}$ could be removed from the above set and hence that a smaller generating set was available. Later, the irreducibility of the above set was proved in [5]. Although the attempt in [10] was unsuccessful, we shall show that we can find a new generating set which is indeed smaller than the above set.

2. A criterion for generating sets

We shall use the following criterion [11] to judge whether a given set of tensor generators is a generating set or not.

Let $\psi_1(\mathbf{A}, \mathbf{W}), \cdots, \psi_r(\mathbf{A}, \mathbf{W})$ *be* r *given symmetric tensor-valued isotropic functions of the symmetric tensor* $\mathbf{A}$ *and the skewsymmetric tensor* $\mathbf{W}$. *Then they form a generating set for the symmetric tensor-valued isotropic functions of the tensors* $\mathbf{A}$ *and* $\mathbf{W}$ *if and only if*

$$\operatorname{rank}\{\psi_1(\mathbf{A}, \mathbf{W}), \cdots, \psi_r(\mathbf{A}, \mathbf{W})\} \geq \dim\, \operatorname{Sym}(g(\mathbf{A}, \mathbf{W})) \tag{2.1}$$

for any symmetric tensor $\mathbf{A}$ *and any skewsymmetric tensor* $\mathbf{W}$.

Here and hereafter, the notations rank S and dim L are used to designate the number of the linearly independent elements in any given set S of tensors and the dimension of any given tensor subspace L, respectively. Moreover, for any given symmetric tensor $\mathbf{A}$ and any given skewsymmetric tensor $\mathbf{W}$, the notation $g(\mathbf{A}, \mathbf{W})$ is used to denote the symmetry group of the tensors $\mathbf{A}$ and $\mathbf{W}$, which includes all orthogonal tensors preserving both $\mathbf{A}$ and $\mathbf{W}$, and for any given orthogonal subgroup g, the notation $\operatorname{Sym}(g)$ is used to represent the tensor subspace that is composed of all tensors invariant under the subgroup g. Specifcally,

$$g(\mathbf{A}, \mathbf{W}) = \{\mathbf{Q} \in \text{Orth} \mid \mathbf{QAQ}^{\mathrm{T}} = \mathbf{A}, \mathbf{QWQ}^{\mathrm{T}} = \mathbf{W}\}, \tag{2.2}$$

$$\mathrm{Sym}(g) = \{\mathbf{B} \in \mathrm{Sym} \mid \mathbf{QBQ}^{\mathrm{T}} = \mathbf{B}, \forall \mathbf{Q} \in g\}, \tag{2.3}$$

where Orth and Sym are the full orthogonal group and the space of symmetric tensors, respectively.

To apply the above criterion, it is required to evaluate the dimension dim Sym($g(\mathbf{A}, \mathbf{W})$) for any symmetric tensor $\mathbf{A}$ and any skewsymmetric tensor $\mathbf{W}$. For $\mathbf{W} \neq \mathbf{O}$ and $\mathbf{A} \in$ Sym, we have

$$\dim \mathrm{Sym}(g(\mathbf{A}, \mathbf{W})) \leq \begin{cases} 2 \ \text{for} \ \mathbf{A} = x\mathbf{I} + y\mathbf{w} \otimes \mathbf{w}, \\ 4 \ \text{for} \ \mathbf{w} \times \mathbf{A}\mathbf{w} = \mathbf{0}, \\ 6 \ \text{for any} \ \mathbf{A} \ \text{and} \ \mathbf{W}. \end{cases} \tag{2.4}$$

Here and hereafter, for each skewsymmetric tensor $\mathbf{W}$ we use the following expression

$$\mathbf{W} = \mathbf{E}\mathbf{w}, \tag{2.5}$$

where $\mathbf{w}$ is an axial vector associated with $\mathbf{W}$, determined by

$$\mathbf{w} = \mathbf{E} : \mathbf{W}, \qquad \text{i.e. } w_i = e_{ijk} W_{jk}. \tag{2.6}$$

Here $\mathbf{E}$ is the third-order Livi – Civita tensor and e_{ijk} the permutation symbol.

The proof for (2.4) is as follows. It is evident that $(2.4)_3$ holds, since Sym($g(\mathbf{A}, \mathbf{W})$) is a subspace of the symmetric tensor space Sym for any given tensors $\mathbf{A}$ and $\mathbf{W}$, and moreover dim Sym$= 6$. In what follows we prove that $(2.4)_{1,2}$ hold. Henceforth, $\mathbf{R}^{\theta}_{\mathbf{W}}$ is used to denote the right-handed rotation through the angle θ about an axis in the direction of the vector $\mathbf{w} \neq \mathbf{0}$.

Suppose that $\mathbf{A} = x\mathbf{I} + y\mathbf{w} \otimes \mathbf{w}$. Then we have

$$\mathbf{Q}(\mathbf{E}\mathbf{w})\mathbf{Q}^{\mathrm{T}} = \mathbf{E}\mathbf{w}, \qquad \mathbf{QAQ}^{\mathrm{T}} = \mathbf{A}$$

for any $\mathbf{Q} = \pm\mathbf{R}^{\theta}_{\mathbf{w}}$. Hence

$$C_{\infty h}(\mathbf{w}) \equiv \{\pm\mathbf{R}^{\theta}_{\mathbf{w}} \mid \theta \in R\} \subset g(\mathbf{A}, \mathbf{E}\mathbf{w}). \tag{2.7}$$

From the latter and

$$\mathrm{Sym}(C_{\infty h}(\mathbf{w})) = \mathrm{span}\{\mathbf{I}, \mathbf{w} \otimes \mathbf{w}\}, \tag{2.8}$$

as well as the fact that for two orthogonal subgroups $g_1, g_2 \subset$ Orth,

$$g_1 \subset g_2 \Longrightarrow \mathrm{Sym}(g_2) \subset \mathrm{Sym}(g_1), \tag{2.9}$$

we infer that $(2.4)_1$ holds.

Suppose that $\mathbf{w} \times \mathbf{A}\mathbf{w} = \mathbf{0}$. Then $\mathbf{w} \neq \mathbf{0}$ is an eigenvector of the symmetric tensor $\mathbf{A}$. Let $(\mathbf{w}, \mathbf{e}_1, \mathbf{e}_2)$ be three mutually orthogonal eigenvectors of the symmetric tensor $\mathbf{A}$. Then we have

$$\mathbf{A} = x\mathbf{w} \otimes \mathbf{w} + y\mathbf{e}_1 \otimes \mathbf{e}_1 + z\mathbf{e}_2 \otimes \mathbf{e}_2, \qquad x, y, z \in R\,,$$

and

$$\mathbf{R}^{\pi}_{\mathrm{w}}\mathbf{w} = \mathbf{w}, \qquad \mathbf{R}^{\pi}_{\mathrm{w}}\mathbf{e}_1 = -\mathbf{e}_1,\ \mathbf{R}^{\pi}_{\mathrm{w}}\mathbf{e}_2 = -\mathbf{e}_2$$

Hence, we derive

$$C_2(\mathbf{w}) \equiv \{\mathbf{I}, \mathbf{R}^{\pi}_{\mathrm{w}}\} \subset g(\mathbf{A}, \mathbf{E}\mathbf{w}).$$

From the latter and

$$\mathrm{Sym}(C_2(\mathbf{w})) = \mathrm{span}\{\mathbf{w}\otimes\mathbf{w}, \mathbf{e}_1\otimes\mathbf{e}_1, \mathbf{e}_2\otimes\mathbf{e}_2, \mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1\}$$

as well as (2.9), we deduce that $(2.4)_2$ holds. *Q. E. D*

3. Smaller generating sets

The main result of this paper is as follows.

THEOREM 1. *The set* $G_0(\mathbf{A}, \mathbf{E}\mathbf{w}) = \{\mathbf{G}_1, \cdots, \mathbf{G}_7\}$ *given by*

$$\begin{aligned}
\mathbf{G}_1 &= \mathbf{I}, \qquad \mathbf{G}_2 = \mathbf{w}\otimes\mathbf{w}, \qquad \mathbf{G}_3 = \mathbf{A}, \qquad \mathbf{G}_4 = \mathbf{A}^2,\\
\mathbf{G}_5 &= \mathbf{A}(\mathbf{E}\mathbf{w}) - (\mathbf{E}\mathbf{w})\mathbf{A}, \mathbf{G}_6 = (\mathbf{w}\times\mathbf{A}\mathbf{w})\otimes(\mathbf{w}\times\mathbf{A}\mathbf{w}),\\
\mathbf{G}_7 &= (\mathbf{w}\times\mathbf{A}\mathbf{w})\otimes(\mathbf{w}\times(\mathbf{w}\times\mathbf{A}\mathbf{w})) + (\mathbf{w}\times(\mathbf{w}\times\mathbf{A}\mathbf{w}))\otimes(\mathbf{w}\times\mathbf{A}\mathbf{w}),
\end{aligned}$$

is a generating set for the symmetric tensor-valued isotropic functions of a symmetric tensor $\mathbf{A}$ *and a skewsymmetric tensor* $\mathbf{W} = \mathbf{E}\mathbf{w}$, *and this set is a minimal one in all possible generating sets consisting of homogeneous polynomial generators.*

In the following, we prove that the set $G_0(\mathbf{A}, \mathbf{E}\mathbf{w})$ given above is a generating set for the isotropic tensor functions at issue. The proof for the minimality of this set will be postponed until the next section.

It can be readily shown that each presented tensor generator $\mathbf{G}_i$ is isotropic with respect to $\mathbf{A}$ and $\mathbf{W} = \mathbf{E}\mathbf{w}$ by means of the facts

$$\begin{aligned}
\mathbf{E} : (\mathbf{Q}\mathbf{W}\mathbf{Q}^{\mathrm{T}}) &= 2(\det\mathbf{Q})\mathbf{Q}\mathbf{w},\\
(\mathbf{Q}\mathbf{e})\times(\mathbf{Q}\mathbf{e}') &= (\det\mathbf{Q})\mathbf{Q}(\mathbf{e}\times\mathbf{e}'),
\end{aligned}$$

for any skewsymmetric tensor $\mathbf{W}$, any orthogonal tensor $\mathbf{Q}$ and any vectors $\mathbf{e}$ and $\mathbf{e}'$. Note that here $\mathbf{w}$ is the axial vector determined by (2.6). Hence, we need only to prove that the presented set $G_0(\mathbf{A}, \mathbf{E}\mathbf{w})$ obeys the criterion (2.1), i.e.

$$\text{(3.1)} \qquad \mathrm{rank}\ G_0(\mathbf{A}, \mathbf{E}\mathbf{w}) = \mathrm{rank}\ \{\mathbf{G}_1, \cdots, \mathbf{G}_7\} \geq \dim\ \mathrm{Sym}(g(\mathbf{A}, \mathbf{E}\mathbf{w}))$$

for any symmetric tensor $\mathbf{A}$ and any vector $\mathbf{w}$. Towards this end, four cases will be discussed.

CASE 1. One of the tensor arguments **A** and **Ew** vanishes

Each tensor generator $\mathbf{G}_i$ for $i = 5, 6, 7$ vanishes. Since the sets $\{\mathbf{I}, \mathbf{w}\otimes\mathbf{w}\}$ and $\{\mathbf{I}, \mathbf{A}, \mathbf{A}^2\}$ provide a generating set for a symmetric tensor **A** and a generating set for a skewsymmetric tensor $\mathbf{W} = \mathbf{Ew}$ respectively, the condition (3.1) are satisfied for the case in question.

CASE 2. $\mathbf{w} \times \mathbf{Aw} \neq \mathbf{0}$

Henceforth, for any two vectors $\mathbf{e}$ and $\mathbf{e}'$ the notation $\mathbf{e} \vee \mathbf{e}'$ is used to signify the symmetric tensor defined by

$$\mathbf{e} \vee \mathbf{e}' = \mathbf{e} \otimes \mathbf{e}' + \mathbf{e}' \otimes \mathbf{e} \,.$$

Let

$$\mathbf{e}_1 = \mathbf{w} \times \mathbf{Aw}, \qquad \mathbf{e}_2 = \mathbf{w} \times \mathbf{e}_1.$$

Then $(\mathbf{w}, \mathbf{e}_1, \mathbf{e}_2)$ is an orthogonal system of vectors. Without loss of generality, we assume that both the vectors **w** and **Aw** are normalized ones, i.e. $|\mathbf{w}| = |\mathbf{Aw}| = 1$, and therefore that the mentioned system of vectors is an orthonormal one. In terms of this system, the symmetric tensors **I** and **A** are expressible as

$$\mathbf{I} = \mathbf{w} \otimes \mathbf{w} + \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2, \tag{3.2}$$

$$\begin{aligned} \mathbf{A} = {} & a_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + a_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + a_3 \mathbf{w} \otimes \mathbf{w} \\ & + a_4 \mathbf{e}_1 \vee \mathbf{e}_2 + a_5 \mathbf{w} \vee \mathbf{e}_1 + a_6 \mathbf{w} \vee \mathbf{e}_2 \,. \end{aligned} \tag{3.3}$$

The condition that $\mathbf{w} \times \mathbf{Aw} \neq \mathbf{0}$ requires

$$(a_5)^2 + (a_6)^2 \neq 0.$$

Hence

$$\begin{aligned} \mathbf{A}(\mathbf{Ew}) - (\mathbf{Ew})\mathbf{A} = {} & (a_1 - a_2)\mathbf{e}_1 \vee \mathbf{e}_2 - a_4(\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2) \\ & + a_6 \mathbf{w} \vee \mathbf{e}_1 - a_5 \mathbf{w} \vee \mathbf{e}_2. \end{aligned} \tag{3.4}$$

In deriving the above, the identity

$$(\mathbf{Ee})\mathbf{e}' = \mathbf{e} \times \mathbf{e}'$$

for any two vectors $\mathbf{e}$ and $\mathbf{e}'$ is used.

Applying the expressions (3.2) – (3.4), we infer

$$\begin{aligned} \mathrm{rank} G_0(\mathbf{A}, \mathbf{Ew}) & \geq \mathrm{rank}\{\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3, \mathbf{G}_5, \mathbf{G}_6, \mathbf{G}_7\} \\ & = \mathrm{rank}\{\mathbf{w} \otimes \mathbf{w}, \mathbf{e}_1 \otimes \mathbf{e}_1, \mathbf{e}_1 \vee \mathbf{e}_2, \mathbf{I}, \mathbf{A}, \mathbf{A}(\mathbf{Ew}) - (\mathbf{Ew})\mathbf{A}\} \\ & = \mathrm{rank}\{\mathbf{w} \otimes \mathbf{w}, \mathbf{e}_1 \otimes \mathbf{e}_1, \mathbf{e}_1 \vee \mathbf{e}_2, \mathbf{e}_2 \otimes \mathbf{e}_2, \\ & \qquad a_5 \mathbf{w} \vee \mathbf{e}_1 + a_6 \mathbf{w} \vee \mathbf{e}_2, a_6 \mathbf{w} \vee \mathbf{e}_1 - a_5 \mathbf{w} \vee \mathbf{e}_2\} \\ & = \mathrm{rank}\{\mathbf{w} \otimes \mathbf{w}, \mathbf{e}_1 \otimes \mathbf{e}_1, \mathbf{e}_1 \vee \mathbf{e}_2, \mathbf{e}_2 \otimes \mathbf{e}_2, \mathbf{w} \vee \mathbf{e}_1, \mathbf{w} \vee \mathbf{e}_2\} \\ & = 6 \geq \mathrm{Sym}(g(\mathbf{A}, \mathbf{Ew})) \,. \end{aligned}$$

In the last step, $(2.4)_3$ is used.

CASE 3. $\mathbf{w} \times \mathbf{A}\mathbf{w} = \mathbf{0}$ and $\mathbf{A}(\mathbf{E}\mathbf{w}) \neq (\mathbf{E}\mathbf{w})\mathbf{A}$

Let $(\mathbf{w}, \mathbf{e}_1, \mathbf{e}_2)$ be an orthonormal system of vectors, where $\mathbf{e}_1$ may be any unit vector in the $\mathbf{w}$-plane and $\mathbf{e}_2 = \mathbf{w} \times \mathbf{e}_1$. Then the expressions (3.2) – (3.3) remain true. The condition that $\mathbf{w} \times \mathbf{A}\mathbf{w} = \mathbf{0}$ implies that $\mathbf{w} \neq \mathbf{0}$ is an eigenvector of the symmetric tensor $\mathbf{A}$. Hence, in (3.3) we have

$$a_5 = a_6 = 0 \ .$$

Moreover, the condition that $\mathbf{A}(\mathbf{E}\mathbf{w}) \neq (\mathbf{E}\mathbf{w})\mathbf{A}$ yields (cf. (3.4))

$$(a_1 - a_2)^2 + (a_4)^2 \neq 0 \ .$$

Thus, we deduce

$$\begin{aligned} \operatorname{rank} G_0(\mathbf{A}, \mathbf{E}\mathbf{w}) &\geq \operatorname{rank}\{\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3, \mathbf{G}_5\} \\ &= \operatorname{rank}\{\mathbf{w} \otimes \mathbf{w}, \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2, a_1\mathbf{e}_1 \otimes \mathbf{e}_1 + a_2\mathbf{e}_2 \otimes \mathbf{e}_2 \\ &\qquad + a_4\mathbf{e}_1 \vee \mathbf{e}_2, (a_1 - a_2)\mathbf{e}_1 \vee \mathbf{e}_2 - a_4(\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2)\} \\ &= \operatorname{rank}\{\mathbf{w} \otimes \mathbf{w}, \mathbf{e}_1 \otimes \mathbf{e}_1, \mathbf{e}_2 \otimes \mathbf{e}_2, \mathbf{e}_1 \vee \mathbf{e}_2\} \\ &= 4 \geq \operatorname{Sym}(g(\mathbf{A}, \mathbf{E}\mathbf{w})) \ . \end{aligned}$$

In the last step, $(2.4)_2$ is used. Moreover, we mention that the determinant of the coefficient matrix of the last three tensors in the second equality with respect to the three independent tensors $\mathbf{e}_1 \otimes \mathbf{e}_1$, $\mathbf{e}_2 \otimes \mathbf{e}_2$ and $\mathbf{e}_1 \vee \mathbf{e}_2$ is given by

$$\Delta = \begin{vmatrix} 1 & 1 & 0 \\ a_1 & a_2 & a_4 \\ -a_4 & a_4 & a_1 - a_2 \end{vmatrix} = -(a_1 - a_2)^2 - 2(a_4)^2 \neq 0,$$

and therefore, that the two sets of three tensors just mentioned are equivalent.

CASE 4. $\mathbf{w} \times \mathbf{A}\mathbf{w} = \mathbf{0}$ and $\mathbf{A}(\mathbf{E}\mathbf{w}) = (\mathbf{E}\mathbf{w})\mathbf{A}$, i.e. $\mathbf{A} = x\mathbf{I} + y\mathbf{w} \otimes \mathbf{w}$

From $(2.4)_1$, it is evident that

$$\operatorname{rank} G_0(\mathbf{A}, \mathbf{E}\mathbf{w}) \geq \operatorname{rank}\{\mathbf{I}, \mathbf{w} \otimes \mathbf{w}\} = 2 \geq \dim \operatorname{Sym}(g(\mathbf{A}, \mathbf{E}\mathbf{w})).$$

Finally, combining the above four cases, we conclude that the presented set $G_0(\mathbf{A}, \mathbf{E}\mathbf{w})$ obeys (3.1) for any tensors $\mathbf{A}$ and $\mathbf{W} = \mathbf{E}\mathbf{w}$, and hence it is a generating set for the symmetric tensor-valued isotropic functions of the symmetric tensor $\mathbf{A}$ and the antisymmetric tensor $\mathbf{W} = \mathbf{E}\mathbf{w}$. *Q. E. D.*

From the proof given above, it can be seen that the tensor generator $\mathbf{G}_4 = \mathbf{A}^2$ comes into play only when $\mathbf{W} = \mathbf{O}$, i.e. $\mathbf{w} = \mathbf{0}$. This fact leads to the following result.

COROLLARY 1. The set

$$G_+(\mathbf{A},\mathbf{w}) = \{\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3, \mathbf{G}_5, \mathbf{G}_6, \mathbf{G}_7\} \tag{3.5}$$

is a generating set for the symmetric tensor-valued isotropic functions of the symmetric tensor $\mathbf{A}$ and the nonvanishing antisymmetric tensor $\mathbf{W} = \mathbf{E}\mathbf{w} \neq \mathbf{O}$, and this set is a minimal one in all possible generating sets for the isotropic tensor functions at issue.

According to [12], a generating set for the symmetric tensor-valued isotropic functions of the symmetric tensor $\mathbf{A}$ and the antisymmetric tensor $\mathbf{E}\mathbf{n}$, where $\mathbf{n}$ is a fixed unit vector, furnishes a generating set for the symmetric tensor-valued transversely isotropic functions of the symmetric tensor $\mathbf{A}$ relative to the transverse isotropy group $C_{\infty h}(\mathbf{n})$ (cf. (2.7)). Thus, by assuming the vector $\mathbf{w}$ as a fixed unit vector $\mathbf{n}$ in (3.5), from the above corollary we derive a generating set for the symmetric tensor-valued transversely isotropic functions of the symmetric tensor $\mathbf{A}$ relative to the transverse isotropy group $C_{\infty h}(\mathbf{n})$ as follows:

$$\begin{aligned} G_0(\mathbf{A}) = \{&\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathbf{A}, \mathbf{A}(\mathbf{E}\mathbf{n}) - (\mathbf{E}\mathbf{n})\mathbf{A}, \\ &(\mathbf{n}\times\mathbf{A}\mathbf{n})\otimes(\mathbf{n}\times\mathbf{A}\mathbf{n}), (\mathbf{n}\times\mathbf{A}\mathbf{n})\vee(\mathbf{n}\times(\mathbf{n}\times\mathbf{A}\mathbf{n}))\}. \end{aligned} \tag{3.6}$$

This generating set, which is composed of six tensor generators only, is smaller than the existing corresponding generating sets (cf. [13, 14]), each of the latter being composed of eight generators. Furthermore, the generating set given by (3.6) is the smallest possible (cf. Theorem 4 given in [15]). In fact, it is equivalent to the corresponding minimal generating set given in [15].

4. The minimality of the generating set $G_0(\mathbf{A}, \mathbf{E}\mathbf{w})$

In forming the generating sets, homogeneous polynomial generators are commonly used (see, e.g., [1 – 9]). Each such generator except the constant vectors and constant tensors is of the property: it vanishes when one of its variables vanishes. Thus, for the tensor functions concerned, each generating sets is of the form

$$G(\mathbf{A}, \mathbf{E}\mathbf{w}) = G_0 \cup G'(\mathbf{A}) \cup G'(\mathbf{E}\mathbf{w}) \cup G'(\mathbf{A}, \mathbf{E}\mathbf{w}), \tag{4.1}$$

where G_0 is a set consisting of constant tensors, $G'(\mathbf{A})$ and $G'(\mathbf{E}\mathbf{w})$ are two sets consisting of tensor generators depending on a single variable $\mathbf{A}$ or $\mathbf{E}\mathbf{w}$ respectively, and $G'(\mathbf{A}, \mathbf{E}\mathbf{w})$ is a set consisting of tensor generators depending on the two tensor variables $\mathbf{A}$ and $\mathbf{E}\mathbf{w}$. Each set given above is assumed to fulfil the properties:

$$\begin{aligned} \psi_0(\mathbf{O}) &= \mathbf{O},\ \forall\psi_0(\mathbf{B}) \in G'(\mathbf{A}) \cup G'(\mathbf{E}\mathbf{w}), \\ \psi(\mathbf{A}, \mathbf{O}) &= \psi(\mathbf{O}, \mathbf{E}\mathbf{w}) = \mathbf{O},\ \forall\psi(\mathbf{A}, \mathbf{E}\mathbf{w}) \in G'(\mathbf{A}, \mathbf{E}\mathbf{w})\ . \end{aligned} \tag{4.2}$$

The main objective of this section is to verify that the generating set $G_0(\mathbf{A}, \mathbf{W})$ given in the last section is minimal in all possible generating sets of the form just stated. To this end, we prove that each generating set of the above-mentioned form must include at least seven tensor generators.

First, by (4.2) we infer that any given generating set $G(\mathbf{A}, \mathbf{Ew})$ in question has the property

$$G(\mathbf{A}, \mathbf{O}) = G_0 \cup G'(\mathbf{A}) \cup \{\mathbf{O}\}, \qquad G(\mathbf{O}, \mathbf{Ew}) = G_0 \cup G'(\mathbf{Ew}) \cup \{\mathbf{O}\} .$$

These indicate that the set $G_0 \cup G'(\mathbf{A})$ (resp. $G_0 \cup G(\mathbf{Ew})$) must be a generating set for the symmetric tensor-valued isotropic functions of a symmetric (resp. skewsymmetric) tensor $\mathbf{A}$ (resp. $\mathbf{Ew}$). According to Theorem 4 given in [15], the set $G_0 \cup G'(\mathbf{A})$ must include at least three generators. Moreover, it is easily understood that the set $G_0 \cup G'(\mathbf{Ew})$ must include at least two generators, such as $\mathbf{I}$ and $\mathbf{w} \otimes \mathbf{w}$ etc. Finally, the set G_0 must include at least one nonvanishing constant tensor, such as $\mathbf{I}$ etc., or else the criterion (2.1) will be violated when $\mathbf{A} = \mathbf{W} = \mathbf{O}$. Thus, by combining these facts we know that the set $G_0 \cup G'(\mathbf{A}) \cup G'(\mathbf{Ew})$ must include at least four generators.

Next, let

$$\mathbf{A}_0 = (\mathbf{n} + \mathbf{e}) \otimes (\mathbf{n} + \mathbf{e}) ,$$

where $\mathbf{e}$ and $\mathbf{n}$ are two orthonormal vectors, i.e.

$$\mathbf{n} \cdot \mathbf{e} = 0, \quad \mathbf{e} \cdot \mathbf{e} = \mathbf{n} \cdot \mathbf{n} = 1 .$$

Then we have

$$\mathbf{Q}(\mathbf{n} + \mathbf{e}) \neq \pm(\mathbf{n} + \mathbf{e})$$

for each $\mathbf{Q} = \pm\mathbf{R}_{\mathbf{n}}^{\theta}$, $\theta \neq 2k\pi$. Hence we infer

$$\mathbf{Q} = \pm\mathbf{R}_{\mathbf{n}}^{\theta} \quad \text{and} \quad \theta \neq 2k\pi \Longrightarrow \mathbf{Q}\mathbf{A}_0\mathbf{Q}^{\mathrm{T}} \neq \mathbf{A}_0.$$

From the latter and the fact (cf. [12])

$$\mathbf{Q}(\mathbf{En})\mathbf{Q}^{\mathrm{T}} = \mathbf{En} \Longleftrightarrow \mathbf{Q} = \pm\mathbf{R}_{\mathbf{n}}^{\theta} \in C_{\infty h}(\mathbf{n}) ,$$

we deduce that

$$g(\mathbf{A}_0, \mathbf{En}) = \{\pm\mathbf{I}\} .$$

Hence,

$$\dim \mathrm{Sym}(g(\mathbf{A}_0, \mathbf{En})) = \dim \mathrm{Sym} = 6 . \tag{4.3}$$

On the other hand, since

$$C_{\infty h}(\mathbf{n} + \mathbf{e}) \subset g(\mathbf{A}_0, \mathbf{O}), \quad C_{\infty h}(\mathbf{n}) = g(\mathbf{O}, \mathbf{En}),$$

we have (cf. (2.7) – (2.9))

$$\mathrm{Sym}(g(\mathbf{A}_0, \mathbf{O})) \subset \mathrm{span}\{\mathbf{I}, \mathbf{A}_0\}, \qquad \mathrm{Sym}(g(\mathbf{O}, \mathbf{En})) = \mathrm{span}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}\}\,.$$

From the latter and the following fact (cf. (1.1) and (2.2) – (2.3); see also Theorem 2.2 given in [11]):

$$\psi_0(\mathbf{B}) \in \mathrm{Sym}(\mathbf{A}, \mathbf{O}) \cup \mathrm{Sym}(\mathbf{O}, \mathbf{Ew}), \qquad \forall\psi_0(\mathbf{B}) \in G_0 \cup G'(\mathbf{A}) \cup G'(\mathbf{Ew}),$$

we derive

$$\text{(4.4)} \qquad \mathrm{rank}(G_0 \cup G'(\mathbf{A}_0) \cup G'(\mathbf{En})) \le \dim(\mathrm{Sym}(g(\mathbf{A}_0, \mathbf{O})) + \mathrm{Sym}(g(\mathbf{O}, \mathbf{En}))) = 3.$$

Finally, applying the criterion (2.1) with the tensors $(\mathbf{A}_0, \mathbf{En})$ given and using (4.3) – (4.4), for each generating set $G(\mathbf{A}, \mathbf{Ew})$ in question we deduce

$$\begin{aligned} \dim\, \mathrm{Sym}(g(\mathbf{A}_0, \mathbf{En})) = 6 \;&\le\; \mathrm{rank} G(\mathbf{A}_0, \mathbf{En}) \\ &\le\; \mathrm{rank}(G_0 \cup G'(\mathbf{A}_0) \cup G'(\mathbf{En})) + \mathrm{rank} G'(\mathbf{A}_0, \mathbf{En}), \end{aligned}$$

i.e.

$$\mathrm{rank} G'(\mathbf{A}_0, \mathbf{En}) \ge 6 - \mathrm{rank}(G_0 \cup G'(\mathbf{A}_0) \cup G'(\mathbf{En})) = 3.$$

The latter shows that in each generating set $G(\mathbf{A}, \mathbf{Ew})$ (cf. (4.1)) with the property (4.2), the set $G'(\mathbf{A}, \mathbf{Ew})$ consisting of generators depending on the two variables $\mathbf{A}$ and $\mathbf{En}$ must include at least three generators.

Finally, from the fact just-proved and the fact proved before we conclude that each generating set $G(\mathbf{A}, \mathbf{Ew})$ at issue (cf. (4.2) – (4.3)) must include at least seven generators.

Thus, the presented generating set $G_0(\mathbf{A}, \mathbf{Ew})$, which is exactly composed of seven tensor generators, is a minimal one in all possible generating sets $G(\mathbf{A}, \mathbf{Ew})$ with the properties (4.1) and (4.2). *Q. E. D.*

5. Conclusions

Although the well-known general results for isotropic functions have been proved to be irreducible, the results given in the previous sections show that some of them could be sharpened. The same is true for the results for transversely isotropic functions and orthotropic functions etc. (cf. Theorem 5 in [15] for the latter). Further considerations require examining the existing results, sharpening some of them and finally arriving at general results that are minimal in a suitable sense. The results will be reported elsewhere [16 – 17].

Acknowledgement

The first author is much obliged to Alexander von Humboldt Foundation for the grant of a research fellowship.

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Received October 24, 1997.