Arch. Mech., **50**, 4, pp. 715–730, Warszawa 1998
FIFTY YEARS OF THE ARCHIVES OF MECHANICS

Magnetohydrodynamic stability of streaming liquid cylinder with doubly perturbed interfaces having a streaming fluid mantle jet

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THE MAGNETOHYDRODYNAMIC (MHD) stability of streaming liquid cylinder with doubly perturbed interfaces coaxial with a streaming fluid mantle jet has been developed. A general dispersion relation is derived, discussed and some reported works are described. The analytical results are confirmed numerically and interpreted physically. The streaming is purely destabilizing while the capillary force is such as for small axisymmetric perturbations only. The magnetic field has a strong stabilizing influence and it gives a measure of rigidity in the fluids. The radii (liquid-fluid) ratio plays an essential role in increasing the MHD stabilizing domains. The densities (liquid-fluid) ratio have a very small stabilizing effect. If the magnetic field strength is so strong that the Alfvèn wave velocity is much greater than the streaming speed, the capillary and streaming destabilizing character is completely suppressed and stability sets in. The present results are in good agreement with the experimental results of Kendall (1986) since we neglect here both the magnetic field influence and the inertia force of the interior fluid jet.

1. Introduction

THE CAPILLARY instability of a full liquid jet in a vacuum has received a considerable attention since a long time ago for its important applications in several domains of science. See PLATEAU [1], RAYLEIGH [2] and CHANDRASEKHAR [3]. The last author [3], Nobel prize winner (1986), summarized the previous reported works and made several extensions for studying the stability of different models. The effect of non-linearities on the capillary instability of a liquid jet has been examined by YUEN [4], WANG [5], NAYFEH [6], NAYFEH and HASSAN [7], and the complete analysis has been carried out by KAKUTANI *et al.* [8].

In the present era (especially in the last decade), the scientific province has turned out for investigating more applicable models than the naive one of a full liquid jet in vacuum. KENDALL [9] made very interesting experiments with modern equipment for different amplitudes and various wavelengths in studying the capillary instability of a fluid jet surrounding a gas cylinder (of negligible inertia), i.e. the annular gas jet. The inertia force of the fluid is considered to be greater than that of the gas mantle jet. KENDALL [9] explained clearly the importance and possible applications of the annular jet in astronomy. Moreover, he [9] did attract our attention to the analytical investigation of stability of such a model. Indeed, the principle and basic physics of the new type of liquid-in-air jet are due to HERTZ and HERMANRUD [10]. The stability of a liquid column with a thin shell endowed with surface tension was studied by PETRYANOV and SHUTOV [11] and SHUTOV [12]. The hydrodynamic and MHD stability of a gas jet (of negligible inertia) immersed in an infinite fluid, i.e. hollow jet, has been elaborated by RADWAN [13]. CHENG [14] has investigated the capillary instability of a streaming gas jet embedded in an infinite liquid, taking into account the inertia forces of both the gas and the liquid media. However, one has to mention here that the results given by CHENG [9], in Eqs. (4) and (5) there, are incorrect in the third term. In fact, the quantity $(1-s^2-k^2a^2)$ must be in the numerator, as it is clear from Eq. (3) derived there. See also Eqs. (3.1) – (3.13) in the present work and Chandrasekhar's results [3], pages 538 – 540 (Eqs. (147) and (155)).

The object of the present work is to study the MHD stability of streaming liquid cylinder with doubly perturbed interfaces having a streaming fluid mantle jet. This instability has been regarded to be important for the origin of the breaking-up of the fluid layers resulting in the appearance of condensation within astronomical objects. Also such a phenomenon of coaxial differently perturbed fluids may be of interest during geological drillings in the crust of the earth, in the case in which we have a fluid column surrounded by a dense gas jet. The present results reduce to those of Refs. [9, 13, 14] under appropriate simplifications.

2. Formulation and eigenvalue relation

We shall consider a fluid cylinder of radius R^i concentric and coaxial with an exterior liquid cylinder of radius R^e with $R^e > R^i$. The fluid and liquid media are assumed to be inviscid, incompressible and perfectly conducting, and the coaxial cylinders are surrounded by vacuum. The model is acting upon the combined effect of inertia, pressure gradient, capillary and electromagnetic forces. The capillary force along the fluid-liquid interface is assumed to be greater than that along the liquid-vacuum interface, and so we shall neglect the latter temporarily in our study here. Both the fluid and liquid are assumed to be streaming with velocity

$$\mathbf{u}_0 = (0,\ 0,\ U_0)\ . \tag{2.1}$$

The interior fluid and exterior liquid are subjected to the magnetic fields

$$\mathbf{H}_0^i = (0,\ 0,\ \alpha,\ H_0), \qquad \alpha \geq 1,$$

(2.2)

$$\mathbf{H}_0^e = (0,\ 0,\ \beta\, H_0), \qquad \beta \geq 1\ ,$$

while the vacuum surrounding the coaxial cylinders is assumed to be penetrated by

$$\mathbf{H}_0^{\text{vac}} = (0,\ 0,\ H_0)\ . \tag{2.2$'$}$$

The components of $\mathbf{u}_0$, $\mathbf{H}_0^i$, and $\mathbf{H}_0^{\text{vac}}$ are written in cylindrical polar coordinates (r, ϕ, z) with the z-axis coinciding with the axis of the coaxial cylinders.

The fundamental MHD equations are derived by combining the ordinary hydrodynamic equations and electrodynamic Maxwell's equations. For the problem at hand, the basic equations in the fluid and liquid media could be written in the form

$$\rho^{i,e}\frac{d\mathbf{u}^{i,e}}{dt} = -\nabla p^{i,e} + \mu^{i,e}(\nabla \wedge \mathbf{H}^{i,e}) \wedge \mathbf{H}^{i,e}\ , \tag{2.3}$$

$$\nabla \cdot \mathbf{u}^{i,e} = 0, \tag{2.4}$$

$$\nabla \cdot \mathbf{H}^{i,e} = 0\ , \tag{2.5}$$

$$\frac{\partial \mathbf{H}^{i,e}}{\partial t} = \nabla \wedge (\mathbf{u} \wedge \mathbf{H})^{i,e}\ , \tag{2.6}$$

and along the fluid-liquid interface

$$p_s = T(\nabla \cdot \hat{\mathbf{n}}_s)\ , \tag{2.7}$$

where the superscipts i and e are pertaining the interior fluid and exterior liquid. Here ρ^i, $\mathbf{u}^i$ and p^i are the fluid mass density, velocity vector and kinetic pressure, μ^i is the magnetic field permeability coefficient, $\mathbf{H}^i$ is the magnetic field intensity; and similarly for the liquid variables with superscript e. p_s is the surface pressure due to the capillary force, T is the surface tension coefficient, and $\hat{\mathbf{n}}_s$ (parallel to the coordinate r) is a unit outward normal vector to the fluid-liquid interface.

In the vacuum (medium of negligible inertia) surrounding the coaxial cylinders, the basic equations are

$$\nabla \wedge \mathbf{H}^{\text{vac}} = 0 \tag{2.8}$$

$$\nabla \cdot \mathbf{H}^{\text{vac}} = 0\ . \tag{2.9}$$

The initial unperturbed state is studied. The balance of the (surface, magnetic and kinetic) pressure across the fluid-liquid interface at $r = R^i$, yields

$$p_0^i = p_0^e + (T/R^i) + (H_0^2/2)(\mu^i\alpha - \mu^e\beta)\ . \tag{2.10}$$

Here the unperturbed quantities are those with index 0, and later on the perturbed quantities will be with index 1. It is found more appropriate to take $\alpha = \beta$ from now on, so the jump of $\mathbf{H}_0$ will be zero at the fluid-liquid interface.

Linearization of Eqs. (2.3) – (2.9) is accomplished by substituting the expansion

$$Q^{i,e} = Q_0^{i,e} + \varepsilon(t) Q_1^{i,e} \ , \tag{2.11}$$

and retaining only the first order terms, in the small fluctuating variable Q_1. The variable Q stands for $\mathbf{u}$, p, $\mathbf{H}$, $\mathbf{n}_s$ and for the perturbed radii of the fluid and liquid cylinders. $\varepsilon(t)$ is the amplitude of perturbation, given by

$$\varepsilon(t) = \varepsilon_0 \exp(\sigma t) \ , \tag{2.12}$$

where $\varepsilon_0 (= \varepsilon$ at $t = 0)$ is the initial amplitude and σ is the growth rate. If $\sigma = i\omega$, $i = (-1)^{1/2}$ is imaginary then $\omega/2\pi$ is the wave oscillation frequency. By considering a simple sinusoidal disturbance waves in z and ϕ, the radii of the fluid and liquid cylinders due to perturbation are

$$r^i = R^i + \varepsilon(t)\eta^i \ , \tag{2.13}$$

$$r^e = R^e + \varepsilon(t)\eta^e \ . \tag{2.14}$$

Here

$$\eta^i = R^i \exp\left(i(kz + m\phi)\right), \tag{2.15}$$

$$\eta^e = R^e \exp\left(i(kz + m\phi)\right) \tag{2.16}$$

are the elevations of the perturbed surfaces wave measured from the initial position, where m (an integer) is the azimuthal wavenumber and k (a real number) is the longitudinal wavenumber. In view of the expansion (2.11), the linearized perturbation equations of the fluid media are given by:

$$\rho^j \left(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial z}\right) \mathbf{u}_1^j = -\nabla p_1^j + \mu^j (\nabla \wedge \mathbf{H}_1^j) \wedge \mathbf{H}_0^j, \tag{2.17}$$

$$\nabla \cdot \mathbf{u}_1^j = 0, \tag{2.18}$$

$$\nabla \cdot \mathbf{H}_1^j = 0, \tag{2.19}$$

$$\frac{\partial \mathbf{H}_1^j}{\partial t} = \nabla \wedge (\mathbf{u}_1^j \wedge \mathbf{H}_0^j) + \nabla \wedge (\mathbf{u}_0^j \wedge \mathbf{H}_1^j), \tag{2.20}$$

with $j = i, e$.

Along the fluid-liquid interface, at $r = R^i$,

$$p_{1s} = \frac{T}{(R^i)^2} \left(\eta^i + \frac{\partial^2 \eta^i}{\partial \phi^2} + R^{i^2} \frac{\partial^2 \eta^i}{\partial z^2}\right) \tag{2.21}$$

and in the region surrounding the coaxial cylinders

$$\nabla \wedge \mathbf{H}_1^{\text{vac}} \ = \ 0, \tag{2.22}$$

$$\nabla \cdot \mathbf{H}_1^{\text{vac}} \ = \ 0. \tag{2.23}$$

By taking the divergence of Eq. (2.17), we get

$$\left(\mathbf{H}_0^j \cdot \nabla\right)\left(\nabla \cdot \mathbf{H}_1^j\right) - \rho^j \left(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial z}\right)\left(\nabla \cdot \mathbf{u}_1^j\right) = \nabla^2 \Pi_1^j \, , \tag{2.24}$$

where Π_1 is the total MHD pressure which is the sum of magnetic and fluid kinetic pressures given by

$$\Pi_1 = (\mu/2)(\mathbf{H}\cdot\mathbf{H}) + p_1. \tag{2.25}$$

Equation (2.24), upon using Eqs. (2.18) and (2.19), yields

$$\nabla^2 \Pi_1^j = 0 \, . \tag{2.26}$$

Equation (2.22) means that the perturbed magnetic field $\mathbf{H}^{\text{vac}}$ in the vacuum region could be derived from a scalar magnetic potential Ψ_1

$$\mathbf{H}_1^{\text{vac}} = \nabla \Psi_1 \, . \tag{2.27}$$

By combining Eqs. (2.23) and (2.27) we get

$$\nabla^2 \Psi_1 = 0 \, . \tag{2.28}$$

Following the surfaces deformation (2.13) – (2.16), as usual for the stability problems of cylindrical models whether with a single perturbed interface or not, and basing on the linear perturbation technique, we assume that any perturbed quantity can be expressed as $\varepsilon(t)\exp\, i(kz+m\phi)$ times an amplitude function of r. Henceforth, the non-singular solutions of the relevant perturbation equations (2.17) – (2.23) are given by

$$\Pi_1^i = A\varepsilon(t) I_m(kr) \exp\left(i(kz+m\phi)\right) \, , \tag{2.29}$$

$$\mathbf{u}_1^i = \frac{-(\sigma + ikU_0)}{((\sigma+ikU_0)^2 + (\Omega_A^i)^2)} \nabla \Pi_1^i \, , \tag{2.30}$$

$$\mathbf{H}_1^i = \frac{ikH_0}{(\sigma + ikU_0)} \mathbf{u}_1^i \, , \tag{2.31}$$

$$p_{1s} = -\varepsilon(t)(T/R^i)(1 - m^2 - k^2 R^{i^2}) \exp\left(i(kz+m\phi)\right) \, , \tag{2.32}$$

$$\Pi_1^e = \varepsilon(t)\{B I_m(kr) + C K_m(kr)\} \exp\left(i(kz+m\phi)\right) \, , \tag{2.33}$$

$$\mathbf{u}_1^e = \frac{-(\sigma + ikU_0)}{((\sigma+ikU_0)^2 + (\Omega_A^e)^2)} \nabla \Pi_1^e \, , \tag{2.34}$$

$$\mathbf{H}_1^e = \frac{ikH_0}{(\sigma + ikU_0)} \mathbf{u}_1^e \, , \tag{2.35}$$

$$\mathbf{H}_1^{\text{vac}} = E\varepsilon(t) \nabla \left\{K_m(kr) \exp\left(i(kz+m\phi)\right)\right\} \, . \tag{2.36}$$

Here A, B, C and E are constants of integration to be determined while I_m and K_m are, respectively, the modified first and second kind Bessel functions of order m, Ω^i_A and Ω^e_A are the Alfvèn wave frequencies of the fluid and liquid cylinders:

$$\Omega^i_A = \left((\mu^i/\rho^i)H_0^2k^2\right)^{1/2}, \tag{2.37}$$

$$\Omega^e_A = \left((\mu^e/\rho^e)H_0^2k^2\right)^{1/2}. \tag{2.38}$$

The solution of the perturbation Eqs. (2.17) - (2.32) given by (2.29) - (2.38) must satisfy apropriate conditions valid at the perturbed interfaces of the fluid-liquid and liquid-vacuum interfaces. Under the present circumstances and for the problem under considerations, these boundary conditions are the folowing.

I. The normal component of the velocity vector of the fluid jet must be compatible with the velocity of the fluid-liquid interface (2.13) at $r = R^i$.

II. The normal component of the velocity $\mathbf{n}\cdot\mathbf{u}^e$ of the (outer) liquid jet must be compatible with the velocity of the free liquid interface (2.14) and simultaneously must comply with the normal component of the velocity $\mathbf{n}\cdot\mathbf{u}^i$ of the fluid at $r = R^e$.

III. The normal component of the magnetic fields $\mathbf{n}\cdot\mathbf{H}^e$ and $\mathbf{n}\cdot\mathbf{H}^i$ must be balanced across the perturbed interface of the fluid-liquid regions described by (2.13), at $r = R^i$.

IV. The normal component of the magnetic fields $\mathbf{n}\cdot\mathbf{H}^{vac}$ and $\mathbf{n}\cdot\mathbf{H}^e$ must be continuous across the perturbed free surface of the liquid jet (2.14) at $r = R^e$.

By applying the foregoing boundary conditions across the perturbed interfaces (2.13) at $r = R^i$ and (2.14) at $r = R^e$, we get

$$A = -\varepsilon_0 R^{i^2}\left((\sigma + ikU_0)^2 + \Omega^{i^2}_A\right)(xI'_m(x))^{-1}, \tag{2.39}$$

$$B = (Y/M_m)(R^{i^2}yK'_m(y) - R^{e^2}xK'_m(x)), \tag{2.40}$$

$$C = (Y/M_m)(R^{e^2}xI'_m(x) - R^{i^2}yI'_m(y)), \tag{2.41}$$

$$E = -ikH_0\left(M_mK'_m(y)\right)^{-1}\left(BI'_m(y) + CK'_m(y)\right), \tag{2.42}$$

$$Y = \left((\sigma + ikU_0)^2 + \Omega^{e^2}_A\right), \tag{2.43}$$

$$M_m = xy\left(I'_m(y)K'_m(x) - I'_m(x)K'_m(y)\right), \tag{2.44}$$

where $x\ (= kR^i)$ and $y (= kR^e)$ are the dimensionless longitudinal wavenumbers.

Finally, we have to apply the compatibility boundary condition which states that the normal component of the stresses (due to the liquid and fluid pressure gradients and the electromagnetic forces acting on the model) must be discontinuous, its jump being equal to the capillary force stresses at $r = R^i$. This condition, at $r = R^i$, is

$$p_{1s} = \rho^i \left(p_1^i + (\mu^i/2)(\mathbf{H}\cdot\mathbf{H})_1^i\right) - \rho^e \left(p_1^e + (\mu^e/2)(\mathbf{H}\cdot\mathbf{H})_1^e\right) + \varepsilon(t)\eta^i\rho^i \frac{\partial}{\partial r}\left(p_0^i + (\mu^i/2)(\mathbf{H}_0\cdot\mathbf{H}_0)^i\right) - \varepsilon(t)\eta^i\rho^e \frac{\partial}{\partial r}\left(p_0^e + (\mu^e/2)(\mathbf{H}_0\cdot\mathbf{H}_0)^e\right). \tag{2.45}$$

Substituting from (2.29) – (2.44) into (2.45) we obtain the following relation:

$$(\sigma + ikU_0)^2 = \frac{T}{\rho^i(R^i)^3} N_m M_m I'_m(x)(1 - m^2 - x^2) - \frac{\mu^i H_0^2}{\rho^i(R^i)^2}\left(yI_m(x)N_m M_m\right) - \frac{\mu^e H_0^2}{\rho^e(R^i)^2}\left(\rho x^2 I'_m(x)(R^2 - yL_y^m)N_m\right) \tag{2.46}$$

with

$$\begin{aligned} R &= R^e/R^i\ , \qquad \rho = \rho^e/\rho^i, \\ L_y^m &= \left(I'_m(y)K_m(x) - I_m(x)K'_m(y)\right), \\ x(N_m)^{-1} &= \{I_m(x)M_m + \rho x(R - yL_y^m)I'_m(x)\}. \end{aligned} \tag{2.47}$$

3. Discussions of the results

Equation (2.46) is the eigenvalue relation of MHD doubly perturbed liquid cylindrical stream having a fluid mantle jet, accounting for the inertia forces of both the liquid and the fluid regions. By means of this relation, the characteristics of the present model can be determined: one can identify the domains of instability (in particular their critical wavenumbers, maximum growth rate values and the corresponding wavenumbers) and those of stability as well.

The eigenvalue relation (2.46) relates the growth rate σ (or rather the oscillation frequency ω) with the value of $\left(\varepsilon T/\rho^i R^{i^2}\right)$, $\left(\mu^i H_0^2/\rho^i R^{i^2}\right)$ as well as $\left(\mu^e H_0^2/\rho^e R^{e^2}\right)$ as a unit of time, the radii ratio R, the densities ratio ρ and the densities ρ^e and ρ^i, the radii of the cylinders R^i and R^e, the surface tension coefficient T, the modified Bessel functions and their combinations N_m, M_m and L_y^m, and with the magnetic field H_0. The marginal or neutral stability could be obtained from (2.46) at $\sigma = 0$, the unstable states are those when σ is real while the stability states are those when σ is imaginary. The eigenvalue relation (2.46) is in reality a simple linear combination of the eigenvalue relations of concentric-coaxial doubly perturbed liquid-fluid cylinders endowed with surface tension only and those with electromagnetic forces only.

Since this problem is somewhat more general, one can obtain other dispersion relations as limiting cases from the present relation (2.46) under suitable assumptions.

If we assume $U_0 = 0$, $H_0 = 0$, $\rho^e = 0$ and $y \to \infty$, Eq. (2.46) gives

$$(3.1) \qquad \sigma^2 = \frac{T}{\rho^i (R^i)^3} (x I'_m(x)/I_m(x))(1 - m^2 - x^2) .$$

The relation (3.1) was established by Chandrasekhar [3], see also reference [2] as $m = 0$.

If we set $U_0 = 0$, $\rho^e = 0$ and, at the same time, $y \to \infty$, Eq. (2.46) reduces to

$$(3.2) \qquad \sigma^2 = \frac{T}{\rho^i (R^i)^3} \frac{x I'_m(x)}{I_m(x)} \left(1 - m^2 - x^2\right) + \frac{\mu^i H_0^2}{\rho^i (R^i)^2} \frac{x}{I_m(x) K'_m(x)} .$$

Equation (3.2) is the MHD eigenvalue of a full liquid jet in vacuum acted on by the electromagnetic Lorentz force and endowed with surface tension. It was Chandrasekhar [3] who derived the simple case of Eq. (3.2) at $m = 0$ of axisymmetric perturbation (Eq. (165), page 545) by means of the energy principle.

If we assume that $\rho^i = 0, H_0 = 0$ and $y \to \infty$, the relation (2.46) degenerates to

$$(3.3) \qquad (\sigma + ikU_0)^2 = -\frac{T}{\rho^e (R^i)^3} \frac{x K'_m(x)}{K_m(x)} \left(1 - m^2 - x^2\right) .$$

This is the eigenvalue relation of a hollow jet (i.e a gas jet of negligible inertia immersed in a streaming liquid which extends radially to infinity), endowed with surface tension. Equation (3.3) coincides with our dispersion relation of the recent result in references [13] as $U_0 = 0$ here and neglecting the viscosity effect there. One has to mention here that for $U_0 = 0$ the relation (2.46) leads to that given by Drazin and Reid [15]. Also Eq. (2.46) with $m = 0$ and $U_0 = 0$ reduced to that reported by Chandrasekhar [3] (page 540).

If we assume $H_0 = 0$ and $y \to \infty$, Eq. (2.46) reduces to

$$(3.4) \qquad (\sigma + ikU_0)^2 = \frac{T}{\rho^i (R^i)^3} \frac{x(1 - m^2 - x^2) I'_m(x) K_m(x)}{(I_m(x) K'_m(X) - \rho I'_m(x) K_m(x)} .$$

This relation might be identical with the relation obtained by Cheng [14] in investigating the instability of a streaming gas jet through an infinite liquid region, under the influence of the capillary force and pressure gradient forces, in addition to the inertia forces of both the gas and the liquid.

Other limiting cases may be obtained from the relation (2.46) with other simplifying assumptions:

1) $H_0 \neq 0$, $T = 0$, $U_0 \neq 0$,
2) $H_0 \neq 0$, $T = 0$, $y \to \infty$, $U_0 \neq 0$, and
3) $H_0 = 0$, $T \neq 0$, $U_0 \neq 0$.

In order to clarify the hydrodynamic, magnetodynamic and MHD (in-)stability states of the present problem, we have to remember the behaviour of the modified Bessel functions and their products for different arguments.

Consider now the recurrence relations (see ABRAMOWITZ and STEGUN [16])

$$2I'_m(x) = I_{m-1}(x) + I_{m+1}(x) \tag{3.5}$$

$$2K'_m(x) = -K_{m-1}(x) - K_{m+1}(x) \,. \tag{3.6}$$

It is known that $I_m(x)$ is always positive and monotonically increasing function, and that $K_m(x)$ is monotonically decreasing but never negative for each non-zero real value of x; hence one can observe that $I'_m(x)$ is positive while $K'_m(x)$ is always negative. On the basis of these arguments, we may prove that for $x \neq 0$ and $y \neq 0$

$$I_m(y) > I_m(x) \,, \tag{3.7}$$

$$K_m(y) < K_m(x) \,, \tag{3.8}$$

and that the function

$$L^m_y > 0 \tag{3.9}$$

is positive definite and never changes sing. We may also prove that

$$(I'_m(x)K'_m(x)) < (I'_m(y)K'_m(x)) \tag{3.10}$$

and consequently, for all $x \neq 0$ and $y \neq 0$ we have

$$M_m = xy\left\{-I'_m(x)\left|K'_m(x)\right| + I'_m(x)\left|K'_m(y)\right|\right\} > 0 \,. \tag{3.11}$$

In a similar manner, on utilizing the inequalities (3.7) – (3.10), we can show that

$$N_m > 0 \tag{3.12}$$

is positive definite and never changing sign for all values of $R > 1$, $\rho \neq 0$, $x \neq 0$, $y \neq 0$ and $m \geq 0$.

3.1. Hydrodynamic stability

In this case we neglect the influence of the electromagnetic forces in each of the fluid, liquid and vacuum regions. The eigenvalue relation of the present case is obtained from the general relation (2.46) as $H_0 = 0$, by

$$(\sigma + ikU_0)^2 = \frac{T}{\rho^i(R^i)^3}\left(1 - m^2 - x^2\right) I'_m(x) M_m N_m \,. \tag{3.13}$$

With the aid of the relations (3.5) and (3.6) together with the inequalities (3.7) – (3.12), the relation (3.13) is investigated and we deduce the following.

The stationary coaxial-concentric cylinders of doubly perturbed interfaces are capillary stable in the non-axisymmetric modes of perturbation $m \geq 1$ for all short and long wavelengths $\lambda(= 2\pi/x,\ x \neq 0)$. In the axisymmetric (sausage) mode $m = 0$ of the perturbation, the model is capillary unstable in the domain $0 < x < 1$ while it is stable in the neighbouring domains $1 \leq x < \infty$ where the equality corresponds to the marginal stability state. This means that present non-streaming model is unstable only in the axisymmetric mode $m = 0$ if the perturbed wavelength λ_i is longer than the circumference of the fluid cylinder $2\pi R^i$, while it is stable in all the remaining states. The streaming has a destabilizing influence and this influence is independent of the kind of the perturbation ($m = 0$ or/and $m \neq 0$) and the perturbed wavelengths. Therefore, the streaming has the property of increasing the capillary unstable domain ($m = 0$, $0 < x < 1$) and, at the same time, it decreases the stable domains ($m = 0, 1 \leq x < \infty$) and ($m \geq 0$, $0 < x < \infty$). Thus, the streaming liquid cylinder with doubly perturbed interfaces having a fluid cylinder mantle is capillary unstable in axisymmetric mode $m = 0$ and non-axisymmetric modes $m \neq 0$ for all short and long wavelengths.

3.2. Magnetodynamic stability

If the effect of the electromagnetic forces acting in the fluid, liquid and vacuum regions is predominant over that of the capillary forces acting along the fluid-liquid and liquid-vacuum interfaces, the general relation (2.46) degenerates to

$$(3.14) \qquad (\sigma + ikU_0)^2 = \left(\mu^e H_0^2\right) / \left(\rho^e R^{i^2}\right) \left\{\rho x^2 I'_m(x) N_m \left(y L_y^m - R^2\right)\right\} - \left(\mu^i H_0^2\right) / \left(\rho^i R^{i^2}\right) \left\{y I_m(x) M_m N_m\right\},$$

where ρ, R, N_m and M_m are defined by (2.47) and (2.44).

The influence of the magnetic field in the fluid region is represented by the term associated with the natural quantity $\mu^i H_0^2 / \left(\rho^i (R^i)^2\right)$. By the use of the inequalities (3.7) – (3.12), we can show that the magnetic field acting in the interior of the fluid cylinder is stabilizing. That stabilizing effect is valid for all short and long wavelengths, not only in the $m = 0$ mode but also in $m \neq 0$ modes. The influence of the magnetic field acting in the liquid cylinder of doubly perturbed interfaces is represented by the terms associated with the quantity $\mu^e H_o^2 / \left(\rho^e (R^i)^2\right)$. In view of the inequalities (3.7) – (3.12) it is clear that $\rho x^2 I'_m(x) N_m$ is positive definite for all non-zero values of ρ, x, y and $m \leq 0$. Therefore the magnetic field in the liquid region is stabilizing in all modes $m \geq 0$ of perturbation if the condition

$$(3.15) \qquad R^2/y > L_y^m$$

is satisfied. On the other hand, if that condition is not satisfied then the electromagnetic force acting in the liquid region is purely destabilizing and moreover, it has no influence at all if $R = yL_y^m$. Therefore, the non-streaming coaxial cylinders are magnetodynamically stable in the $m = 0$ or/and $m \neq 0$ modes according to restrictions (cf. (3.15)).

The streaming has a destabilizing influence. That influence is independent of the kind of perturbation (axisymmetric or non-axisymmetric) and short or long wavelengths. Consequently, the streaming has the property of decreasing the magnetodynamic stable domains and at the same time, it increases those of instability.

We conclude that the streaming coaxial fluid-liquid cylinders are magnetodynamically unstable not only in the axisymmetric mode $m = 0$ but also in the non-axisymmetric modes $m \neq 0$ of perturbation.

3.3. Magnetohydrodynamic stability

In this general case the present model of a liquid cylinder with doubly perturbed interfaces having a fluid jet mantle is acted upon by combined effects of the pressure gradient inertial, capillary and electromagnetic forces. The dispersion relation of this case is given by the relation (2.46) in its general form. With the aid of the discussions and results of the two different cases given above in Subsecs. 3.1 and 3.2, the present general case could be discussed.

As we have seen, streaming has a purely destabilizing influence while the capillary forces have a destabilizing influence only in the axisymmetric mode $m = 0$. The electromagnetic forces have a strong stabilizing influence for all modes $m \geq 0$ of perturbation. The latter plays an important role in stabilizing the present model. When the intensity of the magnetic field is so strong that the Alfvèn wave velocity is greater than the fluid streaming velocity, the capillary instability is suppressed and stability sets in. See the numerical example later on.

The stabilizing influence of the electromagnetic force could be interpreted physically as follows.

Indeed, the stabilizing effects of the magnetic fields in the liquid and fluid regions are due to the presence of the magnetodynamic force $\mu(\nabla \wedge \mathbf{H}) \wedge \mathbf{H}$ in the MHD equations of motion (2.3) and (2.17). This force may be interpreted as following from the action on the fluid of the Maxwell stresses: tension $\mu(\mathbf{H} \cdot \mathbf{H})/2$ per unit area along the magnetic lines of force and an equal pressure acting in all directions in the conducting fluid. Note that the latter is not perpendicular to the magnetic lines of force and acting in all directions since the diffusion terms are neglected in the evolution equation of the magnetic field (2.6). Due to these stresses, the lines of force are able to endow the liquid with a sort of rigidity.

Another important physical explanation may be given here for the stabilizing effects of both magnetic fields in the fluid, liquid and vacuum regions as follows. The magnetic field exerts a strong influence not only on the axisymmetric mode that causes only the bending of the magnetic lines of force, but also on non-axisymmetric modes that leads also to twisting of the lines of force. This is physically plausible since the magnetic fields are conserved (cf. (2.5)) and, in addition, they are uniform, see Eqs. (2.2) and (2.2)′.

In order to clarify the electromagnetic force effect and its influence on the capillary force, it is found more convenient to write down the eigenvalue relation (2.46) in the dimensionless form

$$\text{(3.16)}\qquad \frac{(\sigma+ikU_0)^2}{T/\rho^i\,(R^i)^3}=\left\{\left(1-m^2-x^2\right)M_m-\left(H_0/H_s^i\right)^2\left(M_m\frac{yI_m(x)}{I'_m(x)}\right)\right.$$
$$\left.-\left(H_0/H_s^e\right)^2\left(\rho x^2\left(R^2-yL_y^m\right)\right)\right\}I'_m(x)N_m,$$

where

$$\text{(3.17)}\qquad H_s^i = \left(T/\mu^i\rho^i\right)^{1/2},$$

$$\text{(3.18)}\qquad H_s^e = \left(T/\rho^e\mu^e\right)^{1/2}.$$

The relation (3.16) has been analyzed by the computer for the most dangerous mode $m=0$ and assuming that μ^e is infinitesimally small and tends to zero. The calculations have been carried out for different values of (H_0/H_s), (ρ^i/ρ^e) and (R^e/R^i). For every group of values of the last magnitude, several values of $U(=-ikU_0/(T/\rho^i(R^i)^3)^{1/2})$ are given. The numerical data are classified into two types. The results of the instability states are tabulated and presented for the values of $\sigma/(T/\rho^i(R^i)^3)^{1/2}$ against x while those associated with stability states are illustrated for the values of $\omega/(T/\rho^i(R^i)^3)^{1/2}$ against x. These states are analyzed in various categories as follows.

CATEGORY (i). For $(H_0/H_s,\ \rho^e/\rho^i)=(0.3,\ 0.3)$

As $(R^e/R^i)=1.3$, corresponding to $U=0$, 0.2, 0.5, 0.7 and 1.0, it is found that the unstable domains are $0<x<0.718$, $0<x<0.844$, $0<x<1.0911$, $0<x<1.2447$ and $0<x<1.453$, while those of stability neighbouring the unstable domains are $x\geq 0.718$, $x\geq 0.844$, $x\geq 1.0911$, $x\geq 1.2447$ and $x\geq 1.453$, see Fig. 1.

As $(R^e/R^i)=1.7$, corresponding to $U=0$, 0.2, 0.5, 0.7 and 1.0, it is found that the unstable domains are $0<x<0.8165$, $0<x<0.9101$, $0<x<1.1061$, $0<x<1.2426$ and $0<x<1.4356$ while those of stability are such that $x\geq 0.8165$, $x\geq 0.9101$, $x\geq 1.1061$, $x\geq 1.243$ and $x\geq 1.436$. See Fig 2.

As $(R^e/R^i)=3.0$, corresponding to $U=0$, 0.2, 0.5, 07 and 1.0, it is found that the unstable domains are $0<x<0.8768$, $0<x<0.936$, $0<x<1.111$, $0<x<1.242$ and $0<x<1.431$ while those of stability are such that $x\geq 0.877$, $x\geq 0.936$, $x\geq 1.1111$, $x\geq 1.242$ and $x\geq 1.431$.

CATEGORY (ii). For $(H_0/H_s,\rho^e/\rho^i)=(0.5,\ 0.3)$

As $(R^e/R^i)=1.3$, corresponding to $U=0.2$, 0.5, 0.7 and 1.0, it is found that the unstable domains are $0<x<0.489$, $0<x<0.863$, $0<x<1.047$ and $0<x<1.2823$ while those of stability are $x\geq 0.489$, $x\geq 0.862$, $x\geq 1.047$ and $x\geq 1.282$. Note that there is no unstable domain as $U=0$. See Fig. 2.

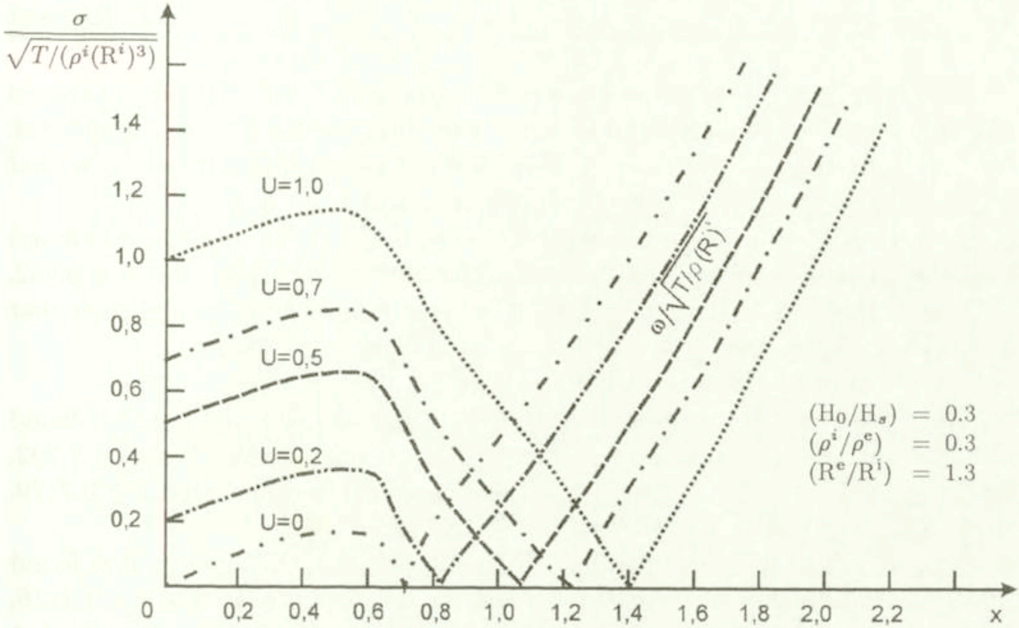
σ
$\sqrt{T/(\rho^i(R^i)^3)}$
1,4
1,2
1,0
0,8
0,6
0,4
0,2
0
U=1,0
U=0,7
U=0,5
U=0,2
U=0
0,2
0,4
0,6
0,8
1,0
1,2
1,4
1,6
1,8
2,0
2,2
x
$(H_0/H_s) = 0.3$
$(\rho^i/\rho^e) = 0.3$
$(R^e/R^i) = 1.3$

FIG. 1.

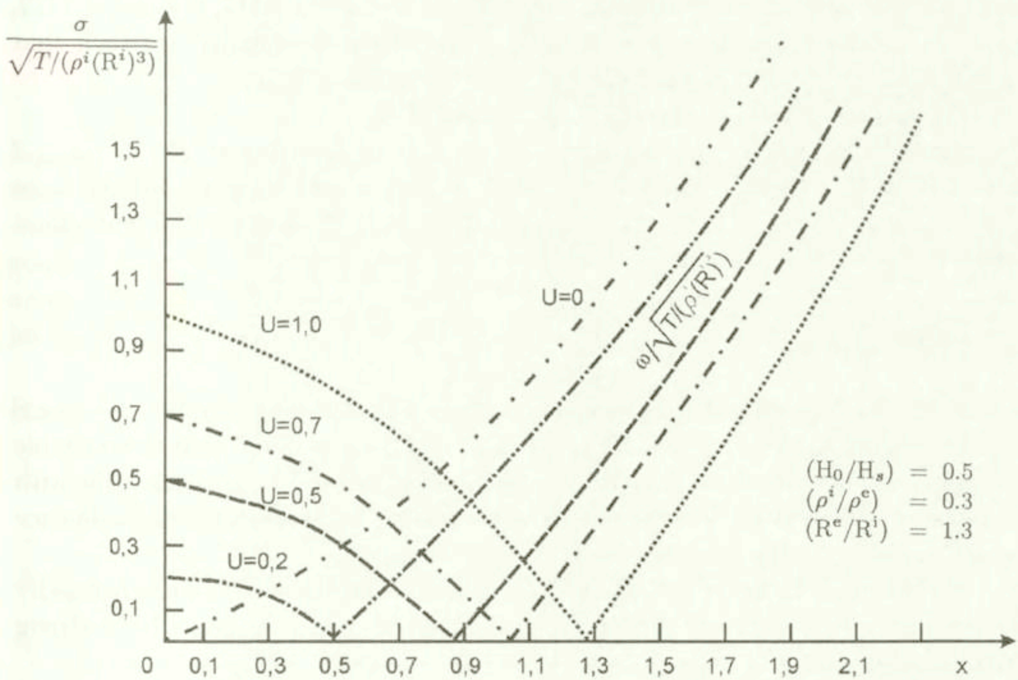
σ
$\sqrt{T/(\rho^i(R^i)^3)}$
1,5
1,3
1,1
0,9
0,7
0,5
0,3
0,1
0
U=0
U=1,0
U=0,7
U=0,5
U=0,2
0,1
0,3
0,5
0,7
0,9
1,1
1,3
1,5
1,7
1,9
2,1
x
$(H_0/H_s) = 0.5$
$(\rho^i/\rho^e) = 0.3$
$(R^e/R^i) = 1.3$

FIG. 2.

As $(R^e/R^i) = 1.7$, corresponding to $U = 0$, 0.2, 0.5, 0.7 and 1.0, it is found that the unstable domains are $0 < x < 0.469$, $0 < x < 0.6477$, $0 < x < 0.934$, $0 < x < 1.0812$ and $0 < x < 1.3045$, while those of stability are such that $x \geq 0.469$, $x \geq 0.6477$, $x \geq 0.934$, $x \geq 1.0812$ and $x \geq 1.3045$.

As $(R^e/R^i) = 3.0$, corresponding to $U = 0$, 0.2, 0.5, 0.7 and 1.0, it is found that the unstable domains are $0 < x < 0.5976$, $0 < x < 0.7198$, $0 < x < 0.962$, $0 < x < 1.1093$ and $0 < x < 1.3112$, while those of stability are such that $x \geq 0.5976$, $x \geq 0.7198$, $x \geq 0.962$, $x \geq 1.1093$ and $x \geq 1.3112$.

CATEGORY (iii). For $(H_0/H_s, \rho^e/\rho^i) = (0.3, 1.0)$

As $(R^e/R^i) = 1.3$, corresponding to $U = 0$, 0.2, 0.5, 0.7 and 1.0, it is found that the unstable domains are $0 < x < 0.7179$, $0 < x < 0.892$, $0 < x < 1.202$, $0 < x < 1.384$ and $0 < x < 1.628$ while those of stability are such that $x \geq 0.7179$, $x \geq 0.892$, $x \geq 1.202$, $x \geq 1.384$ and $x \geq 1.628$.

As $(R^e/R^i) = 1.7$, corresponding to $U = 0$, 0.2, 0.5, 0.7 and 1.0, it is found that the unstable domains are $0 < x < 0.8165$, $0 < x < 0.927$, $0 < x < 1.1616$, $0 < x < 1.3176$ and $0 < x < 1.537$, while those of stability are such that $x \geq 0.8165$, $x \geq 0.927$, $x \geq 1.162$, $x \geq 1.3176$ and $x \geq 1.537$.

As $(R^e/R^i) = 3.0$, corresponding to $U = 0$, 0.2, 0.5, 0.7 and 1.0, it is found that the unstable domains are $0 < x < 0.8769$, $0 < x < 0.9417$, $0 < x < 1.1472$, $0 < x < 1.2959$ and $0 < x < 1.5113$, while those of stability are such that $x \geq 0.8769$, $x \geq 0.9417$, ≥ 1.1472, $x \geq 1.2959$ and $x \geq 1.5113$.

CATEGORY (iv). For $(H_0/H_s, \rho^e/\rho^i) = (0.5, 1.0)$

Similar discussions may be presented for the cases with $(R^e/R^i) = 1.3$, 1.7 and 3.0 for $U = 0$, 0.2, 0.5, 0.7 and 1.0. See Fig. 3, and similarly for the cases $(H_0/H_s, \rho^e/\rho^i) = (1.0, 1.0), (3.0, 3.0), (5, 3)$ and $(1, 3)$, and for different values of (R^e/R^i) and U.

4. Conclusions

From the foregoing discussions we conclude the following results.

For the same values of H_0/H_s, ρ^e/ρ^i and R^e/R^i, it is found that the unstable domains are increasing and simultaneously those of instability are decreasing with increasing values of U. This means that streaming has a destabilizing influence and this confirms the analytical discussions.

For the same values of (ρ^e/ρ^i), (R^e/R^i) and U, the stable domains are rapidly increasing with increasing H_0/H_s values, i.e. the magnetic fields have strong stabilizing effects.

For the same values of H_0/H_s, U and ρ^e/ρ^i, the stable domains are monotonically increasing with increasing R^e/R^i. This means that the liquid-fluid radii ratio plays an important role in stabilizing the annular fluid jet with doubly perturbed interfaces.

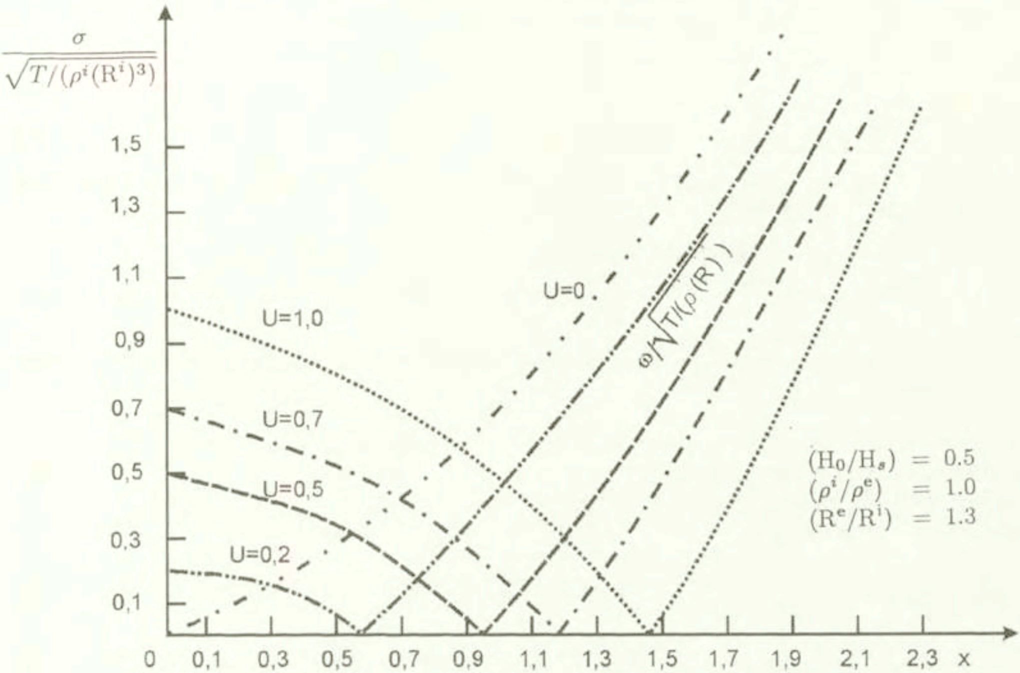

FIG. 3.

For the same values of (H_0/H_s), R^e/R^i and $U(\geq 0)$, the unstable domains are rapidly decreasing with increasing ρ^e/ρ^i values. This means that the liquid-fluid densities ratio is stabilizing.

As $\rho^i = 0, R \to \infty$ and $H_0 = 0$, it is found that the present results are in good agreement with the experimental results of Kendall (1986) that the temporal amplification in the hollow jet is much higher than that appearing in a full liquid jet embedded in a vacuum.

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Received December 5, 1997.