Variational bounds on the effective moduli of anisotropic piezoelectric composites

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INEQUALITIES of Thompson and Dirichlet type have been used to derive the elementary bounds on elastic, piezoelectric and dielectric macroscopic moduli. The Ritz method has been applied to determine approximately the effective moduli.

1. Introduction

PIEZOELECTRIC MATERIALS are often used in modern technical devices. The necessity of modelling the behaviour of intelligent materials significantly influences the interest of many researchers in finding effective properties of piezoelectric composites, cf. [1 – 5] and the references cited therein. We observe that biological materials such like dry bones also exhibit piezoelectric properties, cf. [6]. It seems that the paper by the third author [6] was the first on non-uniform homogenization of piezoelectric composites, which includes periodic homogenization as a particular case. In an accompanying paper [5], a detailed study of Γ-convergence for a class of physically nonlinear piezocomposites has been performed. This class includes the linear case, which is the starting point for the present paper. To determine the effective moduli one has to solve appriopriate periodic boundary value problems posed on the so-called basic cell. For two- and three-dimensional problems such cell problems cannot be exactly solved, hence the need for approximate methods and bounding techniques. In the present contribution we shall consider the problem of variational bounds for the effective moduli. Inequalities of Thompson and Dirichlet type are used to derive elementary bounds on elastic, piezoelectric and dielectric macroscopic moduli. Moreover, for two-phase composites the Ritz method is applied to determine approximately the effective moduli. The results obtained by using Ritz's method compare favourably with upper and lower bounds on the corresponding coefficients.

2. Piezoelectric solid with a microperiodic structure

The elastic, piezoelectric and dielectric moduli are denoted by c_{ijkl}, g_{ijk}, ε_{ij}, respectively, $(i, j, k, l = 1, 2, 3)$. The constitutive relations are given by

$$(2.1) \qquad D_i = \varepsilon_{ij} E_j + g_{ikl} e_{kl},$$

$$(2.2) \qquad \sigma_{ij} = c_{ijkl} e_{kl} - g_{kij} E_k,$$

or equivalently

$$(2.3) \qquad E_k = \kappa_{kj} D_j - h_{kij} e_{ij},$$

$$(2.4) \qquad \sigma_{ij} = a_{ijkl} e_{kl} - h_{kij} D_k,$$

or

$$(2.5) \qquad D_i = \hat{\kappa}_{ij} E_j - \hat{h}_{ikl} \sigma_{kl},$$

$$(2.6) \qquad e_{ij} = \hat{a}_{ijkl} e_{kl} - \hat{h}_{kij} E_k,$$

where

$$(2.7) \qquad \kappa_{ij} = (\varepsilon^{-1})_{ij}, \quad h_{kij} = (\varepsilon^{-1})_{kn} g_{nij}, \quad a_{ijkl} = c_{ijkl} + (\varepsilon^{-1})_{rn} g_{rij} g_{nkl}$$

and

$$(2.8) \qquad \hat{a}_{ijkl} = (\mathbf{c}^{-1})_{ijkl}, \qquad c_{ijkl} = a_{ijkl} - h_{mij}(\kappa^{-1})_{mn} h_{nkl},$$

$$(2.9) \qquad (\hat{\kappa}^{-1})_{ij} = \kappa_{ij} - h_{ikl}(a^{-1})_{klmn} h_{jmn}, \quad \hat{h}_{ijk} = -h_{lmn}(\kappa^{-1})_{il} \hat{a}_{mnjk}.$$

By $(\varepsilon^{-1})_{ij}, (\mathbf{c}^{-1})_{ijkl}$ etc. we denote the components of matrices being inverse of matrices $(\varepsilon_{ij}), (c_{ijkl})$ etc., respectively. Here D_i, E_i, σ_{ij}, e_{ij} are components of the electric displacement vector, the electric field, the stress tensor and the strain tensor, respectively. In the linear case, being the subject of the present contribution, the internal energy $U(y, \mathbf{e}, \mathbf{D})$, $y \in Y$ and its dual $U^*(y, \boldsymbol{\sigma}, \mathbf{E})$, have the following form:

$$(2.10) \qquad U(y, \mathbf{e}, \mathbf{D}) = \frac{1}{2} a_{ijkl}(y) e_{ij} e_{kl} - h_{ijk}(y) D_i e_{jk} + \frac{1}{2} \kappa_{ij}(y) D_i D_j,$$

$$(2.11) \qquad U^*(y, \boldsymbol{\sigma}, \mathbf{E}) = \frac{1}{2} \hat{a}_{ijkl}(y) \sigma_{ij} \sigma_{kl} - \hat{h}_{ijk}(y) E_i \sigma_{jk} + \frac{1}{2} \hat{\kappa}_{ij}(y) E_i E_j.$$

The set $Y \subset \mathbb{R}^3$ is called the basic cell, cf. [2, 3] and the references cited therein. The dual function U^* is calculated as the Fenchel conjugate of U, see [7]

$$(2.12) \qquad U^*(y, \boldsymbol{\sigma}, \mathbb{E}) = \sup \left\{ \sigma_{ij} e_{ij} + E_i D_i - U(y, \mathbf{e}, \mathbf{D}) | \mathbf{e} \in \mathbb{E}_s^3, \mathbf{D} \in \mathbb{R}^3 \right\},$$

where $\mathbb{E}_s^3$ stands for the space of symmetric 3×3 matrices. The functions a_{ijkl}, h_{ijk} and κ_{ij} are Y-periodic. The elastic, piezoelectric and dielectric coefficients satisfy obvious symmetry conditions. The function $U(y, \cdot, \cdot)$ is assumed to be strictly convex, cf. [1, 6].

3. Homogenization

The density of the macroscopic or homogenized potential U^h is expressed as follows [6]:

$$(3.1) \qquad U^h(\mathbf{e}^h, \mathbf{D}^h) = \inf_{\mathbf{v} \in \mathbf{H}_{\mathrm{per}}} \inf_{\mathbf{D} \in \Delta_{\mathrm{per}}} \frac{1}{|Y|} \int_Y \left[\frac{1}{2} a_{ijkl}(y) \left(e_{ij}^y(\mathbf{v}) + e_{ij}^h \right) \right.$$

$$\cdot \left(e_{kl}^y(\mathbf{v}) + e_{kl}^h \right) - h_{kij}(y) \left(D_k(y) + D_k^h \right) \left(e_{ij}^y(\mathbf{v}) + e_{ij}^h \right)$$

$$\left. + \frac{1}{2} \kappa_{ij}(y) \left(D_i(y) + D_i^h \right) \left(D_j(y) + D_j^h \right) \right] dy,$$

where $\mathbf{e}^h \in \mathbb{E}_s^3$, $\mathbf{D}^h \in \mathbb{R}^3$ and $e_{ij}^y(\mathbf{v}) = \frac{1}{2} \left(\dfrac{\partial v_j}{\partial y_i} + \dfrac{\partial v_i}{\partial y_j} \right)$.

The superscript h stands for a homogenized quantity and

$$(3.2) \qquad \mathbf{H}_{\mathrm{per}} = \{ \mathbf{u} = (u_i) \in H^1(Y)^3 | \mathbf{u} \text{ takes equal values}$$

$$\text{on the opposite faces of } Y \},$$

$$(3.3) \qquad \Delta_{\mathrm{per}} = \{ \mathbf{D} = (D_i) \, | D_i \in L^2(Y), \ \mathrm{div}_y \mathbf{D} = 0 \ \text{ in } Y, \ \int_Y \mathbf{D}(y) dy = 0,$$

$$D_i N_i \quad \text{takes opposite values on the opposite sides of } Y \}.$$

The density of the macroscopic dual potential U^{*h} is expressed as follows:

$$(3.4) \qquad U^{*h}(\boldsymbol{\sigma}^h, \mathbf{E}^h) = \inf_{\phi \in H_{\mathrm{per}}^1} \inf_{\sigma \in S_{\mathrm{per}}} \frac{1}{|Y|} \int_Y \left[\frac{1}{2} \hat{a}_{ijkl}(y) \left(\sigma_{ij}(y) + \sigma_{ij}^h \right) \right.$$

$$\cdot \left(\sigma_{kl}(y) + \sigma_{kl}^h \right) - \hat{h}_{kij}(y) \left(E_k^y(\phi) + E_k^h \right) \left(\sigma_{ij}(y) + \sigma_{ij}^h \right)$$

$$\left. + \frac{1}{2} \hat{\kappa}_{ij}(y) \left(E_i^y(\phi) + E_i^h \right) \left(E_j^y(\phi) + E_j^h \right) \right] dy,$$

where $\boldsymbol{\sigma}^h \in \mathbb{E}_s^3$, $\mathbf{E}^h \in \mathbb{R}^3$, $E_i^y(\phi) = -\dfrac{\partial \phi}{\partial y_i}$, and

$$(3.5) \qquad H_{\mathrm{per}}^1 = \left\{ \varphi \in H^1(Y) \,|\, \varphi \text{ is } Y\text{-periodic} \right\},$$

$$(3.6) \qquad S_{\mathrm{per}} = \left\{ \sigma \in L^2(Y, \mathbb{E}_s^2) \,| \mathrm{div}_y \sigma = 0 \ \text{ in } Y, \ \int_Y \sigma(y) dy = 0, \right.$$

$$\sigma_{ij} N_j \text{ takes opposite values on the opposite faces of } Y \}.$$

Here $\mathbf{N}$ is the outward unit vector normal to ∂Y.

Remark. The formula (3.1) results from the theory of Γ-convergence applied to convex periodic homogenization problems [6, 8]. The general case of convex

homogenization of piezoelectric composites with periodic or non-uniformly periodic microstructure has been investigated in our accompanying paper [5].

The form (3.4) of the dual potential U^{*h} is obtained by applying Azé's theory of duality, cf. [5, 6, 9].

4. Elementary bounds

The homogenized internal energy (3.1) can be written in the following form [6]:

$$(4.1) \qquad U^h(\mathbf{e}^h, \mathbf{D}^h) = \frac{1}{2}a^h_{ijkl}e^h_{ij}e^h_{kl}, -h^h_{kij}D^h_k e^h_{ij} + \frac{1}{2}\kappa^h_{ij}D^h_i D^h_j,$$

where

$$a^h_{ijkl} = \frac{\partial^2 U^h}{\partial e^h_{ij}\partial e^h_{kl}}, \qquad h^h_{kij} = \frac{\partial^2 U^h}{\partial e^h_{ij}\partial D^h_k}, \qquad \kappa^h_{ij} = \frac{\partial^2 U^h}{\partial D^h_i \partial D^h_j}.$$

Explicit expressions in terms of a solution $(\tilde{\mathbf{v}}, \tilde{\mathbf{D}}) \in \mathbf{H}_{per} \times \Delta_{per}$ of the minimization problem appearing on the r.h.s of (3.1) were derived in [6]. Supose that the trial fields are: $e^y_{ij}(\mathbf{v}) = 0$, $D_i(y) = 0$, $\sigma_{ij}(y) = 0$, and $\phi(y) = 0$. Then we get

$$(4.2) \qquad U^h(\mathbf{e}^h, \mathbf{D}^h) \le \frac{1}{2} <a_{ijkl}(y)> e^h_{ij}e^h_{kl} - <h_{kij}(y)> D^h_k e^h_{ij}$$

$$+ \frac{1}{2} <\kappa_{ij}(y)> D^h_i D^h_j,$$

and

$$(4.3) \qquad U^{*h}(\sigma^h, \mathbf{E}^h) \le \frac{1}{2} <\hat{a}_{ijkl}(y)> \sigma^h_{ij}\sigma^h_{kl} - <\hat{h}_{kij}(y)> E^h_k \sigma^h_{ij}$$

$$+ \frac{1}{2} <\hat{\kappa}_{ij}(y)> E^h_i E^h_j,$$

where

$$(4.4) \qquad <\cdot> = \frac{1}{|Y|} \int_Y (\cdot)dy.$$

For convex functions f and g the inequality $f(x) \le g(x)$ implies $f^*(x^*) \ge g^*(x^*)$, cf. [7]. Hence

$$(4.5) \qquad U^h(\mathbf{e}^h, \mathbf{D}^h) \ge \frac{1}{2}a^v_{ijkl}e^h_{ij}e^h_{kl} - h^v_{kij}D^h_k e^h_{ij} + \frac{1}{2}\kappa^v_{ij}D^h_i D^h_j,$$

where a^v_{ijkl} and κ^v_{ij} are the components of the matrices inverse to the matrices $\mathbf{A}$ and $\mathbf{K}$ respectively, where

$$(4.6) \qquad A_{ijkl} = <\hat{a}_{ijkl}> - <\hat{h}_{mij}> (<\hat{\kappa}>^{-1})_{mn} <\hat{h}_{nkl}>,$$

$$(4.7) \qquad K_{ij} = <\hat{\kappa}_{ij}> - <\hat{h}_{ikl}> (<\hat{a}>^{-1})_{klmn} <\hat{h}_{jmn}>,$$

and

(4.8) $$h^v_{ijk} = -(< \hat{h}_{lmn} > (< \hat{\kappa} >^{-1})_{il} a^v_{mnjk}.$$

The inequalities (4.2), (4.5) are the elementary bounds on the homogenized internal energy cf. [3]. Taking into account (4.1), (4.2) and (4.5) and taking $\mathbf{D}^h = 0$ or $\mathbf{e}^h = 0$, we obtain two particular bounds

(4.9) $$a^v_{ijkl} e^h_{ij} e^h_{kl} \leq a^h_{ijkl} e^h_{ij} e^h_{kl} \leq < a_{ijkl}(y) > e^h_{ij} e^h_{kl}, \quad \text{for each } \mathbf{e}^h \in \mathbb{E}^3_s,$$

(4.10) $$\kappa^v_{ij} D^h_i D^h_j \leq \kappa^h_{ij} D^h_i D^h_j \leq < \kappa_{ij}(y) > D^h_i D^h_j, \quad \text{for each } \mathbf{D}^h \in \mathbb{R}^3.$$

The inequalities (4.9) and (4.10) are equivalent to positive definiteness of the following quadratic forms:

(4.11) $$(a^h_{ijkl} - a^v_{ijkl}), \; (< a_{ijkl}(y) > -a^h_{ijkl}), \; (\kappa^h_{ij} - \kappa^v_{ij}), \; (< \kappa_{ij}(y) > -\kappa^h_{ij}).$$

By using these inequalities one can derive the elementary bounds on the diagonal elements of Voigt matrices, that is the bounds on these particular homogenized material coefficients. Now they are of the form

(4.12) $$a^v_{ijkl} \leq a^h_{ijkl} \leq < a_{ijkl}(y) >,$$

(4.13) $$\kappa^v_{ij} \leq \kappa^h_{ij} \leq < \kappa_{ij}(y) >,$$

provided that

$$(ijkl) \in \{(1111), (2222), (3333), (1212), (1313), (2323)\},$$
(4.14)
$$(ij) \in \{(11), (22), (33)\}.$$

Every inequality for the homogenized coefficients a^h_{ijkl} and κ^h_{ij} with indices not belonging to (4.14) includes at least two elements from the set of the homogenized moduli a^h_{ijkl} and κ^h_{ij} with indices from (4.14). For instance, for a^h_{1133} the inequalities which include a^h_{1111} and a^h_{3333} have the form:

(4.15) $$|a^h_{1133} - < a_{1133} >| \leq \sqrt{(a^h_{1111} - < a_{1111} >)(a^h_{3333} - < a_{3333} >)},$$

(4.16) $$|a^h_{1133} - a^v_{1133}| \leq \sqrt{(a^h_{1111} - a^v_{1111})(a^h_{3333} - a^v_{3333})}.$$

Any inequality satisfied by h^h_{kij} results directly from (4.2) and (4.5). It always contains at least two elements from the set of the homogenized coefficients a^h_{ijkl} and κ^h_{ij} with indices from the sets defined by (4.14). For instance, the inequalites satisfied by h^h_{113} and involving a^h_{1313} and κ^h_{11} have the following form:

(4.17) $$|h^h_{113} - h^v_{113}| \leq \sqrt{(a^h_{1313} - a^v_{1313})(\kappa^h_{11} - \kappa^v_{11})},$$

(4.18) $$|h^h_{113} - < h_{113} >| \leq \sqrt{(a^h_{1313} - < a_{1313} >)(\kappa^h_{11} - < \kappa_{11} >)}.$$

The bounds obtained in this way are rather wide. Substituting

$$(4.19) \qquad e_{ij}^h = P_{kij} D_k^h, \qquad D_i^h = Q_{ikl} e_{kl}^h,$$

into (4.2) and (4.5) and performing optimization of the expressions thus obtained with respect to P_{kij} and Q_{ikl}, we get

$$(4.20) \qquad \kappa_{ij}^v + (h_{ikl}^h - h_{ikl}^v)((\mathbf{a}^h - \mathbf{a}^v)^{-1})_{klmn}(h_{jmn}^h - h_{imn}^v) \leq \kappa_{ij}^h,$$

$$(4.21) \qquad \kappa_{ij}^h \leq <\kappa_{ij}>$$

$$+(h_{ikl}^h - <h_{ikl}>)((\mathbf{a}^h - <\mathbf{a}>)^{-1})_{klmn}(h_{jmn}^h - <h_{imn}>),$$

and

$$(4.22) \qquad a_{ijkl}^v + (h_{mij}^h - h_{mij}^v)\left((\boldsymbol{\kappa}^h - \boldsymbol{\kappa}^v)^{-1}\right)_{mn}(h_{nkl}^h - h_{nkl}^v) \leq a_{ijkl}^h,$$

$$(4.23) \qquad a_{ijkl}^h \leq <a_{ijkl}>$$

$$+(h_{mij}^h - <h_{mij}>)((\boldsymbol{\kappa}^h - <\boldsymbol{\kappa}>)^{-1})_{mn}(h_{nkl}^h - <h_{nkl}>).$$

For indices from the set (4.14) these bounds are narrower than the Voigt-Reuss bounds (4.12), (4.13).

5. Example: two-phase composite

Let us consider a fibre-reinforced composite, in which the matrix and fibre materials are both homogeneous and transversely isotropic, with the axis of transverse isotropy oriented in the direction of the fibres. The fibres are distributed in such a manner that the symmetry of the homogenized material is the same as that of the plane basic cell.

For convenience, six independent Cartesian components of the strain tensor $\mathbf{e}$ and three Cartesian components of the electric displacement vector $\mathbf{D}$ are arranged in the following manner:

$$(5.1) \qquad \mathbf{F}^T = (e_{11}, e_{22}, e_{33}, 2\,e_{23}, 2\,e_{31}, 2\,e_{12}, D_1, D_2, D_3),$$

where the superscript T denotes the transpose. The free-energy densities of the fibres and of the matrix are quadratic functionals of $\mathbf{F}$, and hence are characterized by two 9×9 symmetric positive-definite matrices, say $\overset{(1)}{\mathbf{A}}$ and $\overset{(2)}{\mathbf{A}}$

$$(5.2) \qquad \overset{(1)}{U}(\mathbf{F}) = \frac{1}{2}\mathbf{F}^T \cdot \overset{(1)}{\mathbf{A}} \cdot \mathbf{F},$$

$$(5.3) \qquad \overset{(2)}{U}(\mathbf{F}) = \frac{1}{2}\mathbf{F}^T \cdot \overset{(2)}{\mathbf{A}} \cdot \mathbf{F}.$$

The material constants for the fibre material PZT-7A and for the matrix material are given by

$$(5.4) \qquad \overset{(1)}{\mathbf{A}} = \left[\begin{array}{c|c} \overset{(1)}{\mathbf{a}} & \overset{(1)}{\mathbf{h}}{}^T \\ \hline \overset{(1)}{\mathbf{h}} & \overset{(1)}{\boldsymbol{\kappa}} \end{array} \right] \quad \text{and} \quad \overset{(2)}{\mathbf{A}} = \left[\begin{array}{c|c} \overset{(2)}{\mathbf{a}} & \overset{(2)}{\mathbf{h}}{}^T \\ \hline \overset{(2)}{\mathbf{h}} & \overset{(2)}{\boldsymbol{\kappa}} \end{array} \right] ,$$

where $\overset{(1)}{\mathbf{a}}$ are the elastic moduli in [GPa]

$$(5.5) \qquad \overset{(1)}{\mathbf{a}} = \begin{bmatrix} 157 & 85.4 & 73 & 0 & 0 & 0 \\ 85.4 & 157 & 73 & 0 & 0 & 0 \\ 73 & 73 & 175 & 0 & 0 & 0 \\ 0 & 0 & 0 & 47.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 47.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 35.8 \end{bmatrix} .$$

By using (2.7) we find

$$(5.6) \qquad \overset{(1)}{\mathbf{c}} = \begin{bmatrix} 154.84 & 83.237 & 82.712 & 0 & 0 & 0 \\ 83.237 & 154.84 & 82.712 & 0 & 0 & 0 \\ 82.712 & 82.712 & 131.39 & 0 & 0 & 0 \\ 0 & 0 & 0 & 25.696 & 0 & 0 \\ 0 & 0 & 0 & 0 & 25.696 & 0 \\ 0 & 0 & 0 & 0 & 0 & 35.8 \end{bmatrix} ,$$

since the piezoelectric coefficients $\overset{(1)}{\mathbf{h}}$ in [V/nm] and $\overset{(1)}{\mathbf{g}}$ in [V/F] are specified by

$$(5.7) \qquad \overset{(1)}{\mathbf{h}} = \begin{bmatrix} 0 & 0 & 0 & 0 & -2.3 & 0 \\ 0 & 0 & 0 & -2.3 & 0 & 0 \\ 1.02 & 1.02 & -4.58 & 0 & 0 & 0 \end{bmatrix} ,$$

$$(5.8) \qquad \overset{(1)}{\mathbf{g}} = \begin{bmatrix} 0 & 0 & 0 & 0 & -9.35 & 0 \\ 0 & 0 & 0 & -9.35 & 0. & 0 \\ 2.121 & 2.121 & -9.52 & 0 & 0 & 0 \end{bmatrix} .$$

Here the dielectric coefficients $\overset{(1)}{\boldsymbol{\kappa}}$ in [m/nF] and $\overset{(1)}{\boldsymbol{\varepsilon}}$ in [nF/m] are given by

$$(5.9) \qquad \overset{(1)}{\boldsymbol{\kappa}} = \begin{bmatrix} 0.246 & 0 & 0 \\ 0 & 0.246 & 0 \\ 0 & 0 & 0.481 \end{bmatrix} ,$$

$$(5.10) \qquad \overset{(1)}{\boldsymbol{\varepsilon}} = \begin{bmatrix} 4.065 & 0 & 0 \\ 0 & 4.065 & 0 \\ 0 & 0 & 2.079 \end{bmatrix} .$$

Let us pass to the specification of the material coefficients of the epoxy matrix. Now the elastic moduli are given by

$$(5.11) \qquad \overset{(2)}{\mathbf{a}} = \overset{(2)}{\mathbf{c}} = \begin{bmatrix} 8 & 4.4 & 4.4 & 0 & 0 & 0 \\ 4.4 & 8 & 4.4 & 0 & 0 & 0 \\ 4.4 & 4.4 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.8 \end{bmatrix}.$$

The piezoelectric coefficients are specified by

$$(5.12) \qquad \overset{(2)}{\mathbf{g}} = \overset{(2)}{\mathbf{h}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

while the dielectric coefficients are given by

$$(5.13) \qquad \overset{(2)}{\boldsymbol{\kappa}} = \begin{bmatrix} 26.9 & 0 & 0 \\ 0 & 26.9 & 0 \\ 0 & 0 & 26.9 \end{bmatrix},$$

$$(5.14) \qquad \overset{(2)}{\boldsymbol{\varepsilon}} = \begin{bmatrix} 0.0372 & 0 & 0 \\ 0 & 0.0372 & 0 \\ 0 & 0 & 0.0372 \end{bmatrix}.$$

Let us introduce the following matrices:

$$(5.15) \qquad \mathbf{A}^s = f_1 \overset{(1)}{\mathbf{A}} + f_2 \overset{(2)}{\mathbf{A}},$$

$$(5.16) \qquad \mathbf{A}^r = \left(f_1 \left(\overset{(1)}{\mathbf{A}} \right)^{-1} + f_2 \left(\overset{(2)}{\mathbf{A}} \right)^{-1} \right)^{-1},$$

where f_1 stands for the fibre volume fraction, and $f_2 = 1 - f_1$.

If $f_1 = 0.4$, $f_2 = 0.6$ then we have

$$(5.17)$$

$$\mathbf{A}^r = \begin{bmatrix} 12.9 & 7.09 & 7.06 & 0 & 0 & 0 & 0 & 0 & 0.042 \\ 7.09 & 12.9 & 7.06 & 0 & 0 & 0 & 0 & 0 & 0.042 \\ 7.06 & 7.06 & 12.9 & 0 & 0 & 0 & 0 & 0 & -0.28 \\ 0 & 0 & 0 & 2.93 & 0 & 0 & 0 & -0.141 & 0 \\ 0 & 0 & 0 & 0 & 2.93 & 0 & -0.141 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.90 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.141 & 0 & 0.349 & 0 & 0 \\ 0 & 0 & 0 & -0.141 & 0 & 0 & 0 & 0.339 & 0 \\ 0.042 & 0.042 & -0.28 & 0 & 0 & 0 & 0 & 0 & 0.669 \end{bmatrix},$$

(5.17)

[cont.]

$$
\mathbf{A}^s = \begin{bmatrix}
67.6 & 36.8 & 31.84 & 0 & 0 & 0 & 0 & 0 & 0.408 \\
36.8 & 67.6 & 31.84 & 0 & 0 & 0 & 0 & 0 & 0.408 \\
31.84 & 31.84 & 74.8 & 0 & 0 & 0 & 0 & 0 & -1.83 \\
0 & 0 & 0 & 19.96 & 0 & 0 & 0 & -0.92 & 0 \\
0 & 0 & 0 & 0 & 19.96 & 0 & -0.92 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 15.4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.92 & 0 & 16.2 & 0 & 0 \\
0 & 0 & 0 & -0.92 & 0 & 0 & 0 & 16.2 & 0 \\
0.408 & 0.408 & -1.83 & 0 & 0 & 0 & 0 & 0 & 16.33
\end{bmatrix}.
$$

The inequalities (4.2) and (4.5) are equivalent to

(5.18) $$\mathbf{A}^r \leq \mathbf{A}^h \leq \mathbf{A}^s$$

where $\mathbf{A}^h$ denotes the effective matrix. The matrix inequalities (5.18) imply immediately upper and lower bounds on the diagonal coefficients. Better estimates can be obtained from (4.20) – (4.23). In this case, however, on account of a weak influence of coupling, the differences are small. At present no explicit bounds are avaiable on the off-diagonal components. BISEGNA and LUCIANO [1, 2] claim that they have provided bounds on the off-diagonal elements by taking appropriate elements of the matrices $\mathbf{A}^r$ and $\mathbf{A}^s$. Such a statement is in general not true since bounds on matrices do not coincide with bounds on their elements. Similar remark concerns Hashin-Shtrikman type bounds given in [1]. We observe that the bounds derived in [11] on off-diagonal coefficients are accurate only for the elastic matrix. Therefore below, by using the formulae (4.20) – (4.23) we provide examples of determination of the bounds on off-diagonal elements for piezoelectric coupling matrix of material coefficients. This seems to be an important novelty of our paper.

Let us summarize these results:

(i) The bounds on the coefficient a^h_{1122} are depicted in Fig. 1. The part of the plane inside the parallelogram contains points with coordinates (a^h_{1111}, a^h_{1122}). In particular, we have $-5.41 \leq a^h_{1122} \leq 49.3$. The bounds lie in the plane because in the example studied it was assumed that a^h_{1111} and a^h_{2222} coincide. The sign + denotes the point determined by the Ritz method [3].

(ii) Figure 2 depicts the surface determined by maximal values of the coefficients a^h_{1133} (the upper bound) as a function of a^h_{1111} and a^h_{3333}. The last two coefficients vary within the Voigt-Reuss bounds. In Fig. 3 the surface of lower bounds is represented. Similar surfaces can be constructed for other off-diagonal homogenized coefficients, cf. Figs. 6 and 7.
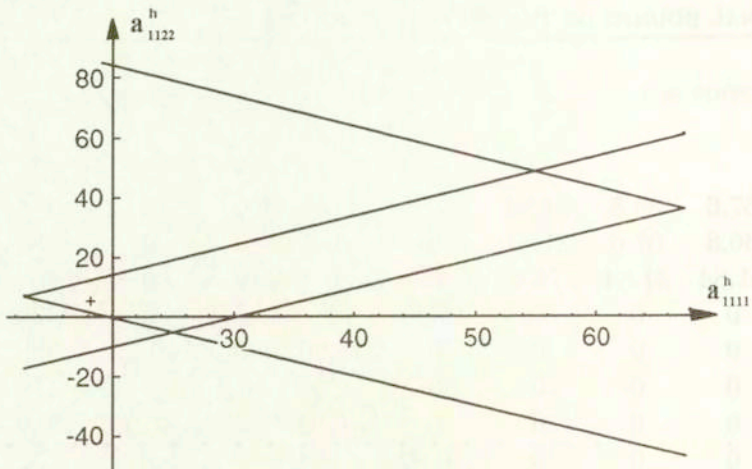
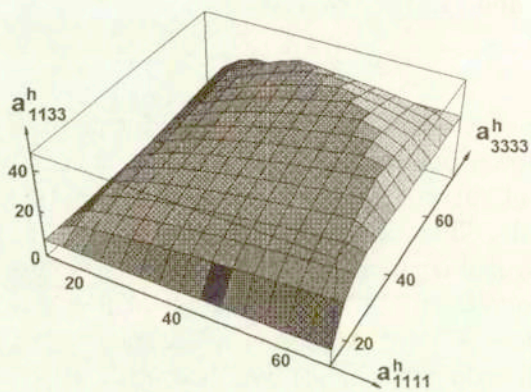http://rcin.org.pl
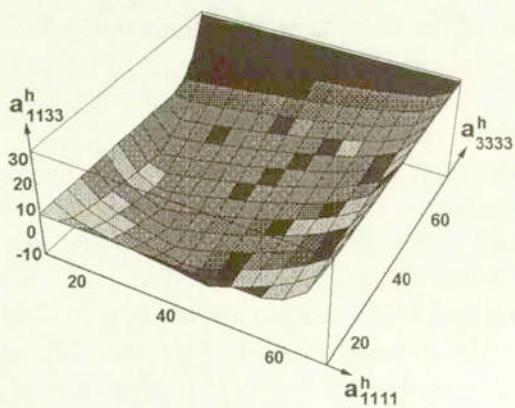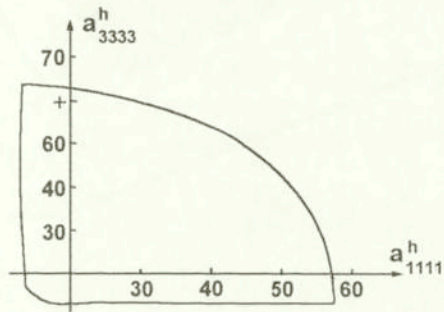
FIG. 1.

FIG. 2.

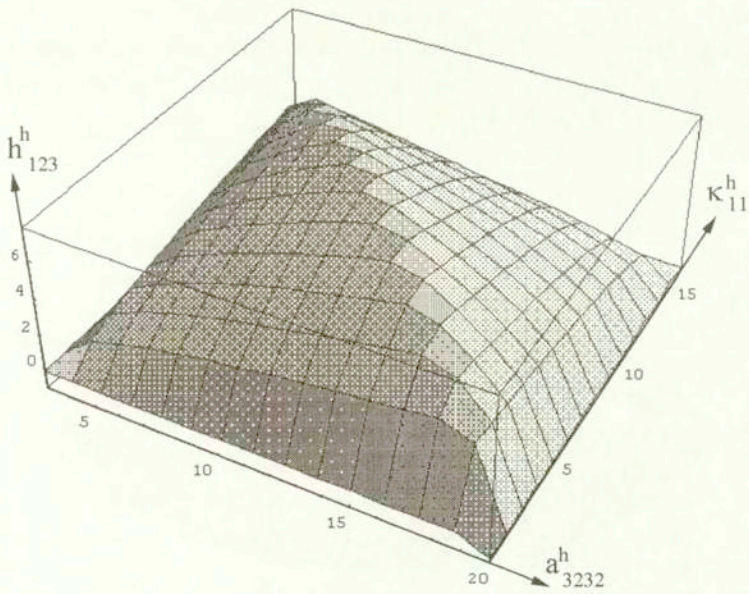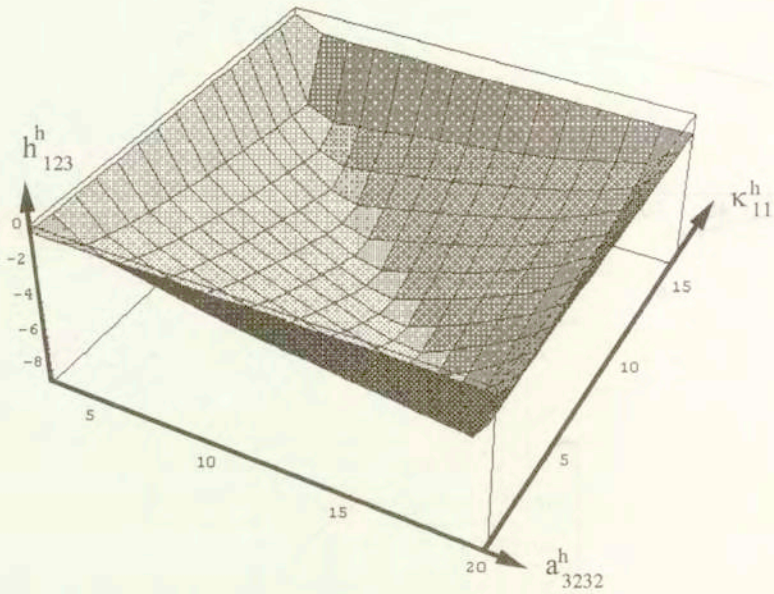
FIG. 3.

[684]
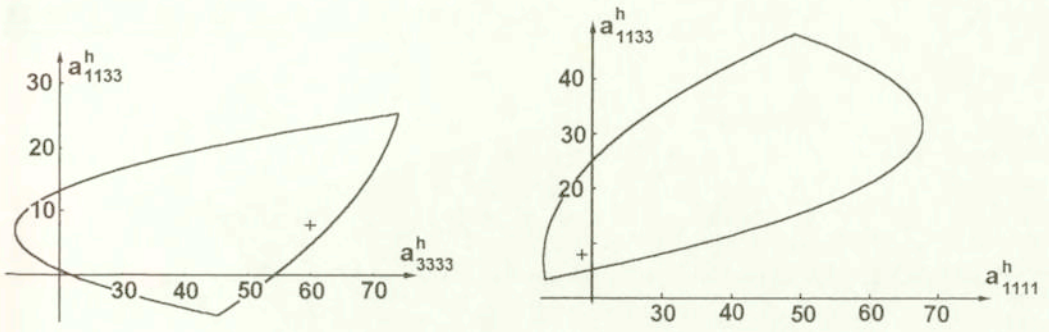
FIG. 4.

FIG. 5.

FIG. 6.

FIG. 7.

The domain contained between these two surfaces is called "the bounds region", cf. Fig. 8. Figures 4a, 4b, 5, 9 and 10 represent cross-sections of the bounds region within planes orthogonal to the coordinate axis and containing the Ritz point, denoted here by +. The coordinates of this point were determined in [3] by using the Ritz method. It is worth noting that the Ritz point lies within the bounds region.
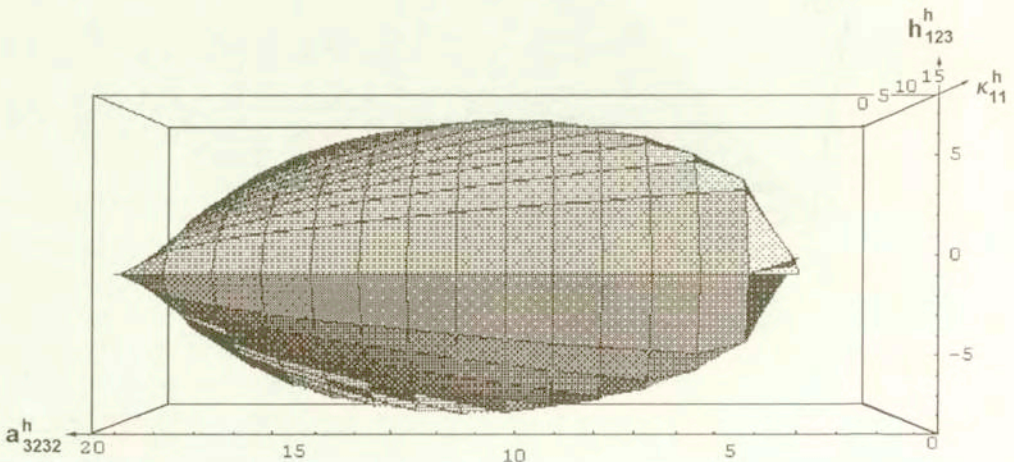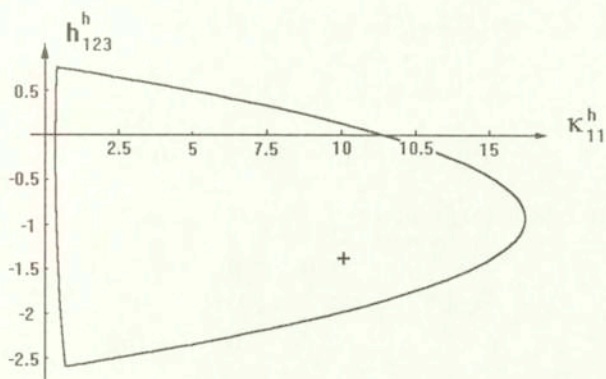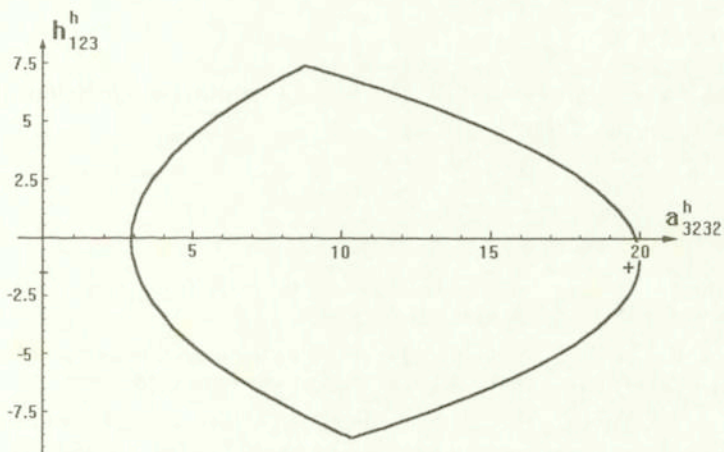

FIG. 8.

FIG. 9.

FIG. 10.

Let us now pass to providing the values of the homogenized coefficients for the same composite as they were obtained by employing the Ritz method [3]. Now they are given by:

a. *The elastic moduli*

$$(5.19) \qquad a^h = \begin{bmatrix} 18.46 & 5.81 & 7.96 & 0 & 0 & 0 \\ 5.81 & 18.46 & 7.96 & 0 & 0 & 0 \\ 7.96 & 7.96 & 59.98 & 0 & 0 & 0 \\ 0 & 0 & 0 & 19.82 & 0 & 0 \\ 0 & 0 & 0 & 0 & 19.82 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.19 \end{bmatrix};$$

b. *The piezoelectric coefficients*

$$(5.20) \qquad \mathbf{h}^h = \begin{bmatrix} 0 & 0 & 0 & 0 & -1.35 & 0 \\ 0 & 0 & 0 & -1.35 & 0 & 0 \\ 0.128 & 0.128 & -4.98 & 0 & 0 & 0 \end{bmatrix} ;$$

c. *The dielectric coefficients*

$$(5.21) \qquad \mathbf{k}^h = \begin{bmatrix} 10.04 & 0 & 0 \\ 0 & 10.04 & 0 \\ 0 & 0 & 1.152 \end{bmatrix} .$$

By comparing the method of bounds with the Ritz method we conclude that in this particular case of two-phase composite, Ritz's method yields results which fall within the upper and lower bounds.

Acknowledgement

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