On the time of existence of weak discontinuity waves in poroelastic materials

Dedicated to Prof. Henryk Zorski
on the occasion of his 70-th birthday

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IN THE PAPER, we consider the possibility of the growth of strong discontinuity waves in the two-component poroelastic materials. We use the model with the hyperbolic set of field equations described in the paper by K. WILMAŃSKI [8]. It is shown that indeed, the critical time (i.e. the maximum time of existence of classical solutions) is finite and it assumes realistic values for real physical systems, such as biological tissues.

1. Introduction

THE PROBLEM of propagation of waves in porous and granular materials bears special features which do not appear in other media with microstructure. The most characteristic one is the presence of additional modes of propagation. For porous materials, such a mode has been described for the first time by M.A. BIOT [1, 2], and it is called the P2-wave in contrast to P1-waves (longitudinal) and S-waves (transversal) appearing in one-component elastic solids. These additional modes are usually connected with a very high attenuation and this yields, in turn, certain difficulties in their experimental observation.

The second characteristic feature is the wave scattering on microscopic heterogeneities. Certainly, this phenomenon appears as well, for instance, in polycrystals. However, in contrast to relatively small grains of polycrystals, the size of the microstructure of typical granular and porous materials is much larger. It means that the strong scattering of sound waves appears in granular and porous materials for much longer waves (waves of much lower frequency) than it is the case for polycrystals. For instance, in the extreme case of a very coarse gravel (the typical size of particles $\sim 4.5\,\mathrm{mm}$), the waves of frequencies higher than app. 300 kHz cannot propagate at all, primarily due to their scattering.

The third feature is connected with the diffusion (relative motion) of components. The influence of the diffusion is particularly dramatic for the value of the attenuation coefficient and it has a smaller effect on the speeds of propagation (e.g. see K. WILMAŃSKI [7, 8, 9]).

Finally, in contrast to linear elastic models of polycrystals or composites, the models of porous materials with the linear elastic skeleton admit the growth of

shock waves, i.e. strong discontinuity waves. This is due to the non-linearity of the contribution of fluid components, which is characteristic for the bulk of such systems appearing in nature.

The literature of the subject contains a rather extensive batch of work on propagation conditions, speeds of propagation and attenuation of acoustic waves in granular and porous materials based on various models (e.g. T. Bourbie, O. Coussy, B. Zinszner [4], R.I. Nigmatulin [5], V.N. Nikolaevskij [6]). On the other hand, scattering of acoustic waves in such materials has been investigated to a very small extent and solely for one-component models. Even less has been done in the field of growth and propagation of shock waves.

In this paper, we investigate the possibility of the growth of the strong discontinuity wave in a porous material, described by the hyperbolic set of field equations and proposed in my papers [8, 11]. In the next section, we present the model. In the third section, we derive the evolution equation for the amplitude of weak discontinuities. As usual, it appears to be the Bernoulli equation along each characteristic. The solution of this equation may become infinite after a finite time. This critical time defines the range of existence of classical solutions of field equations. For times larger than critical, there exist solely weak solutions which define the shock waves. The fourth section is devoted to the dynamical compatibility conditions which describe the propagation of such waves. We show that the present model yields these conditions in the form similar, to a certain extent, to the classical Rankine–Hugoniot conditions for gases. The problem which still remains unsolved is connected with the admissibility (selection) criterion for the shock waves. We shall return to this problem in a separate forthcoming paper.

2. The model

We consider temperature-independent processes in a two-component porous medium described by the following six fields:

$$(2.1) \qquad (\mathbf{x}, t) \mapsto \left\{ \varrho_t^S, \varrho_t^F, \mathbf{v}^S, \mathbf{v}^F, n, \mathbf{e}^S \right\}, \qquad \mathbf{x} \in \mathcal{B}_t, \quad t \in \mathcal{T},$$

where ϱ_t^S denotes the current mass density of the skeleton, ϱ_t^F – the current mass density of the fluid, $\mathbf{v}^S$ – the velocity field of the skeleton, $\mathbf{v}^F$ – the velocity field of the fluid, n – porosity, and $\mathbf{e}^S$ – the symmetric deformation tensor of the skeleton.

The field equations for these fields follow from the balance equations of partial mass and momentum for both components and from the balance equation for porosity. In addition, the velocity field $\mathbf{v}^S$ and the deformation tensor $\mathbf{e}^S$ must satisfy the integrability condition. Under the following assumption of small deformation of the skeleton:

$$(2.2) \qquad \sup_{|\mathbf{n}|=1} \left| \mathbf{e}^S \cdot \mathbf{n} \otimes \mathbf{n} \right| \ll 1,$$

these balance equations have the form presented in the table below. On the left-hand side, we quote the equations in points $\mathbf{x} \in \mathcal{B}_t$ in which the fields are of the class C^1 with respect to time and space variables. On the left-hand side, we quote the dynamical compatibility relations in points of a singular surface, on which the fields may have finite discontinuities.

In addition to the small deformation of the skeleton (2.2), it has been assumed that the speed of the relative motion $|\mathbf{v}^F - \mathbf{v}^S|$ is much smaller than the partial speeds of components. This yields the linearity of the diffusive forces in the momentum balance equations. We assume as well that the deviation of porosity from the equilibrium value $\Delta = n - n_E$ is much smaller than unity.

Table 1.

	regular point $\mathbf{x} \in \mathcal{B}_t$		singular point $\mathbf{x} \in \mathcal{S}_t$
(2.3)	$\dfrac{\partial \varrho_t^S}{\partial t} + \operatorname{div}(\varrho_t^S \mathbf{v}^S) = 0,$	(2.9)	$[[\varrho_t^S(\mathbf{v}^S \cdot \mathbf{n} - c)]] = 0,$
(2.4)	$\dfrac{\partial \varrho_t^F}{\partial t} + \operatorname{div}(\varrho_t^F \mathbf{v}^F) = 0,$	(2.10)	$[[\varrho_t^F(\mathbf{v}^F \cdot \mathbf{n} - c)]] = 0,$
(2.5)	$\varrho_t^F\left(\dfrac{\partial \mathbf{v}^F}{\partial t} + \operatorname{grad}\mathbf{v}^F \mathbf{v}^F\right)$ $= -\operatorname{grad} p^F - \pi(\mathbf{v}^F - \mathbf{v}^S),$	(2.11)	$[[\varrho_t^F(\mathbf{v}^F \cdot \mathbf{n} - c)\mathbf{v}^F]]$ $= -[[p^F]]\,\mathbf{n},$
(2.6)	$\dfrac{\partial \Delta}{\partial t} + \mathbf{v}^S \cdot \operatorname{grad}\Delta + \varphi \operatorname{div}(\mathbf{v}^F - \mathbf{v}^S)$ $= -\dfrac{\Delta}{\tau}, \qquad \Delta := n - n_E.$	(2.12)	$[[\varrho_t^S(\mathbf{v}^S \cdot \mathbf{n} - c)\Delta]]$ $+ \varphi \varrho^S\,[[\mathbf{v}^F - \mathbf{v}^S]] \cdot \mathbf{n} = 0,$
(2.7)	$\varrho^S \dfrac{\partial \mathbf{v}^S}{\partial t} = \operatorname{div}\mathbf{T}^S + \pi(\mathbf{v}^F - \mathbf{v}^S)$	(2.13)	$[[\varrho_t^S(\mathbf{v}^S \cdot \mathbf{n} - c)\mathbf{v}^S]]$ $- [[\mathbf{T}^S]]\,\mathbf{n} = 0,$
(2.8)	$\dfrac{\partial e^S}{\partial t} = \dfrac{1}{2}\left(\operatorname{grad}\mathbf{v}^S + \operatorname{grad}^T\mathbf{v}^S\right),$	(2.14)	$[[e^S]]$ $+ \dfrac{1}{2c}[[\mathbf{v}^S \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{v}^S]] = 0,$

The constitutive relations for poroelastic materials are assumed to have the form

$$(2.15)\,(^1) \qquad p^F = \wp^F(\varrho_t^F; n_E) + \beta\Delta, \qquad \mathbf{T}^S = \lambda^S(e^S \cdot \mathbf{1})\mathbf{1} + 2\mu^S e^S + \beta\Delta\mathbf{1},$$

where all constitutive parameters λ^S, μ^S, φ, β, π and τ depend solely on the equilibrium constant value of the porosity n_E. This is the value of porosity, which

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$(^0)$ In my previous works the material constant describing the coupling of stresses has been denoted by $\mathcal{M}$. It is related to β in the following way: $\beta = \varphi(\mathcal{M}\tau)^{-1}$.

the material reaches after the full relaxation under the constant load. In addition, the propagation of shock waves, considered further in this work, is considered for the following form of the intrinsic pressure $\wp^F$:

$$(2.16) \qquad \frac{d\wp^F}{\wp^F} = \gamma \frac{d\varrho_t^F}{\varrho_t^F}, \quad \Rightarrow \quad U^{F2} := \frac{\partial \wp^F}{\partial \varrho_t^F} = \gamma \frac{\wp^F}{\varrho_t^F},$$

where γ is a constant.

Let us notice that the solution of equation (2.3) for small deformations of the skeleton

$$(2.17) \qquad \varrho_t^S \approx (1 - \mathbf{e}^S \cdot \mathbf{1})\varrho^S \approx \varrho^S,$$

allows us to eliminate ϱ_t^S from the list of fields in regular points. It is not any more the case of the singular surfaces as we will see further in this work.

3. Propagation condition for plane waves, evolution of the amplitude

3.1. Speeds and amplitudes of weak discontinuity waves

In this section we consider a simple case of propagation of plane waves. The purpose of these considerations is primarily the derivation of the so-called evolution equation of the amplitude of the weak discontinuity waves. The solution of this equation leads to the time of existence of classical solutions of the field equations. For a given set of initial conditions, it enables the analysis of the growth of shock waves.

We consider the motion of both components to be described by the single component of velocities v^F and v^S in the direction of x-axis. The deformation $\mathbf{e}^S$ reduces to the extension in the x-direction e^S. Simultaneously, in order to estimate the magnitude of contribution of various effects, we introduce the dimensionless description. In the definition of dimensionless quantities, we use the material parameters U^S, τ and ϱ^S. Namely,

$$(3.1) \qquad \begin{aligned} &\hat{t} = \frac{t}{\tau}, \qquad \hat{x} = \frac{x}{U^S \tau}, \qquad U^S := \sqrt{\frac{\lambda^S + 2\mu^S}{\varrho^S}}, \\[2mm] &\hat{\varrho}^F = \frac{\varrho_t^F}{\varrho^S}, \qquad \hat{v}^F = \frac{v^F}{U^S}, \qquad \hat{v}^S = \frac{v^S}{U^S}, \qquad \hat{\Delta} = \Delta, \qquad \hat{e}^S = e^S, \\[2mm] &\hat{\wp}^F = \frac{\wp^F}{\varrho^S U^{S2}} \Rightarrow \frac{\partial \hat{\wp}^F}{\partial \hat{\varrho}^F} = (U^S)^{-2} \frac{\partial \wp^F}{\partial \varrho_t^F} \equiv \left(\frac{U^F}{U^S}\right)^2, \qquad U^F = \sqrt{\frac{\partial \wp^F}{\partial \varrho_t^F}}, \\[2mm] &\hat{\varphi} = \varphi, \qquad \hat{\beta} = \frac{\beta}{\varrho^S U^{S2}}, \qquad \hat{\pi} = \frac{\pi \tau}{\varrho^S}. \end{aligned}$$

Consequently, the set of unknown fields is as follows:

$$(3.2) \qquad \mathbf{w} := \left(\widehat{\varrho}_t^F, \widehat{v}^F, \widehat{\Delta}, \widehat{v}^S, \widehat{e}^S \right)^T \in \mathcal{V}^5, \qquad (\widehat{x}, \widehat{t}) \mapsto \mathbf{w} \in \mathfrak{R}^5, \quad \widehat{x} \in \mathfrak{R}^1, \quad \widehat{t} \in \mathfrak{R}^1.$$

These fields satisfy the field equations following from (2.4)–(2.8), i.e.

$$\frac{\partial \widehat{\varrho}^F}{\partial \widehat{t}} + \widehat{v}^F \frac{\partial \widehat{\varrho}^F}{\partial \widehat{x}} + \widehat{\varrho}^F \frac{\partial \widehat{v}^F}{\partial \widehat{x}} = 0,$$

$$\frac{\partial \widehat{v}^F}{\partial \widehat{t}} + \widehat{v}^F \frac{\partial \widehat{v}^F}{\partial \widehat{x}} + \left(\frac{1}{\widehat{\varrho}^F} \frac{\partial \widehat{\wp}^F}{\partial \widehat{\varrho}^F} \right) \frac{\partial \widehat{\varrho}^F}{\partial \widehat{x}} + \frac{\widehat{\beta}}{\widehat{\varrho}^F} \frac{\partial \widehat{\Delta}}{\partial \widehat{x}} = -\frac{\widehat{\pi}}{\widehat{\varrho}^F} \left(\widehat{v}^F - \widehat{v}^S \right),$$

$$(3.3) \qquad \frac{\partial \widehat{\Delta}}{\partial \widehat{t}} + \widehat{v}^F \frac{\partial \widehat{\Delta}}{\partial \widehat{x}} + \widehat{\varphi} \frac{\partial}{\partial \widehat{x}} \left(\widehat{v}^F - \widehat{v}^S \right) = -\widehat{\Delta},$$

$$\frac{\partial \widehat{v}^S}{\partial \widehat{t}} - \frac{\partial \widehat{e}^S}{\partial \widehat{x}} - \beta \frac{\partial \widehat{\Delta}}{\partial \widehat{x}} = \widehat{\pi} \left(\widehat{v}^F - \widehat{v}^S \right),$$

$$\frac{\partial \widehat{e}^S}{\partial \widehat{t}} - \frac{\partial \widehat{v}^S}{\partial \widehat{x}} = 0.$$

We write this set of equations in the form

$$(3.4) \qquad \frac{\partial w_\alpha}{\partial \widehat{t}} + A_{\alpha\beta} \frac{\partial w_\beta}{\partial \widehat{x}} = B_\alpha, \qquad \alpha = 1, ..., 5,$$

with the following definitions of the matrices $\mathbf{A}$ and $\mathbf{B}$

$$(3.5) \qquad (A_{\alpha\beta}) = \begin{pmatrix} \widehat{v}^F & \widehat{\varrho}^F & 0 & 0 & 0 \\ \dfrac{1}{\widehat{\varrho}^F} \dfrac{\partial \widehat{\wp}^F}{\partial \widehat{\varrho}^F} & \widehat{v}^F & \dfrac{\widehat{\beta}}{\widehat{\varrho}^F} & 0 & 0 \\ 0 & \widehat{\varphi} & \widehat{v}^F & -\widehat{\varphi} & 0 \\ 0 & 0 & -\widehat{\beta} & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$(B_\alpha) = \left(0 \quad -\frac{\widehat{\pi}}{\widehat{\varrho}^F} \left(\widehat{v}^F - \widehat{v}^S \right) \quad -\Delta \quad \widehat{\pi} \left(\widehat{v}^F - \widehat{v}^S \right) \quad 0 \right).$$

Hyperbolicity of the set of field equations (3.4) means that the eigenvalues of the matrix $\mathbf{A}$ are real and the eigenvectors of this matrix span the space of solutions. We proceed to find these eigenvalues and eigenvectors.

We solve the equation for the eigenvalues

$$(3.6) \qquad \det \left(A_{\alpha\beta} - \lambda \delta_{\alpha\beta} \right) = 0,$$

under the assumption

$$(3.7) \qquad \left| \widehat{v}^F \right| \ll 1 \Rightarrow \lambda \approx \lambda_0 + \widehat{v}^F \lambda_1.$$

For λ_0 we easily obtain the following values:

$$\lambda_0^{(1)} = 0,$$

$$(3.8) \qquad \left(\lambda_0^{(2-5)}\right)^2 = \frac{1}{2}\left\{ 1 + \frac{\partial \widehat{\wp}^F}{\partial \widehat{\varrho}^F} + \left(1 + \frac{1}{\widehat{\varrho}^F}\right)\widehat{\varphi}\widehat{\beta} \right.$$

$$\left. \pm \sqrt{\left[1 - \frac{\partial \widehat{\wp}^F}{\partial \widehat{\varrho}^F} + \left(1 - \frac{1}{\widehat{\varrho}^F}\right)\widehat{\varphi}\widehat{\beta}\right]^2 + 4\frac{\widehat{\varphi}^2\widehat{\beta}^2}{\widehat{\varrho}^{F2}}} \right\}.$$

For the second term λ_1 in the perturbation series (3.7), we have

$$\lambda_1 = \frac{a}{b}, \qquad a := -2\lambda_0^2\left(\lambda_0^2 - 1 - \widehat{\varphi}\widehat{\beta}\right) - \left(\lambda_0^2 - \frac{\partial \widehat{\wp}^F}{\partial \widehat{\varrho}^F} - \frac{\widehat{\varphi}\widehat{\beta}}{\widehat{\varrho}^F}\right)(\lambda_0^2 - 1),$$

$$(3.9) \qquad b := -2\lambda_0^2\left(\lambda_0^2 - 1 - \widehat{\varphi}\widehat{\beta}\right) - \left(\lambda_0^2 - \frac{\partial \widehat{\wp}^F}{\partial \widehat{\varrho}^F}\right)\left[(\lambda_0^2 - 1) + 2\lambda_0^2 - \widehat{\varphi}\widehat{\beta}\right]$$

$$+ \left[(\lambda_0^2 - 1) + 2\lambda_0^2\right]\frac{\widehat{\varphi}\widehat{\beta}}{\widehat{\varrho}^F}.$$

It can be easily checked that this contribution is of the order of magnitude of the unity. This means that we can limit the attention to the zeroth approximation (3.8) of the eigenvalues. Due to the assumption of a small deviation from the thermodynamical equilibrium, we can hardly distinguish between the contributions of $\widehat{v}^F$ and $\widehat{v}^S$ anyway. Consequently, we use the approximation $\lambda \approx \lambda_0$.

Let us notice that relations (3.8) reduce to the classical relations for speeds of propagation of longitudinal waves in the elastic solid and the ideal fluid, respectively, if the coupling coefficient $\varphi\beta$ vanishes. The normalized right and left eigenvectors follow in the form

$$(3.10) \qquad \mathbf{r} = \frac{\bar{\mathbf{r}}}{\bar{r}}, \qquad \bar{\mathbf{r}} := \left(\frac{\lambda^2 - 1}{\lambda^2 - \dfrac{\partial \widehat{\wp}^F}{\partial \widehat{\varrho}^F}}, \frac{\lambda}{\widehat{\varrho}^F}, \frac{\lambda^2 - 1}{\lambda^2 - \dfrac{\partial \widehat{\wp}^F}{\partial \widehat{\varrho}^F}}, \frac{1}{\widehat{\beta}}(\lambda^2 - 1), -\lambda, 1\right)^T,$$

$$\bar{r} := \sqrt{\bar{\mathbf{r}}\cdot\bar{\mathbf{r}}},$$

$$\mathbf{l} = \frac{\bar{\mathbf{l}}}{\bar{l}}, \qquad \bar{\mathbf{l}} := \left(\frac{1}{\widehat{\varrho}^F}\frac{\partial \widehat{\wp}^F}{\partial \widehat{\varrho}^F}\frac{\lambda^2 - 1}{\lambda^2 - \dfrac{\partial \widehat{\wp}^F}{\partial \widehat{\varrho}^F}}, \lambda\frac{\lambda^2 - 1}{\lambda^2 - \dfrac{\partial \widehat{\wp}^F}{\partial \widehat{\varrho}^F}}, \frac{1}{\widehat{\varphi}}(\lambda^2 - 1), -\lambda, 1\right),$$

$$\bar{l} := \sqrt{\bar{\mathbf{l}}\cdot\bar{\mathbf{l}}},$$

Evaluation of the jump of field equations (3.4) yields easily the following property of the amplitude of the weak discontinuity, i.e. the jump of the gradient of the field $\mathbf{w}$. We have

$$(3.11) \qquad [[\mathbf{w}]] = 0 \Rightarrow \left[\left[\frac{\partial \mathbf{w}}{\partial \hat{t}}\right]\right] + \mathbf{A}\left[\left[\frac{\partial \mathbf{w}}{\partial \hat{x}}\right]\right] = (\mathbf{A} - c\mathbf{1})\left[\left[\frac{\partial \mathbf{w}}{\partial \hat{x}}\right]\right] = 0$$

$$\Rightarrow \hat{c} = \lambda \,\&\, \left[\left[\frac{\partial \mathbf{w}}{\partial \hat{x}}\right]\right] = \mathsf{a}\,\mathbf{r}, \qquad \mathsf{a} := \sqrt{\left[\left[\frac{\partial \mathbf{w}}{\partial \hat{x}}\right]\right] \cdot \left[\left[\frac{\partial \mathbf{w}}{\partial \hat{x}}\right]\right]}, \qquad \hat{c} := \frac{c}{U^S},$$

i.e. the speeds of propagation are identical with the eigenvalues of the matrix $\mathbf{A}$, and the amplitude of discontinuity is parallel to the right eigenvector of the matrix $\mathbf{A}$.

3.2. Evolution of the amplitude

We proceed to investigate the time changes of the magnitude a of the amplitude. In order to simplify the calculations we assume that the medium ahead of the wave (i.e. on the positive side of the surface which is defined, in general, by the direction of the normal vector $\mathbf{n}$) is in the static undeformed state. Then the differentiation of the field equations with respect to x, multiplication by the right eigenvector and evaluation of the jump on the singular surface yields the following equation:

$$(3.12) \qquad \frac{\delta \mathsf{a}}{\delta \hat{t}} - \Gamma_1 \mathsf{a}^2 + \Gamma_2 \mathsf{a} = 0, \qquad \frac{\delta}{\delta \hat{t}} := \frac{\partial}{\partial \hat{t}} + \lambda \frac{\partial}{\partial \hat{x}},$$

where

$$(3.13) \qquad \Gamma_1 := \frac{1}{\mathbf{l}\cdot\mathbf{r}}l_\alpha \frac{\partial A_{\alpha\beta}}{\partial w_\gamma} r_\gamma r_\beta, \qquad \Gamma_2 := \frac{1}{\mathbf{l}\cdot\mathbf{r}}l_\alpha \frac{\partial B_\alpha}{\partial w_\gamma} r_\gamma,$$

and the standard kinematical compatibility conditions have been used. The eigenvectors and the coefficients Γ_1 and Γ_2 are known because they are evaluated for the fields in the static undeformed state ahead of the wave, i.e.

$$(3.14) \qquad \mathbf{w}^+ = \left(\hat{\varrho}^{F+} = \frac{\varrho_0^F}{\varrho^S}, \; \hat{v} = 0, \; \hat{\Delta} = 0, \; \hat{v}^S = 0, \; e^S = 0\right)^T.$$

The evolution equation of the amplitude (3.12) is of the Bernuolli type and it is typical for the quasi-linear hyperbolic systems of equations (e.g. see: M.F. McCarthy [3]). Certainly, it can be immediately solved. In the most general case, we have

$$(3.15) \qquad \frac{1}{\mathsf{a}} = \frac{1}{\mathsf{a}_0}\exp\left(\Gamma_2(\hat{t} - \hat{t}_0)\right) - \int_{\hat{t}_0}^{\hat{t}} \Gamma_1 \exp\left(\Gamma_2(\hat{t} - \eta)\right) d\eta, \qquad \mathsf{a}_0 := \mathsf{a}(\hat{t} = \hat{t}_0).$$

Let us mention a few most important properties of the above evolution equation. First of all, it is obvious that the nonlinearity of the field equations is necessary for the existence of the non-zero coefficient Γ_1. If this is not the case, the evolution of the amplitude has an exponential character. Such an amplitude is constant if the sources **B** are absent and it grows or decays exponentially otherwise.

The situation changes in the case of presence of Γ_1. Then it may happen that the amplitude a grows to infinity in a finite time – say t_c. According to the relation (3.15), it can happen iff

$$(3.16) \qquad \exp\left(\Gamma_2(\widehat{t}_c - \widehat{t}_0)\right) = \mathsf{a}_0 \int_{\widehat{t}_0}^{\widehat{t}_c} \Gamma_1 \exp\left(\Gamma_2(\widehat{t}_c - \eta)\right) d\eta\,.$$

Consequently, the existence of the critical time t_c depends on the initial data, on the signs of coefficients Γ_1, Γ_2, as well as on the choice of the characteristic, i.e. on the choice of the eigenvalue of **A** and of the corresponding eigenvector. We shall investigate this problem for the system (3.4).

Bearing in mind the form (3.5) of the matrix **A**, we obtain after easy manipulations the following form of the coefficients Γ_1, Γ_2:

$$\Gamma_1 = \frac{\lambda^+}{\varrho_0^{F2}\mathbf{l}\cdot\mathbf{r}} \left(\frac{\lambda^{+2} - 1}{\lambda^{+2} - \left(\frac{U^{F+}}{U^{S+}}\right)^2}\right)^3$$

$$\cdot\left[\varrho_0^F \left.\frac{\partial^2 \wp^F}{\partial \varrho^{F2}}\right|^+ + 2\left(\frac{U^{F+}}{U^{S+}}\right)^2 + \frac{\varrho_0^F}{\varphi\beta}\left(\lambda^{+2} - \left(\frac{U^{F+}}{U^{S+}}\right)^2\right)\right],$$

$$\Gamma_2 = \frac{1}{\mathbf{l}\cdot\mathbf{r}}\left\{\lambda^{+2}\left(\frac{\lambda^{+2} - 1}{\lambda^{+2} - \left(\frac{U^{F+}}{U^{S+}}\right)^2}\right)^2 \frac{\pi}{\varrho_0^{F2}}\right.$$

$$(3.17) \qquad\qquad + 2\lambda^{+2}\frac{\lambda^{+2} - 1}{\lambda^{+2} - \left(\frac{U^{F+}}{U^{S+}}\right)^2}\frac{\pi}{\varrho_0^F} + \frac{(\lambda^{+2} - 1)^2}{\varphi\beta} + \lambda^{+2}\pi\bigg\},$$

$$\lambda^{+2} \equiv \frac{1}{2}\left\{1 + \left(\frac{U^{F+}}{U^{S+}}\right)^2 + \left(1 + \frac{1}{\varrho_0^F}\right)\varphi\beta\right.$$

$$\pm\sqrt{\left[1 - \left(\frac{U^{F+}}{U^{S+}}\right)^2 - \left(1 + \frac{1}{\varrho_0^F}\right)\varphi\beta\right]^2 + 4\varphi\beta\left(1 - \left(\frac{U^{F+}}{U^{S+}}\right)^2\right)}\bigg\},$$

where we have skipped the hat for the typographical reasons. The plus sign means that the quantity should be evaluated in the state $\mathbf{w}^+$ given by (3.14).

For the zero value of λ both coefficients vanish. Consequently, the amplitude of discontinuity a does not change in time along this characteristic.

It remains to establish the initial conditions. We have in general

$$(3.18) \qquad \left[\!\left[\frac{\partial \mathbf{w}}{\partial \widehat{x}}\right]\!\right] = -\left(\frac{\partial \mathbf{w}}{\partial \widehat{x}}\right)^- = a\,\mathbf{r},$$

i.e.

$$\frac{\partial \widehat{\varrho}^F}{\partial \widehat{x}}\bigg|^- = -a\,r_1, \qquad \frac{\partial \widehat{v}^F}{\partial \widehat{x}}\bigg|^- = -a\,r_2, \qquad \frac{\partial \widehat{\Delta}}{\partial \widehat{x}}\bigg|^- = -a\,r_3,$$

$$\frac{\partial \widehat{v}^S}{\partial \widehat{x}}\bigg|^- = -a\,r_4, \qquad \frac{\partial \widehat{e}^S}{\partial \widehat{x}}\bigg|^- = -a\,r_5.$$

Let us consider the case of a given initial acceleration of the fluid component. For a given a_0^F, we have

$$(3.19) \qquad \frac{\partial \widehat{v}^F}{\partial \widehat{t}}\bigg|^-_{t=t_0} = \frac{\tau}{U^S}a_0^F, \qquad \text{i.e.} \qquad \frac{\partial \widehat{v}^F}{\partial \widehat{t}}\bigg|^-_{t=t_0} = -\frac{1}{\lambda^+}\frac{\tau}{U^S}a_0^F = -a|_{t=t_0}r_2.$$

Bearing in mind the relation (3.10) for the eigenvector, we get finally

$$(3.20) \qquad a_0 := a|_{t=t_0} = \frac{\widehat{\varrho}^{F+}}{\lambda^{+2}}\,\frac{\lambda^{+2} - \left(\dfrac{U^{F+}}{U^{S+}}\right)^2}{\lambda^{+2} - 1}\,\frac{\tau}{U^S}a_0^F.$$

This choice of the initial conditions has no essential influence on the qualitative properties of the amplitude a. This is due to the fact that the initial disturbance of any quantity appearing in (3.18) yields the disturbances of all the remaining gradients. Consequently, it is solely the problem of proper normalization of initial conditions.

It is seen that the coefficients Γ_1, Γ_2 and the eigenvectors $\mathbf{r}^+$, $\mathbf{l}^+$ are constant in this example. Hence we can easily integrate in (3.16). It follows that

$$(3.21) \qquad \widehat{t}_c = \widehat{t}_0 - \frac{1}{\Gamma_2}\ln\left(1 - \frac{\Gamma_2}{a_0\Gamma_1}\right).$$

Then the existence of positive critical times requires that the initial amplitude must be greater than the critical value given by the relation

$$(3.22) \qquad a_c := \frac{\Gamma_2}{\Gamma_1} > 0.$$

If this is the case, the classical solution of field equations ceases to exist and the shock wave is created. It shall happen on the characteristic with the shortest critical time. Consequently, in order to find it, we have to check four characteristic critical times for the present example.

We illustrate these conditions by a numerical example. We choose the following data of material parameters:

$$U^S = 1.5 \times 10^3 \text{m/s}, \qquad U_0^F = 0.9 \times 10^3 \text{m/s}, \qquad n_E = 0.4, \qquad \varphi = 0.4,$$

(3.23) $\quad \varrho^S = 1.2 \times 10^3 \text{kg/m}^3, \qquad \varrho_0^F = 0.4 \times 10^3 \text{kg/m}^3,$

$$\pi = 10^7 \text{kg/m}^3\text{s}, \qquad\qquad \tau = 10^{-5}\text{s}, \qquad \beta = 0.3 \times 10^9 \text{m}^2/\text{s}^2.$$

They describe a rather soft skeleton (the relatively small value of the speed U^S) with the coefficient of permeability π smaller than the typical values for, say, rocks. These data have the order of magnitude typical for, for instance, biological tissues.

For these data, the critical initial acceleration, defined by the relation (3.22), has the value $a_0^F = 11.43 \text{ m/s}^2$. Consequently, one can expect the growth of the shock wave in this two-component system provided the initial acceleration is large enough.
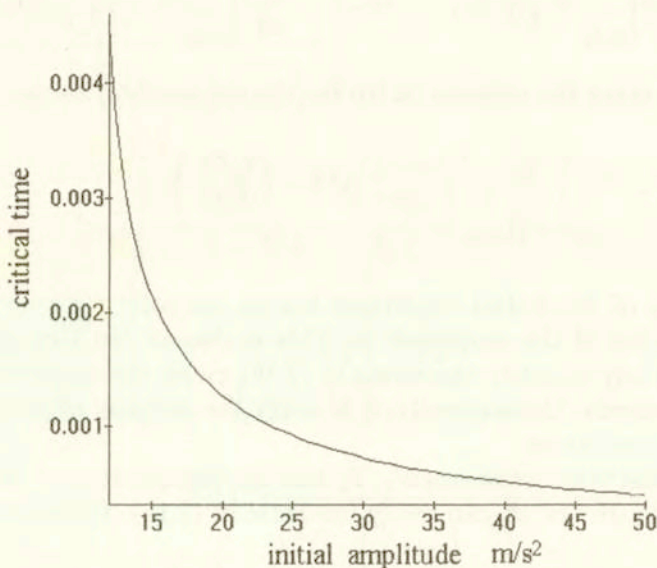

FIG. 1. Critical time in [s] vs. initial amplitude in [m/s^2] for the data (3.23).

In Fig. 1, we present the critical time as the function of the initial amplitude. As expected, this time becomes shorter for larger amplitudes. Also its order of magnitude seems to be realistic even though we are not aware of the experimental data, from which this time could be estimated.

4. On strong discontinuity waves

As pointed out in the analysis of the critical time for the weak discontinuity waves, the nonlinearity of the constitutive relation for the intrinsic part of the pressure in the fluid $\wp^F(\varrho_t^F; n_E)$ indicates the possibility of growth of the shock waves, i.e. the waves carrying strong discontinuities. They are not described by the local field equations (2.3)–(2.8) any more. We have to construct weak solutions of those equations. Very little has been done in this respect for the case of multicomponent continua. In this work, we show solely some basic properties of shock waves for the simple model under consideration which follow from the jump conditions (2.9)–(2.14). As before, we assume the body to be undisturbed ahead of the shock wave, i.e. its state is supposed to be $\mathbf{w}^+$ given by (3.14). Bearing in mind the relations

(4.1)
$$\left[\!\left[e^S \cdot \mathbf{1} \right]\!\right] = \left[\!\left[e^S \cdot (\mathbf{n} \otimes \mathbf{n}) \right]\!\right] = -\frac{1}{c}\left[\!\left[\mathbf{v}^S \cdot \mathbf{n} \right]\!\right],$$
$$\left[\!\left[e^S - e^S \cdot \mathbf{n}\,\mathbf{1} \right]\!\right]\mathbf{n} = -\frac{1}{2c}\left[\!\left[\mathbf{v}^S - \mathbf{v}^S \cdot \mathbf{n}\,\mathbf{n} \right]\!\right],$$

which follow from the jump condition (2.14), the remaining jump conditions yield the following relations:

(4.2)
$$\left[\!\left[e^S \cdot \mathbf{1} \right]\!\right] = \varrho^S \left[\!\left[\frac{1}{\varrho_t^S} \right]\!\right], \qquad \left[\!\left[\mathbf{v}^F \cdot \mathbf{n} \right]\!\right] = -\varrho_0^F \left[\!\left[\frac{1}{\varrho_t^F} \right]\!\right]\mathbf{c},$$

$$\left[\!\left[\Delta \right]\!\right] = \frac{\varphi}{c^3}\left[\!\left[\wp^F \right]\!\right] + \varphi\left[\!\left[e^S \cdot \mathbf{1} \right]\!\right],$$

$$\left[\!\left[e^S \cdot \mathbf{1} \right]\!\right] = \frac{\varphi\beta}{c^3\varrho^S}\left(c^2 - \frac{\lambda^S + 2\mu^S}{\varrho^S} - \frac{\varphi\beta}{c\varrho^S} \right)^{-1}\left[\!\left[\wp^F \right]\!\right],$$

$$\left[\!\left[\mathbf{v}^F - \mathbf{v}^F \cdot \mathbf{n}\,\mathbf{n} \right]\!\right] = 0, \qquad \left(c^2 - \frac{\mu^S}{\varrho^S} \right)\left[\!\left[\mathbf{v}^S - \mathbf{v}^S \cdot \mathbf{n}\,\mathbf{n} \right]\!\right] = 0,$$

$$\varrho_0^{F\,2}c^2 = -\frac{\left[\!\left[\wp^F \right]\!\right]}{\left[\!\left[\frac{1}{\varrho_t^F} \right]\!\right]}.$$

These relations should be read in the following way. The formulae (4.1) together with (4.2)$_{1-4}$ determine the jumps of ϱ_t^S, $e^S \cdot \mathbf{1}$, $\mathbf{v}^S \cdot \mathbf{n}$, $\mathbf{v}^F \cdot \mathbf{n}$ and Δ in terms of the jumps of the mass density of the fluid ϱ_t^F and of the intrinsic pressure p^F which are, in turn, connected by the barotropic constitutive relation (2.16). According to (4.2)$_5$, the admissible discontinuity of the fluid velocity $\mathbf{v}^F$ reduces to the component in the direction of propagation $\mathbf{n}$. The tangential component must be continuous. On the other hand, according to (4.2)$_6$, the tangential component of velocity of the skeleton $\mathbf{v}^S$ can be discontinuous but its discontinuity propagates

with the velocity of transversal sound wave equal to $\sqrt{\mu^S/\varrho^S}$. Hence, it can be solely due to initial conditions and does not grow during the motion of the body.

The last relation $(4.2)_7$ determines the speed of propagation of such waves if the jump of the mass density ϱ_t^F is given. This relation is identical with the classical Rankine–Hugoniot condition for shock waves in gases.

The investigation of the one-dimensional problem of construction of weak solutions accounting for these relations and the analogy to the theory of shock waves in gases is in progress and it shall be presented in a forthcoming paper.

5. Concluding remark

In spite of the practical – in particular, medical – applications of shock waves in porous materials, the theoretical research in this field has not made much progress since many years. This is due to the two following main reasons. First of all, it is very difficult to extend the methods of solutions of one-dimensional hyperbolic problems to higher spatial dimensions. Unfortunately, such an extension is necessary in practical applications of shock waves in porous materials (e.g. in lithotripsy). Secondly, in the case of more than two fields, very little is known about the criteria of choice of a weak solution. It is known that the usual entropy criterion does not yield uniqueness of solution and an alternative is not yet known. This problem was only indicated in the present work.

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