A note on kinematics of surfaces

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THE QUESTION how to describe effectively the motion of a deformable surface in ordinary Euclidean space is discussed. Two alternative formulations are supplied, both based on the assumption that the Riemannian metric of the moving surface must appear explicitly in the system describing the motion. The motivation for this assumption is to divide the variables responsible for the evolution of the intrinsic geometry (strains) of the surface from those responsible for the evolution of its extrinsic geometry (bending). Exemplary application of these results to the large deflection/small strain class of deformations of thin shells is considered.

1. Introduction

IN MECHANICS surfaces appear generally in two contexts:

(i) as boundaries of three-dimensional domains;

(ii) as an idealization of thin structures, that is objects with one dimension negligible in comparison with the other two.

The first of the above categories includes all phenomena that can be modeled by three-dimensional topological manifolds with boundary, such like interface dynamics, crystal growth and contact problems, whereas the second – all those that can be directly modeled by two-dimensional manifolds like soap films, membranes, thin plates and shells. Consequently, one faces surfaces of any topological type: most often – with boundary, frequently - closed compact, and occasionally – complete.

For problems belonging to category (i) it frequently happens that it is only the evolution of the boundary itself, and not the ambient domain, that is of main interest (capillarity, crystal growth, solidification, friction and wear). Yet, to find how the boundary evolves in time requires solving a three-dimensional differential or even integro-differential problem for the entire domain, with input data sometimes difficult to come by due to the limitations and/or obstacles in applying the measurement techniques. A remedy is to introduce a coarser, yet simpler, model founded on a geometrical approach, wherein the velocity field of the particles on the boundary or the surface energy are expressed as functionals of the physical quantities driving the evolution process:

see the monograph by Kosiński [1] on propagation of singularities in media, the article by Blinowski and Trzęsowski [2] on generalized theory of capillary phenomena, the article by Brower, Kessler, Koplik and Levine [3] for discussion in the context of interface dynamics and papers by Zmitrowicz [4], Strömberg [5] and Starmans, Brekelmans and Janssen [6] for friction and wear problems. This step converts the original three-dimensional problem into a two-dimensional one for a surface, which allows to move it to the second category.

Problems from (ii) are posed ab initio as two-dimensional by suitable approximations of different forms of energy appearing in the phenomenon and subsequent construction of some two-dimensional constitutive equations. If the surface is deformable (and usually it is), these constitutive equations must contain the kinematical variables accounting for the influence of the actual configuration of the surface on the physical state of the body the surface models.

Once all the physical factors are enclosed in mathematical formulae, one has to cast the problem into its final form, which will then become an evolution problem on a two-dimensional topological manifold. The evolution parameter may be the time (interface dynamics), forces (statics of shells) or some other variables depending on the nature of the problem. In the course of this evolution not only the geometry but also the topology of the surface may vary (cracks, ruptures, bifurcations, branching of soap films). However, the common background is the kinematics, which in this case is the theory of immersions of two-dimensional manifolds into R^3. Thus, the optimal, from the analytical point of view, final formulation of the problem arises as a compromise between the group of equations responsible for the physics and those accounting for kinematics. At this stage the key element is the choice of primary variables, which subsequently will become the unknown functions in the systems of PDE's to be solved for some initial and/or boundary conditions. The right choice may not only facilitate the analysis by setting the problem in a form as compact as possible (say, for the fewest number of unknowns) or exposing the group of parameters dominating in the problem; it may also reveal interdisciplinary connections and analogies, which help in understanding the underlying phenomenon.

Due to the above, kinematics of a moving surface has been treated from different aspects in various places in the literature: the most recent references are [1, 3, 8, 7, 10, 9, 14, 15]. Specifically, the theory of shells contributed to the subject so abundantly that it is impossible to cite here even the most important references (take, for instance, [11] as the starting point). All the same, as the recent advances in mathematical theory of surfaces indicate [13], the subject is far from being exhausted. Below I discuss possible novel formulations of the evolution problem under the assumption that the surface metric must appear as an explicit primary variable and show how they can be applied to the large deflection/small strain theory of thin shells. This material has been treated in more detail in [9, 15].

2. Intrinsic and reduced formulations; evolution

Let our surface, moving in $\mathbf{R}^3$, be given at time t by the position vector

$$(2.1) \qquad\qquad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \,,$$

where $x = x(\vartheta^\alpha, t)$, $y = y(\vartheta^\alpha, t)$, $z = z(\vartheta^\alpha, t)$ are the Cartesian coordinates in some fixed in time Cartesian frame $\mathbf{i}, \mathbf{j}, \mathbf{k}$, and ϑ^α, $\alpha = 1, 2$, are convective curvilinear coordinates (i.e. $\dot\vartheta^\alpha \equiv 0$). Then, the local basis consists of the two tangent vectors $\mathbf{r}_{,\alpha}$ (commas denote partial derivatives with respect to the coordinates ϑ^α) and the normal to the surface $\mathbf{n}$. Let the velocity field on the surface be $\dot{\mathbf{r}} = V^\alpha \mathbf{r}_{,\alpha} + U\mathbf{n}$. For V^α and U as known functionals on the surface, the latter equation yields a quasi-linear evolution system of three PDE's for the three unknown Cartesian coordinates of the immersion $\mathbf{r}$ (see [9]). This type of formulation corresponds to the displacement formulations. Although being simple, it gives, however, no insight into what is happening with physically measurable quantities characterizing the surface (like internal distances between adjacent points or curvatures), as the motion progresses.

The intrinsic formulations lie at the other extremity. They abstract the problem of the evolution from the ambient Euclidean space. The fundamental theorem of the theory of surfaces establishes a local one-to-one correspondence between the space of non-congruent surfaces and sets of six functions $a_{\alpha\beta}$ and $b_{\alpha\beta}$ (coefficients of the metric and the second fundamental form) satisfying the compatibility conditions: the Gauss–Mainardi–Codazzi equations

$$(2.2) \qquad\qquad \frac{1}{2}\varepsilon^{\alpha\lambda}\varepsilon^{\beta\mu}b_{\alpha\beta}b_{\lambda\mu} = K \,,$$

$$(2.3) \qquad\qquad b_{\alpha\beta}|_\lambda\varepsilon^{\beta\lambda} = 0 \,,$$

where $(.)|_\alpha$ denotes the covariant derivative, $\varepsilon^{\alpha\lambda}$ are the contravariant components of the permutation tensor and K is the Gaussian curvature. Formally, Eqs. (2.2) and (2.3), written down in maps from some atlas, form a nonlinear underdetermined system of three partial differential equations for six unknown functions of points on some two-dimensional differentiable manifold representing the topology of the surface. For a given topological type of the manifold, solutions of these equations correspond to all possible mutually homeomorphic and non-congruent configurations of the surface. By choosing any one-parameter family of such solutions (with some parameter t) and interpreting this parameter as time, we obtain a full description of a topology-preserving evolution corresponding to some motion of a surface in $\mathbf{R}^3$.

The fact that the system is underdetermined leaves a gap to be filled with additional three equations whose connotation lends a definite meaning to the whole system. In geometry these additional equations are predominantly of finite type, like in the problem of isometric bending: $\dot{a}_{\alpha\beta} \equiv 0$. In physics they are

usually systems of PDE that additionally relate the coefficients of both forms, thus closing the system. For instance, in theory of shells these are the equations of motion and in interfacial dynamics – the velocity functionals (see [3]).

The connected evolution system (see [1, 3, 7, 8] for discussion and derivation) consists of the following six PDE's:

$$(2.4) \qquad \dot{a}_{\alpha\beta} = -2b_{\alpha\beta}U + V_\alpha|_\beta + V_\beta|_\alpha \,,$$

$$(2.5) \qquad \dot{b}_{\alpha\beta} = U|_{\alpha\beta} - U(2Hb_{\alpha\beta} - Ka_{\alpha\beta}) + b_{\alpha\lambda}V^\lambda|_\beta + b_{\beta\lambda}V^\lambda|_\alpha + V^\lambda b_{\alpha\beta}|_\lambda$$

for six unknown functions $a_{\alpha\beta} = a_{\alpha\beta}(\vartheta^\lambda, t)$ and $b_{\alpha\beta} = b_{\alpha\beta}(\vartheta^\lambda, t)$ of points on the manifold and time. H in Eq. (2.5) denotes the mean curvature of the surface.

Although mathematically satisfactory, this system suffers from excessive, for physical applications, number (six) of equations and unknowns. A question arises: does the information about the kinematics of a moving surface have to be scattered on so many functions? Besides, the evolution of the metric informs us about the variation of the local distances between the adjacent points of the surface, but no such clear interpretation may be ascribed to the evolution of the second fundamental form.

As a remedy, we may replace the coefficients of the second fundamental form with its invariants: the mean and Gaussian curvatures, and its principal directions. The curvatures certainly bear more physical meaning than abstract functions $b_{\alpha\beta}$ (in support of this statement see [2], where the authors used this idea in connection with their considerations about the changes of surface energy under transitions of the Euler–Poincaré characteristic of the interfacial surface). Then, the Gaussian curvature may be eliminated from the system with the use of the Gauss equation (2.2). This step leads to replacement of the compatibility equations (2.2), (2.3) with a system of two nonlinear PDE's for five unknown functions: the coefficients of the metric, the mean curvature and the angle φ between a fixed convective field of directions on the surface and one of the principal directions. In [9] I have shown that if one picks the directions tangent to ϑ^1 coordinate lines to be the reference field of directions, then the corresponding evolution system consists of the following five equations:

$$(2.6) \qquad \dot{a}_{\alpha\beta} = -2HU\,a_{\alpha\beta}$$
$$- 2\sqrt{H^2 - K}\,U(\cos 2\varphi\,\delta_\alpha^\mu - \sin 2\varphi\,\varepsilon_{\alpha\lambda}\,g^{\lambda\mu}) \left(\frac{1}{a^{11}}\,\delta_\mu^1 \delta_\beta^1 - \frac{1}{a_{22}}\,a_{2\mu}a_{2\beta} \right)$$
$$+ V_\alpha|_\beta + V_\beta|_\alpha \,,$$

$$(2.7) \qquad \dot{H} = H_{,\alpha}\,V^\alpha + (2H^2 - K)U + \frac{1}{2}\,\Delta U \,,$$

$$(2.8) \qquad \dot{\varphi} = (\varphi_{,\alpha} + k_\alpha^{\vartheta^1})V^\alpha + \frac{\sqrt{a}}{a_{22}}V^1|_2 + \frac{1}{2a_{22}}\sqrt{\frac{a}{H^2 - K}}\,a^{1\alpha}U|_{\alpha 2}\cos 2\varphi$$
$$+ \frac{1}{2\sqrt{H^2 - K}}\left[\frac{1}{a_{22}}\,U|_{22} - \frac{1}{2}\Delta U + 2(H^2 - K)U \right]\sin 2\varphi \,.$$

In the above formulae Δ denotes the Laplace–Beltrami operator, $\dot{\varphi}$ is the angular velocity of the principal directions of the second fundamental form, $a = \det(a_{\alpha\beta})$ and $k_\alpha^{\vartheta^1}$ are the components of the connection vector (the co-form of the connection form in the Maurer–Cartan equations) given by the formulae:

$$(2.9) \qquad \mathbf{k}^{\vartheta^1} = \frac{1}{2\sqrt{a}} \left[\left(\frac{a_{12}}{a_{22}} a_{22,1} - a_{11,2} \right) \mathbf{a}^1 + \left(\frac{a_{12}}{a_{22}} a_{22,2} + a_{22,1} - 2a_{12,2} \right) \mathbf{a}^2 \right].$$

Locally, away from an umbilical point, the evolution system (2.6)–(2.8) is completely equivalent to the original system (2.4)-(2.5). Above all, it preserves the bijective nature of the relation between the space of its solutions and the space of non-rigid motions. It still contains, however, five unknown functions.

To obtain any further reduction of the evolution system and, thus, a more compact description of motion of a surface, one may choose to decompose the motion into the part responsible for the evolution of its intrinsic geometry and the other, describing the evolution of the extrinsic geometry. This approach is particularly useful in mechanics of deformable material surfaces. Then, evolution of the intrinsic geometry is described by the strain rate, and that of extrinsic geometry corresponds to the isometric bending (the other terms in common use for this kind of motion are: pure bending, geometric bending, inextensional deformation). There are exactly two alternative ways to make description of this kind as compact as possible. Both are based on the so-called Darboux equations – two nonlinear second-order PDE derived independently by Darboux and Bianchi in connection with the problem of finding all isometric immersions of a given two-dimensional Riemannian manifold into the ordinary Euclidean space. For a fixed metric they become equations of the Monge–Ampère type, i.e. equations whose leading term is the determinant of the second covariant derivative of the unknown function.

The unknown function in the case of the first of the two equations has the interpretation of a distance function from some fixed, but otherwise arbitrary, plane in R^3. Suppose this plane coincides with the $z = 0$ plane. Then, the unknown function is exactly the z Cartesian coordinate of the immersion sought and the corresponding equation is:

$$(2.10) \qquad d_z - K(1 - z_{,\alpha} z_{,\beta} a^{\alpha\beta}) = 0,$$

where $d_z = 1/2\varepsilon^{\alpha\lambda}\varepsilon^{\beta\mu} z|_{\alpha\beta} z|_{\lambda\mu}$ is the Monge–Ampère operator of the function z and K should be replaced with the Gauss formula expressing the Gaussian curvature of a given metric via its coefficients. Upon solving this equation, the second fundamental form of the related immersion is furnished by the formula:

$$(2.11) \qquad b_{\alpha\beta} = (1 - z_{,\lambda} z_{,\mu} a^{\lambda\mu})^{-1/2} z|_{\alpha\beta},$$

and the remaining two Cartesian coordinates x, y follow from the coefficients of the metric and the solution z via quadratures (see [9, 12]). Note that the above

set of relations provides complete information about the surface at a fixed time t in terms of just four quantities. By letting z and the metric vary with time we obtain systematic account of its evolution. Then, evolution of the metric describes the evolution of its intrinsic geometry, whereas evolution of z supplies information about the evolution of the second fundamental form and, thus, the isometric bending, as desired.

As follows from [9], the corresponding evolution system is:

$$(2.12) \qquad \dot{a}_{\alpha\beta} = -\frac{2U}{\sqrt{1 - z_{,\lambda}\, z_{,\mu}\, a^{\lambda\mu}}}\, z|_{\alpha\beta} + V_\alpha|_\beta + V_\beta|_\alpha\,,$$

$$(2.13) \qquad \dot{z} = V^\alpha z_{,\alpha} + U\sqrt{1 - z_{,\lambda}\, z_{,\mu}\, a^{\lambda\mu}}\,.$$

An alternative description of the evolution problem may be based on the second of the two Darboux equations after minor modifications. The unknown function in this case is the function $r = r(\vartheta^\alpha, t)$ describing the distance of the points of the surface at some moment t from some fixed, but otherwise arbitrary, point O in $\mathbf{R}^3$. Take O to be the origin of the Cartesian system. Then r is the length of the position vector and the equation is:

$$(2.14) \qquad d_{r^2} - 2\Delta r^2 + K(r^2_{,\alpha}\, r^2_{,\beta}\, a^{\alpha\beta} - 4r^2) + 4 = 0\,.$$

Again, upon solving this equation the second fundamental form follows from the formula

$$(2.15) \qquad b_{\alpha\beta} = \frac{1}{\sqrt{r^2 - \frac{1}{4}(r^2)_{,\alpha}\,(r^2)_{,\beta}\, a^{\alpha\beta}}}\left(\frac{1}{2}r^2|_{\alpha\beta} - a_{\alpha\beta}\right).$$

The related evolution system is:

$$(2.16) \qquad \dot{a}_{\alpha\beta} = \frac{1}{r\sqrt{1 - r_{,\alpha}\, r_{,\beta}\, a^{\alpha\beta}}}(2a_{\alpha\beta} - r^2|_{\alpha\beta})U + V_\alpha|_\beta + V_\beta|_\alpha\,,$$

$$(2.17) \qquad \dot{r} = V^\alpha r_{,\alpha} + U\sqrt{1 - r_{,\lambda}\, r_{,\mu}\, a^{\lambda\mu}}\,.$$

3. Application to large deflection/small strain type of thin shell theories

For purposes of demonstration let us consider how the selection of a proper description of surface kinematics may be helpful in modification of the nonlinear theory of thin shells within the class of large deflection/small strain deformations. The practical significance of this class follows from the fact that for metals, most frequently used materials in construction of thin-walled structures, the admissible strain is approximately a quantity of the order 10^{-3}. This limitation on the admissible deformations the middle surface of the shell may assume, considerably

reduces the practically accessible states of its equilibrium (see [14, 15]). Yet, this fact is nowhere visible in the framework of the classical shell theory, due to the fact that it is based on description of motion via displacements.

As I have argued in [15], to pose the problem in a form convenient for qualitative as well as quantitative analysis, it is necessary to employ one of the two descriptions of motion based on the Darboux equations. With this choice the strains appear as the primary variables in the theory. Besides, the whole extrinsic geometry of the shell's middle surface is described by a single function z or r. Thus, the shell problem is governed by a determined system of four nonlinear PDE for four unknown functions, one of them being the Darboux equation and the remaining three – the equations of motion. In view of "smallness" of strains, this system may be subsequently linearized in these three variables. This step simplifies the whole problem, because we then have to deal with a system linear in three functions and nonlinear in only one, although the whole class of nonlinear large deflection/small strain deformations is covered effectively. Besides, upon solving this system, the displacements may be computed *via* quadratures.

For demonstrative purposes, let us see how this procedure works in the simplest case of statics of thin linearly elastic shells (see [15] for a more detailed discussion). To distinguish the deformed configuration of the shell middle surface from the undeformed one, the objects pertaining to the deformed configuration are marked by overbars, e.g. $\overline{a}_{\alpha\beta}$, $\overline{b}_{\alpha\beta}$, etc. That is, for this whole section and in contradistinction to the notation from the previous one, $a_{\alpha\beta}$ are now coefficients of the metric *only* at some initial time t_0, and $\overline{a}_{\alpha\beta}$ at some later time t. Then, $\gamma_{\alpha\beta} = (1/2)(\overline{a}_{\alpha\beta} - a_{\alpha\beta})$ is the surface strain tensor, $\kappa_{\alpha\beta} = -\overline{b}_{\alpha\beta} + b_{\alpha\beta}$ is the curvature change measure and $J = \dfrac{d\overline{A}}{dA} = \sqrt{\dfrac{\overline{a}}{a}}$. It may be shown (see [15]) that with the above notation we have:

$$(3.1) \qquad \overline{a}^{\alpha\beta} = \frac{1}{J}\left[(1 - 2\gamma)a^{\alpha\beta} + 2\gamma^{\alpha\beta}\right],$$

where $\gamma = \operatorname{tr}(\boldsymbol{\gamma})$,

$$(3.2) \qquad \overline{\varepsilon}_{\alpha\beta} = J\,\varepsilon_{\alpha\beta}, \qquad \overline{\varepsilon}^{\alpha\beta} = J^{-1}\varepsilon^{\alpha\beta}, \qquad J^2 = 4\det(\boldsymbol{\gamma}) + 2\gamma + 1.$$

Besides, the Christoffel symbols in both configurations are related by the formula

$$(3.3) \qquad \overline{\Gamma}^{\lambda}_{\alpha\beta} - \Gamma^{\lambda}_{\alpha\beta} = \overline{a}^{\lambda\mu}\gamma_{\mu\alpha\beta}, \qquad \text{where} \quad \gamma_{\mu\alpha\beta} = \gamma_{\mu\alpha}|_{\beta} + \gamma_{\mu\beta}|_{\alpha} - \gamma_{\alpha\beta}|_{\mu}.$$

Now use the above to pull the Darboux equation (2.10) from the deformed configuration to the undeformed one and subsequently, to linearize the result in $\gamma_{\alpha\beta}$. This leads to the equation

$$(3.4) \qquad d_{\overline{z}} - \left[\overline{z}|_{\alpha\beta} - (\Delta\overline{z})a_{\alpha\beta}\right]\overline{z}_{,\lambda}\,\gamma^{\lambda\alpha\beta} - \left(\gamma^{\lambda\mu}|_{\lambda\mu} - \Delta\gamma\right)\left(1 - \overline{z}_{,\alpha}\,\overline{z}_{,\beta}\,a^{\alpha\beta}\right)$$
$$- K\left\{1 - \overline{z}_{,\alpha}\,\overline{z}_{,\beta}\left[a^{\alpha\beta}(1 + \gamma) - 2\gamma^{\alpha\beta}\right]\right\} = 0,$$

which is a second-order PDE, linear in strains and nonlinear in the function $\bar{z}$ – a Cartesian coordinate of the deformed shell middle surface. By assumption, this equation approximates the kinematics of the shell middle surface in the neighborhood of its undeformed state for sufficiently small strains and their derivatives, but arbitrary bendings. To obtain a determined system of equations, we need three further equations relating our four unknown functions. These are granted by the balance conditions.

Suppose we have derived the balance conditions from the virtual work principle by routine variational methods, starting from the strain energy density of the form $\mathcal{W} = \mathcal{W}(\gamma_{\alpha\beta}, \kappa_{\alpha\beta})$ (if $\mathcal{W}$ is a quadratic function of its arguments, we get the Koiter–Sanders first approximation theory). Then, the resulting Lagrangian equilibrium equations take the form

$$(3.5) \quad \left(-\frac{\partial \mathcal{W}}{\partial \gamma_{\alpha\beta}} + \bar{b}^\alpha_\lambda \frac{\partial \mathcal{W}}{\partial \kappa_{\lambda\beta}} \right) \Big|_\beta + \bar{b}^\alpha_\lambda \frac{\partial \mathcal{W}}{\partial \kappa_{\lambda\beta}} \Big|_\beta + \left[\left(-\frac{\partial \mathcal{W}}{\partial \gamma_{\alpha\mu}} + 2\bar{b}^\alpha_\lambda \frac{\partial \mathcal{W}}{\partial \kappa_{\lambda\mu}} \right) \bar{a}^{\beta\nu} \right.$$

$$\left. + \left(-\frac{\partial \mathcal{W}}{\partial \gamma_{\mu\beta}} + \bar{b}^\mu_\lambda \frac{\partial \mathcal{W}}{\partial \kappa_{\lambda\beta}} \right) \bar{a}^{\alpha\nu} + \bar{b}^\alpha_\lambda \frac{\partial \mathcal{W}}{\partial \kappa_{\lambda\beta}} \bar{a}^{\lambda\nu} \right] \gamma_{\nu\beta\mu} + p^\alpha = 0,$$

$$(3.6) \quad -\frac{\partial \mathcal{W}}{\partial \kappa_{\alpha\beta}} \Big|_{\alpha\beta} - \left[\left(\frac{\partial \mathcal{W}}{\partial \kappa_{\alpha\mu}} \bar{a}^{\beta\lambda} + \frac{\partial \mathcal{W}}{\partial \kappa_{\mu\beta}} \bar{a}^{\alpha\lambda} \right) \gamma_{\lambda\alpha\mu} \right] \Big|_\beta$$

$$+ \left(-\frac{\partial \mathcal{W}}{\partial \gamma_{\alpha\beta}} + \bar{b}^\alpha_\lambda \frac{\partial \mathcal{W}}{\partial \kappa_{\lambda\beta}} \right) \bar{b}_{\alpha\beta}$$

$$- \left[\frac{\partial \mathcal{W}}{\partial \kappa_{\lambda\beta}} \bar{a}^{\alpha\mu} + \left(\frac{\partial \mathcal{W}}{\partial \kappa_{\lambda\psi}} \bar{a}^{\beta\varphi} + \frac{\partial \mathcal{W}}{\partial \kappa_{\psi\beta}} \bar{a}^{\lambda\varphi} \right) \gamma_{\varphi\lambda\psi} \bar{a}^{\alpha\mu} \right] \gamma_{\mu\alpha\beta} + p = 0,$$

where p^α and p account for the contributions from the external loads. Now, replace in the above $\bar{a}^{\alpha\beta}$ and $\bar{b}^{\alpha\beta}$ with the right-hand sides of (3.1) and (2.11), respectively. The two equations (3.5) are of the order three in $\bar{z}$ and two in $\gamma_{\alpha\beta}$, and the equation (3.6) – of the order four in $\bar{z}$ and three in $\gamma_{\alpha\beta}$. Therefore, linearization in $\gamma_{\alpha\beta}$ transforms them into the following three PDE:

$$(3.7) \quad \mathcal{H}^\alpha + \mathcal{H}^{\alpha\kappa\lambda} \gamma_{\kappa\lambda} + \mathcal{H}^{\alpha\kappa\lambda\mu} \gamma_{\kappa\lambda}|_\mu = 0,$$

$$(3.8) \quad \mathcal{V} + \mathcal{V}^{\alpha\beta} \gamma_{\alpha\beta} + \mathcal{V}^{\alpha\beta\lambda} \gamma_{\alpha\beta}|_\lambda + \mathcal{V}^{\alpha\beta\lambda\mu} \gamma_{\alpha\beta}|_{\lambda\mu} = 0.$$

The tensor coefficients $\mathcal{H}$ and $\mathcal{V}$ in the above are nonlinear functions of: the derivatives of $\bar{z}$ (up to the third order in the case of $\mathcal{H}$ and fourth of $\mathcal{V}$), the parameters of the undeformed configuration $a_{\alpha\beta}$ and $b_{\alpha\beta}$, material constants, and the external loads (for large deformations they may depend on the configuration and, therefore, they may enter all of the coefficients). Their precise form will depend on the physical model of the shell material and the external loads applied to the shell.

Acknowledgments

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