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FIFTY YEARS OF THE ARCHIVES OF MECHANICS

System symmetries and inverse variational problems in continuum theory

Dedicated to Prof. Henryk Zorski
on the occasion of his 70-th birthday

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THE AIM of the conventional Inverse Problem in Lagrange formalism is to find a Lagrangian, the associated Euler–Lagrange equations of which are equivalent to a given set of partial differential equations of a physical system. In contrast, I am dealing with a different type of an inverse problem. I look for a Lagrangian which is associated with a given set of balance equations. My approach is based on general relations between symmetry groups (geometrical and gauge symmetries) and its associated balance equations. I follow two different mathematical lines: The first one is *Noether's theorem*: Universal Lie symmetry groups like translations (spatial and temporal), rotations and Galilei transformation are connected with the fundamental conservation laws for energy, linear momentum, angular momentum and center of mass motion. All of these balances are of the "volume-type". The second line takes account of a relationship between non-Lie symmetry groups (e.g. regauging of potentials) and balances of the "area-type". These are physically associated with *line-shaped objects* like vortex lines and dislocations. Following both lines in an inverse manner I derive the relevant symmetry properties of a yet unknown Lagrangian for a given set of balance equations of volume- and area-types. Consequently, a rough scheme for the analytical structure of the Lagrangian can be given. As an example, a Lagrangian for the elastic deformation of a body with eigenstresses due to fixed dislocations is constructed.

Notations

Subsequently I use the following notations for temporal and spatial coordinates and their corresponding first order derivatives:

$$x = (x^\alpha) = (t, \mathbf{x}) = \left(x^0 = t, x^1, \cdots, x^3\right),$$
$$\partial = (\partial_\alpha) = (\partial_t, \nabla) = \left(\partial_0 = \partial_t = \frac{\partial}{\partial t}, \partial_1 = \frac{\partial}{\partial x^1}, \cdots, \partial_3 = \frac{\partial}{\partial x^3}\right).$$

The *substantial time derivative* is the operator

$$\mathrm{D}_t := \partial_t + \mathbf{v}\cdot\nabla,$$

where $\mathbf{v}$ denotes the velocity field of the medium. Furthermore, the three canonical Euclidean base vectors $\mathbf{e}_\kappa$ of the laboratory frame are defined as

$$\mathbf{e}_1 = (1,0,0), \qquad \mathbf{e}_2 = (0,1,0), \qquad \mathbf{e}_3 = (0,0,1).$$

Let $\ell = \ell(\psi, \partial\psi, x)$ be a first order Lagrangian of a system based on N independent fields

$$\psi = \left(\psi^i(x)\right) = \left(\psi^1(x), \cdots, \psi^N(x)\right).$$

Then, the conjugated canonical field momenta $\pi^\alpha = (\pi_i^\alpha) = (\pi_1^\alpha(\psi, \partial\psi, x), \cdots, \pi_N^\alpha(\psi, \partial\psi, x))$ are given by

$$\pi_i^\alpha(\psi, \partial\psi, x) := \frac{\partial \ell}{\partial(\partial_\alpha \psi^i)}. \tag{0.1}$$

We additionally define $\pi = \left(\pi^1, \pi^2, \pi^3\right)$.

Einstein's summation convention is implied, whenever two indices occur twice in a product, exept for special cases, where for clarity the summation is indicated explicitly.

1. The inverse problem of the second kind

IN TRADITIONAL continuum theories a system is regarded to be physically defined, if a set of relevant balance equations is established together with an associated set of constitutive equations for the densities, flux densities and production rates involved in the balance equations.

Apart from this method, Lagrange formalism (LF) gives rise to an alternative formulation of the dynamics of the system: as the main feature, all information on the processes of a particular system is contained in one function only, namely its Lagrangian. All balance equations can be derived from the Lagrangian.

However, in many cases the Lagrangian is unknown. Then, to establish an adequate Lagrangian formulation of the system, one has to start from the established set of phenomenological balance equations in order to construct the suitable Lagrangian. This is a rough description of the "inverse problem in LF". An exact mathematical definition of the inverse problem in LF is much more difficult; it depends on the viewpoint, how the phenomenological equations should be derived from the Lagrangian:

1. From the traditional viewpoint, either the phenomenological equations themselves or an equivalent self-adjoint set of equations are expected to be the *Euler–Lagrange equations* of the Lagrangian. We call this traditional concept the *inverse problem of the first kind* (IP1).

2. Noether's theorem shows us, how to derive the balance equations from a given Lagrangian. Taking universal symmetries like time- and space translation, rigid rotation and Galilei-transformation into account, the phenomenological balances for energy, linear momentum, angular momentum and center of mass motion can be derived by means of the wellknown "Noether machinery". Thus, if one interprets the relevant phenomenological balances of the system as Noether balances, a different kind of inverse problem is defined which I call the *inverse problem of the second kind* (IP2).

Taking the second viewpoint, it is practicable to treat different physical systems in the same universal way, i.e. apart from their individual constitutive laws. Lagrange formalism is a *unifying concept* for quite different physical problems.

However, not all relevant phenomenological equations can be interpreted as Noether balances, e.g. the fundamental dynamical equations for *vortices* and *dislocations* which are "area-type" balances. In order to define IP2 completely we first have to find an adequate way to treat such balances in a way which is quite analogous to Noether's theorem.

2. Symmetries and balances beyond Noether's theorem

2.1. Two types of balance equations

We refer to the wellknown homogenous balance equation of an observable A,

$$\partial_t a(\mathbf{x},t) + \nabla\cdot\mathbf{j}_a(\mathbf{x},t) = 0\,, \tag{2.1}$$

which we call a *volume-type balance equation*. a is the local density and $\mathbf{j}_a$ the flux density of the observable A. This local form of the balance equation is associated with a global form by integrating Eq. (2.1) over a fixed test volume V:

$$\frac{dA}{dt} = \Phi_A\,, \tag{2.2}$$

where

$$A(t) := \int\limits_V a(\mathbf{x},t)\,d^3x \tag{2.3}$$

is the total amount of the observable A in the volume V, and

$$\Phi_A(t) := -\int\limits_V \nabla\cdot\mathbf{j}_a(\mathbf{x},t)\,d^3x = -\int\limits_{\partial V} \mathbf{j}_a(\mathbf{x},t)\cdot d\mathbf{S} \tag{2.4}$$

is the total flux of A across the boundary ∂V of the test volume.

In the case of *point-like objects*, the quantities A and Φ_A are obviously associated with the number of objects within the test volume V and with the number of objects passing the boundary ∂V, respectively. In Eq. (2.4) we took account of Gauss' theorem. Thus the term "volume-type balance" is sufficiently motivated.

We further refer to the homogenous balance equation of the type

$$\partial_t\mathbf{w}(\mathbf{x},t) + \nabla\times\mathbf{J}_w(\mathbf{x},t) = 0 \tag{2.5}$$

which, with regard to the subsequent theory and applications, is supplemented by the equation

$$\nabla\cdot\mathbf{w} = 0. \tag{2.6}$$

Obviously Eq. (2.5) is formally different from (2.1); we call it an *area-type balance equation*. It is associated with an observable Γ. Let us call $\mathbf{w}$ a *generalized vortex density* and $\mathbf{J}_w$ the associated *generalized vortex flux density*. The local form

(2.5) of the balance equation is related to a global form by integrating over a fixed test area F:

$$\frac{d\Gamma}{dt} = \Phi_\Gamma \,, \tag{2.7}$$

where

$$\Gamma(t) := \int_F \mathbf{w}(\mathbf{x},t)\cdot d\mathbf{F} \tag{2.8}$$

is called the total *generalized circulation* of the observable Γ *referred to the test area* F.

$$\Phi_\Gamma(t) := -\int_F \nabla\times\mathbf{J}_w(\mathbf{x},t)\cdot d\mathbf{F} = -\int_{\partial F}\mathbf{J}_w(\mathbf{x},t)\cdot d\mathbf{s} \tag{2.9}$$

is called the total *generalized circulation flux* across the boundary line ∂F of the test area.

It is a wellknown fact that Eqs. (2.5), (2.6) are frequently physically realized by means of *line-shaped objects* which may be counted in a test cross-section. In this case the vortex density $\mathbf{w}$ and the vortex flux density $\mathbf{J}_w$ are associated with the number of objects piercing the test area F and with the number of objects passing the boundary line ∂F. In Eq. (2.9) we took account of Stokes' theorem. Thus the term "area-type balance" is sufficiently motivated.

Examples for two simultanous equations of the types (2.5), (2.6) are:

• *Helmholtz' equations* for the vorticity $\boldsymbol{\omega} = \frac{1}{2}\nabla\times\mathbf{v}$ of an ideal fluid ($\mathbf{v}$: velocity field):

$$\partial_t\boldsymbol{\omega} + \nabla\times[-\mathbf{v}\times\boldsymbol{\omega}] = 0, \tag{2.10}$$

$$\nabla\cdot\boldsymbol{\omega} = 0. \tag{2.11}$$

• Fundamental kinematical equations of the *dislocation theory*:

$$\partial_t\underline{\underline{\alpha}} + \nabla\times\underline{\underline{J}}_\alpha = 0, \tag{2.12}$$

$$\nabla\cdot\underline{\underline{\alpha}} = 0. \tag{2.13}$$

$\underline{\underline{\alpha}}$ is the second rank dislocation tensor and $\underline{\underline{J}}_\alpha$ the associated dislocation flux density. Equation (2.13) is related to the fact that dislocation lines cannot end within the crystal, whereas Eq. (2.12) is related to the number of dislocation lines and its balance.

These area-type balances cannot be interpreted as Noether balances! Nevertheless the question arises if the area-type balances (2.5), (2.6) can be obtained within the Lagrange formalism by means of a straightforward and unified formalism similar to the Noether theorem.

2.2. An alternative symmetry-balance-theorem

Let $\psi = (\psi^1, \cdots, \psi^N)$ be a set of fundamental field variables of a physical system, e.g. a set of potentials. Let further $\ell(\psi, \partial\psi, x)$ be the Lagrangian of the system. Then let us look at a set $\mathbb{F} = \left\{(F^1(\psi), \cdots, F^N(\psi))\right\}$ of functions $F^i \in C^2(\mathbb{R}^N)$ which induce a *regauging* of the fields ψ (potentials):

$$\begin{aligned} x^\alpha &\longrightarrow x^\alpha, \\ \psi^i &\longrightarrow F^i(\psi), \qquad F \in \mathbb{F}. \end{aligned} \tag{2.14}$$

Let us assume that these transformations (2.14) fulfil the *strict symmetry criterion*:

$$\ell(F(\psi), \partial F(\psi), x) = \ell(\psi, \partial\psi, x). \tag{2.15}$$

Then they are called *symmetry transformations of the Lagrangian*. We further assume that the set $\mathbb{F}$ is a *non-Lie group*, i.e. there are no group parameters available with the consequence that Noether's theorem is not applicable to (2.14). Nevertheless, I shall establish an alternative method to get a set of balance equations associated with the symmetry group $\mathbb{F}$: Deriving (2.15) with respect to $\partial_\alpha \psi^i$, I obtain a system of $4N$ equations

$$\left(\pi_j^\alpha\right)_{\psi \to F} \frac{\partial F^j}{\partial \psi^i} = \pi_i^\alpha, \qquad \alpha = 0, \cdots, 3, \quad i = 1, \cdots, N, \tag{2.16}$$

with the field momenta π_i^α defined in Eq. (0.1). This *differential symmetry criterion* is a system of necessary conditions for the set $\mathbb{F}$ of symmetry transformations. These $4N$ equations are not *linearly independent* in general. However, there always exists a *basic representation* of the field momenta

$$\pi^\alpha = \sum_{p=1}^{M} \mu_p^\alpha B^p \tag{2.17}$$

with *coefficients* $\mu_p^\alpha(\psi, \partial\psi, x)$ which are invariant under to the gauge transformation (2.14), and with a set of $M \leq 4$ *linearly independent basis elements* $\mathbb{B} := \left\{B^1 = (B_1^1, \cdots, B_N^1), \cdots, \ B^M = \left(B_1^M, \cdots, B_N^M\right)\right\}$. Their linear independence means that

$$\sum_{p=1}^{M} \lambda_p B^p = 0 \quad \Longleftrightarrow \quad \lambda_p = 0 \quad \forall p \tag{2.18}$$

with coefficients $\lambda_p(\psi, \partial\psi, x)$ *invariant* with respect to (2.14). The basis elements B^p are functions of $\psi, \partial\psi$ and x, in general. By means of this basis representation, the set of Eqs. (2.16) can be simplified; it can be substituted by the *reduced symmetry criterion*

$$\left(B_j^p\right)_{\psi \to F} \frac{\partial F^j}{\partial \psi^i} = B_i^p, \tag{2.19}$$

a system of $M \cdot N$ linearly independent equations which contains the same information about the non-Lie regauging group $\mathbb{F}$ as (2.16) does.

The basis elements B^p represent characteristic features of the non-Lie group $\mathbb{F}$; they are of the same importance as the infinitesimal generators for Lie-groups are.

THEOREM. *Let $\mathbb{B} = \{B^p \,|\, p = 1, \cdots, M \leq 4\}$ be a proper set of linearly independent basis elements due to Eq. (2.17). From this basis $\mathbb{B}$ we define the vortex densities $\mathbf{w}^p$ and the associated flux densities $\mathbf{J}^p$ as*

$$\mathbf{w}^p := \nabla B_i^p \times \nabla \psi^i, \tag{2.20}$$

$$\mathbf{J}^p := \partial_t \psi^i \nabla B_i^p - \partial_t B_i^p \nabla \psi^i. \tag{2.21}$$

Then, these quantities are invariant under regauging (2.14) and fulfil the M homogenous area-type balances ($p = 1, \cdots, M$)

$$\partial_t \mathbf{w}^p + \nabla \times \mathbf{J}^p = 0, \tag{2.22}$$

$$\nabla \cdot \mathbf{w}^p = 0. \tag{2.23}$$

P r o o f: 1. The gauge-invariance of the balance quantities $\mathbf{w}^p$ and $\mathbf{J}^p$ is a consequence of the reduced symmetry criterion (2.19). Looking at

$$\begin{aligned} \mathbf{w}^p &= \nabla B_i^p \times \nabla \psi^i = \nabla \left[\left(B_j^p \right)_{\psi \to F} \frac{\partial F^j}{\partial \psi^i} \right] \times \nabla \psi^i \\ &= \nabla \left(B_j^p \right)_{\psi \to F} \times \left(\frac{\partial F^j}{\partial \psi^i} \nabla \psi^i \right) + \left(B_j^p \right)_{\psi \to F} \frac{\partial^2 F^j}{\partial \psi^i \partial \psi^k} \nabla \psi^k \times \nabla \psi^i \end{aligned}$$

and keeping in mind that the last term vanishes [1], the gauge-invariance of the vortex densities can be easily shown:

$$\mathbf{w}^p = \nabla \left(B_j^p \right)_{\psi \to F} \times \left(\frac{\partial F^j}{\partial \psi^i} \nabla \psi^i \right) = \nabla \left(B_j^p \right)_{\psi \to F} \times \nabla F^j(\psi) = \left(\mathbf{w}^p \right)_{\psi \to F} .$$

The gauge-invariance of the vortex flux densities $\mathbf{J}^p$ can be proven in the same way.

2. Obviously, by definition (2.20), (2.21) the two identities

$$\begin{aligned} \partial_t \mathbf{w}^p &= (\partial_t \nabla B_i^p) \times \nabla \psi^i + \nabla B_i^p \times \left(\partial_t \nabla \psi^i \right) \\ &= \nabla \times \left[\partial_t B_i^p \nabla \psi^i - \partial_t \psi^i \nabla B_i^p \right] - \partial_t B_i^p \nabla \times \nabla \psi^i + \partial_t \psi^i \nabla \times \nabla B_i^p = -\nabla \times \mathbf{J}^p \\ \nabla \cdot \mathbf{w}^p &= \nabla \cdot \left[\nabla \times \left(B_i^p \nabla \psi^i \right) - B_i^p \nabla \times \nabla \psi^i \right] = 0 \end{aligned}$$

are fulfilled.

[1] This term contains a contraction of an expression symmetric with respect to the index change $i \leftrightarrow k$ and an expression antisymmetric with respect to the same operation.

The above theorem supplements Noether's theorem – it operates with another kind of symmetries and another kind of balances. For further details I refer to the forthcomming paper [8].

2.3. Helmholtz' equation of the ideal fluid as an example

It is well-known that the Lagrangian for an ideal fluid takes the form [2]

$$\ell = -\varrho\left[\partial_t\Phi + \gamma\partial_t\vartheta + \frac{1}{2}(\nabla\Phi + \gamma\nabla\vartheta)^2 + u(\varrho)\right] \tag{2.24}$$

depending on the 4 independent fields $\psi = (\psi^1, \cdots, \psi^4) = (\varrho, \Phi, \gamma, \vartheta)$, namely the mass density ϱ and the so-called Clebsch potentials which give rise to the potential representation [3, 4]

$$\mathbf{v} = \nabla\Phi + \gamma\nabla\vartheta \tag{2.25}$$

of the velocity field $\mathbf{v}$. As a consequence, the vortex field $\boldsymbol{\omega}$ takes the form

$$\boldsymbol{\omega} = \frac{1}{2}\nabla\times\mathbf{v} = \frac{1}{2}\nabla\gamma\times\nabla\vartheta\,. \tag{2.26}$$

The function $u(\varrho)$ denotes the elastic energy of the fluid. Now we apply the alternative symmetry-balance-theorem for area-type balances to this example: the conjugate canonical field momenta take the form

$$\begin{aligned}\pi^0 &= \left(\pi^0_1, \cdots, \pi^0_4\right) = (0, -\varrho, 0, -\varrho\gamma)\,,\\ \boldsymbol{\pi} &= (\boldsymbol{\pi}_1, \cdots, \boldsymbol{\pi}_4) = (0, -\varrho\mathbf{v}, 0, -\varrho\gamma\mathbf{v})\,.\end{aligned}$$

Obviously, the field momenta can be represented by means of one basis element B^0 only:

$$\begin{aligned}\pi^0_i &= -\varrho B^0_i\,,\\ \boldsymbol{\pi}_i &= -\varrho\mathbf{v}B^0_i\,,\end{aligned} \tag{2.27}$$

with

$$B^0 = \left(B^0_1, \cdots, B^0_4\right) = (0, 1, 0, \gamma). \tag{2.28}$$

According to (2.20), (2.21), this basis element gives rise to a line density $\mathbf{w}$ and its flux density $\mathbf{J}$ via

$$\mathbf{w}^0 = \nabla B^0_i \times \nabla\psi^i = \nabla\gamma\times\nabla\vartheta = 2\boldsymbol{\omega}, \tag{2.29}$$

$$\mathbf{J}^0 = \partial_t\psi^i\nabla B^0_i - \partial_t B^0_i\nabla\psi^i = -\mathbf{v}\times\mathbf{w}^0, \tag{2.30}$$

which are gauge invariant and fulfil the homogenous balance equation of the area type (2.22), (2.23). In (2.30) we took account of the two Euler–Lagrange equations [2]

$$\mathrm{D}_t\vartheta = 0, \qquad \mathrm{D}_t\gamma = 0,$$

of (2.24). Obviously, the associated balance equations (2.22), (2.23) turn out to be *Helmholtz' equations* (2.10), (2.11). Thus, the regauging group of the Clebsch potentials Φ, γ, ϑ determined by the basis element B^0 is associated with the vortex dynamics of the fluid.

3. A Lagrangian description for the dynamical theory of elasticity with eigenstresses

3.1. The general analytical form of the Lagrangian

For the ideal fluid we found that the conjugate field momenta can be represented by means of one basis element B^0 only (see Eq. (2.27), and that it is associated with the dynamics of *vortices*. Passing from the fluid to the solid we take account of *dislocations* in an analogous way. Dislocations are line-shaped objects like vortices. Let us extend Eq. (2.27) towards a generalized basis representation

$$
\begin{aligned}
\pi_i^0 &= -\varrho B_i^0 + \sum_{\kappa=1}^{3} \sigma_\kappa^0 B_i^\kappa , \\
\boldsymbol{\pi}_i &= -\varrho \mathbf{v} B_i^0 + \sum_{\kappa=1}^{3} \boldsymbol{\sigma}_\kappa B_i^\kappa ,
\end{aligned}
\tag{3.1}
$$

with *four basis elements* B^0, B^1, B^2, B^3, the dimension N of which is still open. B^0 corresponds to the *vortex dynamics*, whereas B^1, B^2, B^3 correspond to the *dislocation dynamics*.

The associated area-type balances (2.22), (2.23) are based on the quantities

$$
2\boldsymbol{\omega} = \nabla B_i^0 \times \nabla \psi^i = \nabla \times \left(B_i^0 \nabla \psi^i \right) , \tag{3.2}
$$

$$
2\mathbf{J}_\omega = \partial_t \psi^i \nabla B_i^0 - \partial_t B_i^0 \nabla \psi^i , \tag{3.3}
$$

and

$$
\underline{\underline{\alpha}} = \sum_{\kappa=1}^{3} \left(\nabla B_i^\kappa \times \nabla \psi^i \right) \otimes \mathbf{e}_\kappa = \sum_{\kappa=1}^{3} \nabla \times \left(B_i^\kappa \nabla \psi^i \right) \otimes \mathbf{e}_\kappa , \tag{3.4}
$$

$$
\underline{\underline{J}}_\alpha = \sum_{\kappa=1}^{3} \left[\partial_t \psi^i \nabla B_i^\kappa - \partial_t B_i^\kappa \nabla \psi^i \right] \otimes \mathbf{e}_\kappa , \tag{3.5}
$$

with an open set $\psi = \left(\psi^1, \cdots, \psi^N \right)$ of fundamental field variables. $\boldsymbol{\omega}$, $\mathbf{J}_\omega$ are the *vortex field* and the *vortex flux density*, and $\underline{\underline{\alpha}}$, $\underline{\underline{J}}_\alpha$ are the *dislocation density* and the associated flux density. In the theory of *generalized Cosserat-continua* [6, 7],

the dislocation density in a crystal lattice is defined from the three (reciprocal) *Cosserat-directors* $\mathbf{a}^1$, $\mathbf{a}^2$, $\mathbf{a}^3$ via

$$\underline{\underline{\alpha}} = \sum_{\kappa=1}^{3} (\nabla \times \mathbf{a}^\kappa) \otimes \mathbf{e}_\kappa \,. \tag{3.6}$$

The Cosserat-directors are associated with the crystal lattice vectors. Comparing (3.2) with (2.26) and (3.4) with (3.6), I obtain the identifications

$$\mathbf{v} = B_i^0 \nabla \psi^i, \tag{3.7}$$

$$\mathbf{a}^\kappa = B_i^\kappa \nabla \psi^i. \tag{3.8}$$

Aditionally, we have to take account of the *Noether balances*. E.g. the momentum balance [1]

$$\partial_t \mathbf{p} + \nabla \cdot \underline{\underline{\Sigma}} = 0 \tag{3.9}$$

which is due to space translations, is defined via Noether's theorem by the constitutive equations

$$\mathbf{p} = -\pi_i^0 \nabla \psi^i = \varrho B_i^0 \nabla \psi^i + \sum_{\kappa=1}^{3} \sigma_\kappa^0 B_i^\kappa \nabla \psi^i = \varrho \mathbf{v} + \mathbf{p}^*, \tag{3.10}$$

$$\underline{\underline{\Sigma}} = \ell \underline{\underline{1}} - \pi_i \otimes \nabla \psi^i = \ell \underline{\underline{1}} + \varrho \mathbf{v} \otimes B_i^0 \nabla \psi^i - \sum_{\kappa=1}^{3} \boldsymbol{\sigma}_\kappa \otimes B_i^\kappa \nabla \psi^i$$
$$= \varrho \mathbf{v} \otimes \mathbf{v} - \underline{\underline{\sigma}}, \tag{3.11}$$

where I have made use of the basis representation (3.1) and of the following abbreviations:

$$\mathbf{p}^* := \sum_{\kappa=1}^{3} \sigma_\kappa^0 \mathbf{a}^\kappa, \tag{3.12}$$

$$\underline{\underline{\sigma}} := \sum_{\kappa=1}^{3} \boldsymbol{\sigma}_\kappa \otimes \mathbf{a}^\kappa - \ell \underline{\underline{1}}. \tag{3.13}$$

$\mathbf{p}^*$ is the *quasi-momentum density* and $\underline{\underline{\sigma}}$ the *stress tensor*. Furthermore the mass density ϱ is coupled with the Cosserat directors by means of

$$\varrho = \varrho(\mathbf{a}^\kappa) = \varrho_0 \, \mathbf{a}^1 \cdot (\mathbf{a}^2 \times \mathbf{a}^3). \tag{3.14}$$

In order to simplify the subsequent procedure, I assume that the basis elements B^0, B^κ depend on the field variables ψ only and that $\sigma_\kappa^0 = 0$, i.e. the

quasi-momentum density $\mathbf{p}^*$ is assumed to vanish. Consequently, the system of equations (3.1) takes the form

$$
\begin{aligned}
&\frac{\partial \ell}{\partial(\partial_t \psi^i)} = \pi_i^0 = -\varrho B_i^0(\psi), \\
&\frac{\partial \ell}{\partial(\nabla \psi^i)} = \pi_i = -\varrho B_i^0(\psi) + \sum_{\kappa=1}^{3} \boldsymbol{\sigma}_\kappa(\psi, \partial_t\psi, \nabla\psi) B_i^\kappa(\psi),
\end{aligned} \tag{3.15}
$$

with the quantities $\varrho, \mathbf{v}, \mathbf{a}^\kappa$ determined by (3.14), (3.7), (3.8). Substituting $\ell = \varrho\tilde{\ell}$ and taking

$$
\begin{aligned}
\frac{\partial \ell}{\partial(\partial_t \psi^i)} &= \varrho \frac{\partial \tilde{\ell}}{\partial(\partial_t \psi^i)}, \\
\frac{\partial \ell}{\partial(\nabla \psi^i)} &= \varrho \frac{\partial \tilde{\ell}}{\partial(\nabla \psi^i)} + \tilde{\ell} \sum_{\kappa=1}^{3} \frac{\partial \varrho(\mathbf{a}^\lambda)}{\partial \mathbf{a}^\kappa} \frac{\partial \mathbf{a}^\kappa}{\partial(\nabla \psi^i)}
\end{aligned}
$$

into consideration, the system (3.15) simplifies after division by the factor ϱ to

$$
\begin{aligned}
&\frac{\partial \tilde{\ell}}{\partial(\partial_t \psi^i)} = -B_i^0(\psi), \\
&\frac{\partial \tilde{\ell}}{\partial(\nabla \psi^i)} = -\frac{\partial}{\partial(\nabla \psi^i)} \left[\frac{1}{2}\mathbf{v}^2\right] + \sum_{\kappa=1}^{3} \left[\frac{\boldsymbol{\sigma}_\kappa}{\varrho} - \frac{\tilde{\ell}}{\varrho} \frac{\partial \varrho(\mathbf{a}^\lambda)}{\partial \mathbf{a}^\kappa}\right] \frac{\partial \mathbf{a}^\kappa}{\partial(\nabla \psi^i)}.
\end{aligned} \tag{3.16}
$$

This system of partial differential equations for $\tilde{\ell}$ can be integrated if and only if the condition

$$
\frac{\boldsymbol{\sigma}_\kappa}{\varrho} - \frac{\tilde{\ell}}{\varrho} \frac{\partial \varrho(\mathbf{a}^\lambda)}{\partial \mathbf{a}^\kappa} = -\frac{\partial u(\mathbf{a}^\lambda, \psi)}{\partial \mathbf{a}^\kappa} \tag{3.17}
$$

is fulfilled with an adequate scalar function $u = u(\mathbf{a}^\lambda, \psi)$. Then, we result in the *general analytical form* of the Lagrangian:

$$
\ell = \varrho\tilde{\ell} = -\varrho(\mathbf{a}^\kappa) \left[B_i^0(\psi)\partial_t\psi^i + \frac{1}{2}\mathbf{v}^2 + u(\mathbf{a}^\kappa, \psi)\right]. \tag{3.18}
$$

Making use of Noether's theorem with respect to time translation, the term $\varrho u(\mathbf{a}^\kappa, \psi)$ is identified as the *internal energy density*. It should be mentioned that the rough scheme (3.18) of the Lagrangian can be applied to many different physical systems in continuum mechanics!

3.2. The concrete Lagrangian for elasticity with eigenstresses

Having established the general scheme (3.18) for the Lagrangian, we have to go into more details: How many independent field variables ψ^i are necessary to

determine the state of the system and which analytical form of the yet unknown structures $B^0, B^\kappa, u(\mathbf{a}^\kappa, \psi)$ in the Lagrangian (3.18) should be used? In this paper I prefer a more heuristic treatment:

At first, we may choose the internal energy density u according to *Hooke's law* of an isotropic material

$$u = u(\mathbf{a}^\kappa) = K \ln \varrho + \frac{\mu}{2} \varrho^{-2/3} \sum_{\kappa=1}^{3} (\mathbf{a}^\kappa)^2 \tag{3.19}$$

with the *bulk modulus* K and the *shear modulus* μ. By means of Eq. (3.13), this implies the well-known form of the stress tensor:

$$\underline{\underline{\sigma}}(\mathbf{a}_\kappa) = K \varrho \underline{\underline{1}} + \mu \varrho^{1/3} \sum_{\kappa=1}^{3} \left[\mathbf{a}^\kappa \otimes \mathbf{a}^\kappa - \frac{1}{3} (\mathbf{a}^\kappa)^2 \underline{\underline{1}} \right]. \tag{3.20}$$

This formula is Hooke's law rewritten in terms of Cosserat-directors.

Secondly, in analogy with (2.25) I make a *Clebsch-ansatz*

$$\mathbf{a}^\kappa = \nabla \varphi^\kappa + \zeta^\kappa \nabla \vartheta^\kappa \tag{3.21}$$

for each of the three Cosserat directors $\mathbf{a}^\kappa$ with 9 independent potentials φ^κ, ζ^κ, ϑ^κ; $\kappa = 1, 2, 3$. Comparing (3.8) with (3.21), the three basis elements B^1, B^2, B^3 are determined. Consequently, by means of (3.4), the *dislocation density tensor* reads

$$\underline{\underline{\alpha}} = \sum_{\kappa=1}^{3} (\nabla \zeta^\kappa \times \nabla \vartheta^\kappa) \otimes \mathbf{e}_\kappa , \tag{3.22}$$

whereas according to (3.5) the *dislocation flux density tensor* takes the form

$$\underline{\underline{J}} = -\mathbf{v} \times \underline{\underline{\alpha}} + \sum_{\kappa=1}^{3} [\mathrm{D}_t \vartheta^\kappa \nabla \zeta^\kappa - \mathrm{D}_t \zeta^\kappa \nabla \vartheta^\kappa] \otimes \mathbf{e}_\kappa . \tag{3.23}$$

Until this point the considerations took only account of the symmetry properties. Equation (3.23) will be used for fitting the ansatz for the Lagrangian to real dislocation dynamics.

The model is now restricted to purely convective dislocation dynamics. Then, the first term in (3.23) is sufficient and the second term has to vanish according to a particular ansatz for the Lagrangian. Supplementing the already introduced field variables φ^κ, ζ^κ, ϑ^κ by an additional set Φ, χ_κ, γ_κ, ξ_κ of field variables, I make the ansatz

$$\ell = -\varrho \left[\partial_t \Phi + \sum_{\kappa=1}^{3} (\chi_\kappa \partial_t \varphi^\kappa + \gamma_\kappa \partial_t \vartheta^\kappa + \xi_\kappa \partial_t \zeta^\kappa) + \frac{1}{2} \mathbf{v}^2 + u(\mathbf{a}^\kappa) \right] \tag{3.24}$$

for the Lagrangian of the elastically deformed crystal with fixed dislocations. In (3.24) the mass density has to be replaced by (3.14), the quantities $\mathbf{a}^i$ and $u(\mathbf{a}^\kappa)$ by (3.21), (3.19). Finally, the quantity $\mathbf{v}$ has to be understood as

$$\mathbf{v} = \nabla\Phi + \sum_{\kappa=1}^{3} \left[\chi_\kappa \nabla\varphi^\kappa + \gamma_\kappa \nabla\vartheta^\kappa + \xi_\kappa \nabla\zeta^\kappa\right]. \tag{3.25}$$

This expression is in accordance with Eqs. (3.7) and (3.15). Thus, it turns out that $\mathbf{v}$ is the velocity field. As compared with the original Clebsch-ansatz (2.25) in (3.25), there are involved 10 potentials Φ, χ_κ, γ_κ, ξ_κ. We should not worry about this unusual potential representation: (3.25) is a straightforvard result of the theory. The essence of it is its *regauging group* which is associated with the *vortex dynamics* by means of the symmetry-balance theorem developed in Sec 2.2..

Variation with respect to $\Phi, \chi_\kappa, \gamma_\kappa, \xi_\kappa, \varphi^\kappa, \vartheta^\kappa, \zeta^\kappa$ – the free and independent fields – results in the Euler–Lagrange equations

$$\delta\Phi : \partial_t\varrho + \nabla\cdot(\varrho\mathbf{v}) = 0, \tag{3.26}$$

$$\delta\varphi^\kappa : \partial_t(\varrho\chi_\kappa) + \nabla\cdot\left[\varrho\chi_\kappa\mathbf{v} - \boldsymbol{\sigma}_\kappa\right] = 0, \tag{3.27}$$

$$\delta\vartheta^\kappa : \partial_t(\varrho\gamma_\kappa) + \nabla\cdot\left[\varrho\gamma_\kappa\mathbf{v} - \zeta^\kappa\boldsymbol{\sigma}_\kappa\right] = 0 \tag{3.28}$$

$$\delta\zeta^\kappa : \partial_t(\varrho\xi_\kappa) + \nabla\cdot\left[\varrho\xi_\kappa\mathbf{v}\right] + \boldsymbol{\sigma}_\kappa\cdot\nabla\vartheta^\kappa = 0, \tag{3.29}$$

$$\delta\chi_\kappa : -\varrho\mathrm{D}_t\varphi^\kappa = 0, \tag{3.30}$$

$$\delta\gamma_\kappa : -\varrho\mathrm{D}_t\vartheta^\kappa = 0, \tag{3.31}$$

$$\delta\xi_\kappa : -\varrho\mathrm{D}_t\zeta^\kappa = 0. \tag{3.32}$$

$\boldsymbol{\sigma}_\kappa$ is determined by Eq. 3.17). Equation (3.26) is the *mass balance* which can also be obtained via Noether's theorem with respect to the gauge transformation $\Phi \longrightarrow \Phi+\epsilon$, whereas Eqs. (3.27)–(3.32) are the essential dynamical field equations for the potentials. Making use of (3.31), (3.32) the dislocation flux density tensor (3.23) simplifies to

$$\underline{\underline{J}} = -\mathbf{v}\times\underline{\underline{\alpha}}. \tag{3.33}$$

Thus, the dynamical equations (2.12), (2.13) for the dislocations can be obtained by means of the *theorem for area-type balances* stated in Sec. 2.2.:

$$\begin{aligned} \partial_t\underline{\underline{\alpha}} + \nabla\times\left[-\mathbf{v}\times\underline{\underline{\alpha}}\right] &= 0 \\ \nabla\cdot\underline{\underline{\alpha}} &= 0. \end{aligned} \tag{3.34}$$

Obviously I am dealing with a purely convective dislocation dynamics: in terms of a microscopic picture, the dislocations are fixed at their initial lattice position. Thus, the deformations of the crystal are purely elastic ones and the Lagrangian (3.24) describes a *dynamical generalization of Kröner's static theory of eigenstresses* [5] *due to dislocations.*

4. Conclusion

By means of this paper I have shown the method of determination of a Lagrangian along the line of the inverse problem of the second kind. Noether's theorem associated with volume-type balances has been used as well as a new theorem associated with area-type balances. A Lagrangian for the convective generalization of the eigenstress theory of (fixed) dislocations opens perspectives towards a theory of moving dislocations, i.e. of plastic deformations. Investigations are in progress.

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