

## A theory of the elastic-viscoplastic Cosserat continuum

*Dedicated to Prof. Henryk Zorski  
on the occasion of his 70-th birthday*

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BASED on the multiplicative decomposition of the stretch tensor and the additive decomposition of the second Cosserat deformation tensor into elastic and inelastic parts, a theory of the elastic-viscoplastic Cosserat continuum is formulated. It is stressed that the rotation field is to be treated as a kinematical variable which can not be decomposed into elastic and inelastic parts. A thorough discussion of the configuration space by relying on basic concepts of Lie groups is provided and the field equations are derived from a variational statement. The flow rules are specified by means of the postulate of maximum dissipation paralleling some developments of the classical theory.

### 1. Introduction

THE COSSERAT continuum belongs to the class of the so-called generalized continua where, in addition to the displacement field, further fields are considered which specify the micro-structure of the continuum under consideration. In the case of the Cosserat continuum, it is a rotation field which is considered to be independent of displacements. Accordingly, to every point of the continuum, a displacement vector and a rotation tensor (an element of the special orthogonal group) are attached. Since the work of ERICKSEN and TRUESDELL [1] and ERINGEN and KAFADAR [2], the Cosserat continuum attracted the interest of many researchers. Specifically within the shell theory, the philosophy of the Cosserat continuum proved to be very helpful (GREEN *et al.* [3], COHEN and DE SILVA [4], NAGHDI [5], ZHILIN [6], SANSOUR and BEDNARCZYK [7]). In three dimensions, the interest in the Cosserat continuum is increasing since the observation that already the geometric linear Cosserat continuum can prevent ill-conditioning of the field equations within classical elasto-plasticity. Specifically the loss of ellipticity of the governing equations and the observation of mesh-dependence of the finite element solutions can be circumvented if the formulation is a Cosserat-based one (see MÜHLHAUS [8], DE BORST [9], STEINMANN [10]). Hence, the formulation of the elasto-plastic Cosserat continuum can be understood as a regularization method. The so-called internal length needed for such a regularization corresponds to the micro-structure of the continuum which is provided by the Cosserat continuum in a completely natural way.

Early geometrically linear formulations of the elasto-plastic Cosserat continuum are due to LIPPMANN [11] and BESDO [12]. Recently, using different assumptions, geometrically exact formulations have been given by SIEVERT [13] and STEINMANN [14]. The reader is also referred to a recent review article by LIPPMANN [15] where experimental observations in conjunction with the plastic spin are discussed and further references can be found.

The aim of the paper is to give a formulation of the geometrically exact elastic-viscoplastic Cosserat Continuum. A fundamental aspect of the formulation is the crucial understanding that the rotations constitute a kinematical field which, together with the displacements, defines the configuration space. A basic feature of kinematical variables is the existence of field equations corresponding to them (the Euler-Lagrange equations of an appropriate functional) and, correspondingly, the fact that they can not be decomposed into elastic and inelastic parts. Exactly this statement stands in contradistinction to some attempts to develop the theory by means of a decomposition of the rotation field itself (see e.g. STEINMANN [14]).

Another aspect relates to the choice of the strain measures which have to be decomposed in an appropriate way in elastic and inelastic parts. The theory is based on the decomposition of the first and second Cosserat deformation tensors. The decomposition is multiplicative for the first Cosserat deformation tensor (the stretch tensor), and is additive for the second one.

The paper is organized as follows. In Sec. 2, fundamentals of the Cosserat continuum are presented. We focus on the structure of the configuration space and basics of Lie groups are incorporated in the discussion. Specifically the relations between variations and time derivatives within the orthogonal group is discussed. In Sec. 3 the elastic-viscoplastic Cosserat continuum is presented. First the assumed decompositions are introduced and possible derivation of the field equations from a variational statement is discussed. Finally, the flow rules are specified by making use of the postulate of maximum dissipation. Although this postulate does not constitute a physical law, it is helpful due to the lack of sufficient experimental data needed for the formulation of alternative evolution equations. The paper closes with conclusions. In the Appendix, the linearization of the field equations is treated. Hereby the structure of the configuration space is further discussed focusing on the rule of a Killing metric defined on it.

## 2. Kinematics

### 2.1. Preliminaries

Let  $\mathbb{R}$  denote the real numbers. With a set  $\mathcal{B} \subset \mathbb{R}^3$  we define a material body as a three-dimensional manifold. The map  $\varphi(t) : \mathcal{B} \rightarrow \mathbb{R}^3$  is an embedding depending on a time-like parameter  $t \in \mathbb{R}$ . Hereby,  $\varphi_0 = \varphi(t = t_0)$  defines a reference configuration which we use to identify the material points. Accordingly,

we choose  $\varphi_0$  to be the identity map. Writing  $\mathcal{B}$  for  $\varphi_0\mathcal{B}$  and  $\mathcal{B}_t$  for  $\varphi(t)\mathcal{B}$  we get  $\varphi(t) : \mathcal{B} \rightarrow \mathcal{B}_t$ . For  $\mathbf{X} \in \mathcal{B}$  and  $\mathbf{x} \in \mathcal{B}_t$  we have  $\mathbf{x}(t) = \varphi(\mathbf{X}, t)$  and  $\mathbf{X}(t) = \varphi^{-1}(\mathbf{x}, t)$ .

We consider  $\vartheta^i, i = 1, 2, 3$ , as coordinate charts in  $\mathcal{B}$  which we choose to be attached to the body (convected). The tangent spaces of  $\mathcal{B}$  and  $\mathcal{B}_t$  are denoted by  $\mathcal{TB}$  and  $\mathcal{TB}_t$ , respectively. Accordingly, the covariant base vectors are

$$(2.1) \quad \mathbf{G}_i = \frac{\partial \mathbf{X}}{\partial \vartheta^i} \quad \text{with} \quad \mathbf{G}_i \in \mathcal{TB},$$

and

$$(2.2) \quad \mathbf{g}_i = \frac{\partial \mathbf{x}}{\partial \vartheta^i} \quad \text{with} \quad \mathbf{g}_i \in \mathcal{TB}_t.$$

The Riemannian metric in either configuration is denoted by  $\mathbf{G}, \mathbf{g}$  respectively, their components are given by  $G_{ij} = \mathbf{G}_i \cdot \mathbf{G}_j$  and  $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ , where scalar products of vectors are denoted by a dot. The corresponding determinants of the metrics are denoted by  $G$  and  $g$ , their inverse as usual by  $G^{ij}$  and  $g^{ij}$ , respectively. Further we denote the basic skew-symmetric three-dimensional Levi-Civita tensor (permutation tensor) by  $e_{ijk}$  where we have  $e_{ijk} = e^{ijk}$  by its Euclidean structure. Further we define  $\varepsilon_{ijk} := \sqrt{G} e_{ijk}, \varepsilon^{ijk} := 1/\sqrt{G} e^{ijk}$ , and later on make use of the absolute notation  $\boldsymbol{\epsilon} \equiv \varepsilon^{ijk} \mathbf{G}_i \otimes \mathbf{G}_j \otimes \mathbf{G}_k$ .

In addition to the base system  $\mathbf{G}_i$  we consider a Cartesian frame denoted by  $\mathbf{e}_i, i = 1, 2, 3$  and define, for later use, the matrices

$$(2.3) \quad c_{ij} = \mathbf{G}_i \cdot \mathbf{e}_j$$

which relate the two base systems to one another since we have  $\mathbf{G}_i = c_{ij} \mathbf{e}_j$  and  $\mathbf{e}_i = c_{ij} \mathbf{G}^j$ .

The Cosserat continuum is characterized by a rotation field understood as independent of  $\varphi$ . To every point in  $\mathcal{B}$  we attach a tensor  $\mathbf{R} \in SO(3)$ , where  $SO(3)$  denotes the special orthogonal group, parameterised with the help of the exponential map as follows (CHOQUET *et al.* [16], DUBROVIN *et al.* [17]):

$$(2.4) \quad \mathbf{R} = \exp(\mathbf{A}) = \mathbf{1} + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \frac{\mathbf{A}^3}{3!} + \dots = \mathbf{1} + \frac{\sin |\boldsymbol{\alpha}|}{|\boldsymbol{\alpha}|} \mathbf{A} + \frac{1 - \cos |\boldsymbol{\alpha}|}{|\boldsymbol{\alpha}|^2} \mathbf{A}^2,$$

with the skew-symmetric tensor  $\mathbf{A} = -\mathbf{A}^T$ , the axial vector of which is denoted by  $\boldsymbol{\alpha}$ . Accordingly we have  $\mathbf{R} \boldsymbol{\alpha} = \boldsymbol{\alpha}$ . Using the permutation tensor one may write directly

$$(2.5) \quad \boldsymbol{\alpha} = -\frac{1}{2} \boldsymbol{\epsilon} : \mathbf{A}.$$

Here a double contraction is denoted by  $(:)$  (for two second order tensors  $\mathbf{A}, \mathbf{B}$  the relation holds  $\mathbf{A} : \mathbf{B} = \text{tr}(\mathbf{A} \mathbf{B}^T)$  with  $\text{tr}$  denoting the trace operation). The fact that  $\boldsymbol{\epsilon}$  is a three-dimensional tensor reveals the product  $\boldsymbol{\epsilon} : \mathbf{A}$  to be a vector.

## 2.2. Strain measures

The deformation gradient is the tangent of the map  $\varphi : \mathcal{T}\varphi \equiv \mathbf{F}$  with  $\mathcal{T}\varphi : \mathcal{TB} \rightarrow \mathcal{TB}_t$ , or  $\mathbf{F} : \mathbf{G}_i \rightarrow \mathbf{g}_i$ . It is given as the tensor product

$$(2.6) \quad \mathbf{F} = \mathbf{g}_i \otimes \mathbf{G}^i.$$

By introducing the displacement field  $\mathbf{u} = \mathbf{x} - \mathbf{X}$  and denoting partial derivatives by a comma, we get from (2.1) and (2.2)

$$(2.7) \quad \mathbf{g}_i = \mathbf{G}_i + \mathbf{u}_{,i}$$

and from (2.6)

$$(2.8) \quad \mathbf{F} = (\mathbf{G}_i + \mathbf{u}_{,i}) \otimes \mathbf{G}^i.$$

By the relation

$$(2.9) \quad \mathbf{R} \mathbf{R}^T = \mathbf{1}$$

we have

$$(2.10) \quad \mathbf{R}^T \mathbf{R}_{,i} + \mathbf{R}_{,i}^T \mathbf{R} = \mathbf{0}.$$

The relation shows that the products  $\mathbf{R}^T \mathbf{R}_{,i}$  are skew-symmetric. We denote the corresponding axial vectors by  $\mathbf{k}_i$ . Between  $\boldsymbol{\alpha}$  and  $\mathbf{k}_i$  the relation holds (see e.g. PIETRASZKIEWICZ and BADUR [18])

$$(2.11) \quad \mathbf{k}_i = \frac{\sin |\boldsymbol{\alpha}|}{|\boldsymbol{\alpha}|} \boldsymbol{\alpha}_{,i} + \frac{1 - \cos |\boldsymbol{\alpha}|}{|\boldsymbol{\alpha}|^2} \boldsymbol{\alpha}_{,i} \times \boldsymbol{\alpha} + \left( \frac{1}{|\boldsymbol{\alpha}|} - \frac{\sin |\boldsymbol{\alpha}|}{|\boldsymbol{\alpha}|^2} \right) \frac{(\boldsymbol{\alpha} \cdot \boldsymbol{\alpha}_{,i})}{|\boldsymbol{\alpha}|} \boldsymbol{\alpha}.$$

The strain measures we are considering are the first Cosserat deformation tensor (ERINGEN and KAFADAR [2], HJALMARS [19])

$$(2.12) \quad \mathbf{U} := \mathbf{R}^T \mathbf{F}$$

and the second Cosserat deformation tensor

$$(2.13) \quad \mathbf{K} := \mathbf{k}_i \otimes \mathbf{G}^i.$$

Alternatively,  $\mathbf{K}$  may be written down in terms of the rotation tensor directly as

$$(2.14) \quad \mathbf{K} = -\frac{1}{2} \boldsymbol{\epsilon} : \mathbf{R}^T \mathbf{R}_{,i} \otimes \mathbf{G}^i.$$

For the sake of completeness we include explicit expressions for the strain measures where it is convenient to underline the following decompositions

$$(2.15) \quad \mathbf{U} = U_{ji} \mathbf{G}^i \otimes \mathbf{G}^j, \quad \mathbf{K} = K_{ji} \mathbf{G}^i \otimes \mathbf{G}^j,$$

$$(2.16) \quad \mathbf{u} = u_k \mathbf{e}_k, \quad \boldsymbol{\alpha} = \alpha_k \mathbf{e}_k.$$

With (2.4), (2.8), (2.11)–(2.16) we get the following expressions for the strain measures

$$(2.17) \quad U_{rs} = G_{rs} + c_{sk} u_{k,r} + (c_{rk} + u_{k,r})c_{sj} \times \left[ \frac{\sin |\boldsymbol{\alpha}|}{|\boldsymbol{\alpha}|} e_{ijk} \alpha_i + \frac{1 - \cos |\boldsymbol{\alpha}|}{|\boldsymbol{\alpha}|^2} (\alpha_k \alpha_j - \alpha_i \alpha_i \delta_{jk}) \right],$$

$$(2.18) \quad K_{rs} = c_{sk} \left( \frac{\sin |\boldsymbol{\alpha}|}{|\boldsymbol{\alpha}|} \alpha_{k,r} + \frac{1 - \cos |\boldsymbol{\alpha}|}{|\boldsymbol{\alpha}|^2} e_{ijk} \alpha_{i,r} \alpha_j + \frac{|\boldsymbol{\alpha}| - \sin |\boldsymbol{\alpha}|}{|\boldsymbol{\alpha}|^2} |\boldsymbol{\alpha}|_{,r} \alpha_k \right).$$

2.3. Rates and variations

The deformation of the Cosserat continuum is completely described in terms of the pair  $(\mathbf{u}, \mathbf{R})$  attached to every point of the continuum. This motivates the definition of the configuration space as the set  $\mathcal{C}$  consisting of all admissible configurations of the body  $\mathcal{B}$ . A precise definition of it is given by

$$(2.19) \quad \mathcal{C}(\mathcal{B}) = \{U = (\mathbf{u}, \mathbf{R}) \mid U : \mathcal{B} \longrightarrow \mathbb{R}^3 \times SO(3)\}$$

with  $\mathbf{R} = \exp(\mathbf{A})$  and  $\mathbf{A} = -\mathbf{A}^T$ .

The deformation gradient  $\mathbf{F}$  and the rotation tensor  $\mathbf{R}$  define fields over  $\mathbf{X}$ . Pointwise, they take values parameterized by the real time  $t$  or by a time-like parameter where the set of all admissible values of  $\mathbf{F}$  and  $\mathbf{R}$ , related to one and the same particle, constitute a Lie group. In fact, it proves to be very fruitful to understand  $\mathbf{F}$  as well as  $\mathbf{R}$  as elements of a Lie group. This becomes crucial in conjunction with the linearization process where variations or time derivatives should be understood as vectors in the tangent space of an appropriate Lie group. First let  $\mathcal{G}$  be a linear Lie group. Of special interest for us is the group of invertable matrices with positive determinants  $GL^+(3)$  since we have  $\mathbf{F} \in GL^+(3)$  and the special orthogonal group  $SO(3)$  where we have  $\mathbf{R} \in SO(3)$ . We consider a curve in  $\mathcal{G}$ . By the very definition of a group, the identity  $\mathbf{1}$  is an element of  $\mathcal{G}$ . Any element  $\mathbf{Z} \in \mathcal{G}$  in the neighbourhood of the identity can be reached by the exponential map according to

$$(2.20) \quad \mathbf{Z} = \exp(\mathbf{b}) = \mathbf{1} + \mathbf{b} + \frac{\mathbf{b}^2}{2!} + \frac{\mathbf{b}^3}{3!} + \dots,$$

where  $\mathbf{b}$  is an element of the Lie algebra which defines the tangent space of  $\mathcal{G}$  at the identity. In the case of  $GL^+(3)$ , the Lie algebra, which is denoted by  $gl^+(3)$ , consists of quadratic matrices. In the case of  $SO(3)$ , the Lie algebra is denoted by  $so(3)$  and consists of skew-symmetric matrices.

Consider now any  $\mathbf{Z} \in \mathcal{G}$ . A curve in  $\mathcal{G}$  parametrized by  $t$  and going through  $\mathbf{Z}$  at  $t = t_0$  is given by the one-parameter subgroup

$$(2.21) \quad \mathcal{V}(t) = [\exp((t - t_0)\widehat{\mathbf{b}})\mathbf{Z}].$$

$\widehat{\mathbf{b}}$  is then an element of the corresponding Lie algebra of the group. Tangent vector fields in  $\mathcal{G}$  are given by means of derivation with respect to the time-like parameter  $t$ . Explicitly we have

$$(2.22) \quad D\mathbf{Z} = \frac{D}{Dt}\mathcal{V}|_{t=t_0} = \frac{D}{Dt}[\exp((t-t_0)\widehat{\mathbf{b}})\mathbf{Z}]|_{t=t_0} = \widehat{\mathbf{b}}\mathbf{Z}.$$

$\widehat{\mathbf{b}}\mathbf{Z}$  is understood as a tangent vector in  $\mathcal{G}$ ; a right invariant tangent vector, strictly spoken. The above tangents have been derived by considering the so-called left action of the group. Alternatively, a curve in  $\mathcal{G}$  and a corresponding derivative with respect to  $t$  can be defined by means of a right group action according to

$$(2.23) \quad D\mathbf{Z} = \frac{D}{Dt}\mathcal{V}|_{t=t_0} = \frac{D}{Dt}[\mathbf{Z}\exp((t-t_0)\mathbf{b})]|_{t=t_0} = \mathbf{Z}\mathbf{b}.$$

Here,  $\mathbf{Z}\mathbf{b}$  is a left invariant tangent vector. The relation holds

$$(2.24) \quad \mathbf{b} = \mathbf{Z}^{-1}\widehat{\mathbf{b}}\mathbf{Z}.$$

We apply now these concepts directly to  $\mathbf{F}$  and  $\mathbf{R}$  by understanding them as elements of the Lie groups  $GL^+(3)$  and  $SO(3)$ , respectively. One has

$$(2.25) \quad \dot{\mathbf{F}} = \mathbf{1}\mathbf{F}$$

with  $\mathbf{1}$  as the left rate, and

$$(2.26) \quad \dot{\mathbf{F}} = \mathbf{F}\mathbf{L}$$

with  $\mathbf{L}$  as the right rate. In terms of continuum mechanics,  $\mathbf{1}$  is the rate defined at the actual configuration whereas  $\mathbf{L}$  is its material counterpart. From (2.25) and (2.26) we get directly

$$(2.27) \quad \mathbf{L} = \mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}.$$

In the same spirit we have

$$(2.28) \quad \dot{\mathbf{R}} = \widehat{\mathbf{\Omega}}\mathbf{R}$$

with  $\widehat{\mathbf{\Omega}} \in so(3)$  (that is  $\widehat{\mathbf{\Omega}}$  is skew-symmetric) as the left rate, and

$$(2.29) \quad \dot{\mathbf{R}} = \mathbf{R}\mathbf{\Omega}$$

with  $\mathbf{\Omega}$  as the corresponding right rate. Here again we have

$$(2.30) \quad \mathbf{\Omega} = \mathbf{R}^T\dot{\mathbf{R}}\mathbf{R}.$$

Further, it is useful to consider the material rate related to  $\mathbf{1}$  by means of a rotation. From (2.12), (2.25) and (2.29) we conclude

$$(2.31) \quad \mathbf{1} = \mathbf{R}\widehat{\mathbf{L}}\mathbf{R}^T, \quad \widehat{\mathbf{L}} = \dot{\mathbf{U}}\mathbf{U}^{-1} + \mathbf{\Omega}.$$

Instead of the time derivatives we can consider variations in the same way. These are explicitly needed for the rotation space. We consider for this purpose again the configuration space consisting of the pairs  $(\mathbf{u}, \mathbf{R})$  considered as a Cartesian product. The space is understood as a Lie group where for two elements  $(\mathbf{u}, \mathbf{R})$  and  $(\mathbf{w}, \mathbf{Q})$  the group operation is defined by the direct product, namely  $(\mathbf{u}, \mathbf{R}) \circ (\mathbf{w}, \mathbf{Q}) = (\mathbf{u} + \mathbf{w}, \mathbf{QR})$ . To derive variational formulas we rely on the above concepts and consider the neighbourhood of an element say  $\mathcal{U} = (\mathbf{u}, \mathbf{R})$ . Again a curve  $\mathcal{V}(s)$  in  $\mathcal{C}$  passing through  $\mathcal{U} = (\mathbf{u}, \mathbf{R})$  with  $\mathcal{V}(s = s_0) = \mathcal{U}$  is given as the one-parameter subgroup

$$(2.32) \quad \mathcal{V}(s) = [\mathbf{u} + (s - s_0)\hat{\mathbf{u}}, \exp((s - s_0)\widehat{\mathbf{W}})\mathbf{R}].$$

The variation now is defined as the tangent at  $\mathcal{U}$  which is given by the derivation with respect to the parameter  $s$

$$(2.33) \quad D\mathcal{U} = \frac{D}{Ds}\mathcal{V}|_{s=s_0} = \frac{D}{Dt}[\mathbf{u} + (s - s_0)\hat{\mathbf{u}}, \exp((s - s_0)\widehat{\mathbf{W}})\mathbf{R}]|_{s=s_0} \\ = (\hat{\mathbf{u}}, \widehat{\mathbf{W}}\mathbf{R}) \quad \text{with } \widehat{\mathbf{W}} \in so(3).$$

The pair  $(\hat{\mathbf{u}}, \widehat{\mathbf{W}}) = \delta\mathcal{U}$  defines the infinitesimal deformation to be superimposed on a given admissible state. That is, with (2.33) a neighbourhood of  $\mathcal{U}$  is given as

$$(2.34) \quad (\mathbf{u}, \mathbf{R}) \circ (\hat{\mathbf{u}}, \widehat{\mathbf{W}}) = (\mathbf{u} + \hat{\mathbf{u}}, \widehat{\mathbf{W}}\mathbf{R}).$$

To make it more convenient we make use of the notation

$$(2.35) \quad \delta\mathbf{R} = \widehat{\mathbf{W}}\mathbf{R}.$$

The crucial point now is the fact that  $\delta\mathbf{R}$  is to be understood as a tangent vector in the space  $SO(3)$ . This fact will play a dominant role when taking the linearization of these vectors, that is considering the second variations, issues to be addressed later on. Already by considering the time derivatives we have seen that one can define two derivatives, a left one and a right one. Here again and completely in the same way, one can define a variation on the basis of the right group action as

$$(2.36) \quad \delta\mathbf{R} = \mathbf{R}\mathbf{W}.$$

Comparison of (2.35) with (2.36) gives

$$(2.37) \quad \mathbf{W} = \mathbf{R}^T\widehat{\mathbf{W}}\mathbf{R}.$$

Here again,  $\widehat{\mathbf{W}}$  is the variation defined at the actual configuration whereas  $\mathbf{W}$  is that defined at the reference one.

Having established the time derivatives as well as the variations of  $\mathbf{R}$ , a relation can be constructed between the variations of the time derivatives and the

time derivatives of the variations. We exercise these aspects by considering the material rates only. By considering the equality

$$(2.38) \quad (\delta \mathbf{R})^\cdot = \delta \dot{\mathbf{R}}$$

and by making use of (2.29) and (2.36), we get

$$(2.39) \quad \dot{\mathbf{R}} \mathbf{W} + \mathbf{R} \dot{\mathbf{W}} = \mathbf{R} \mathbf{W} \Omega + \mathbf{R} \delta \Omega$$

or

$$(2.40) \quad \delta \Omega = \Omega \mathbf{W} - \mathbf{W} \Omega + \dot{\mathbf{W}}.$$

In terms of the axial vectors  $\boldsymbol{\omega}$  and  $\mathbf{w}$  of  $\Omega$  and  $\mathbf{W}$ , respectively, we have the relation

$$(2.41) \quad \delta \boldsymbol{\omega} = \boldsymbol{\omega} \times \mathbf{w} + \dot{\mathbf{w}}.$$

For a complete discussion we need relations for the time derivatives and the first variations of the strain measures. A useful equation relating the variation of  $\mathbf{k}_i$  to  $\mathbf{w}$ , the axial vector of  $\mathbf{W}$  and to its derivative, is obtained by making use of (2.13), (2.14) and (2.36). After some algebraic manipulations, which we omit for the sake of shortness, we get

$$(2.42) \quad \delta \mathbf{k}_i = \mathbf{w}_{,i} + \mathbf{k}_i \times \mathbf{w}.$$

A similar relation holds for the time derivatives

$$(2.43) \quad \dot{\mathbf{k}}_i = \boldsymbol{\omega}_{,i} + \mathbf{k}_i \times \boldsymbol{\omega}.$$

### 3. The elastic-viscoplastic Cosserat continuum

#### 3.1. Stress tensors, equilibrium equations and external power

Let  $\boldsymbol{\sigma}$  be the Cauchy stress tensor. We define the moment tensor (the couple stress tensor)  $\boldsymbol{\gamma}$  which is expected to fulfill the Cauchy lemma with respect to external moments. For a field of external moments  $\mathbf{m}_s$  acting on  $\partial \mathcal{B}_{t_s}$ , the boundary of  $\mathcal{B}_t$  with prescribed tractions, and with  $\boldsymbol{\nu}$  being the actual normal vector at that boundary, we have

$$(3.1) \quad \mathbf{m}_s = \boldsymbol{\gamma} \boldsymbol{\nu}.$$

The equilibrium equations read (ERINGEN and KAFADAR [2])

$$(3.2) \quad \frac{D}{Dt} \int_{\mathcal{B}_t} \rho \dot{\mathbf{u}} dv = \int_{\mathcal{B}_t} \mathbf{f} dv + \int_{\partial \mathcal{B}_{t_s}} \mathbf{f}_s da,$$

$$(3.3) \quad \frac{D}{Dt} \int_{\mathcal{B}_t} \rho (\mathbf{x} \times \dot{\mathbf{u}} + \hat{\Theta} \hat{\boldsymbol{\omega}}) dv = \int_{\mathcal{B}_t} (\mathbf{x} \times \mathbf{f} + \mathbf{m}) dv + \int_{\partial \mathcal{B}_{t_s}} (\mathbf{x} \times \mathbf{f}_s + \mathbf{m}_s) da.$$



Here,  $\mathbf{f}$ ,  $\mathbf{f}_s$  mean external forces acting in the field and at the boundary,  $\mathbf{m}$ ,  $\mathbf{m}_s$  are the corresponding external moments,  $\hat{\Theta}$  is the rotational inertia,  $dv$  and  $da$  are the volume and area elements. By straightforward calculations the above equations can be localized leading to the following field equations:

$$(3.4) \quad \varrho \ddot{\mathbf{u}} = \operatorname{div} \boldsymbol{\sigma} + \mathbf{f},$$

$$(3.5) \quad \varrho \frac{D}{Dt} (\hat{\Theta} \hat{\boldsymbol{\omega}}) = -\boldsymbol{\epsilon} : \boldsymbol{\sigma} + \operatorname{div} \boldsymbol{\gamma} + \mathbf{m},$$

where  $\operatorname{div}$  means the divergence operation at the actual configuration. To recast the above equations in a material setting, we define by the following isometric material stress and moment tensors

$$(3.6) \quad \boldsymbol{\Sigma} = \frac{\varrho_{\text{ref}}}{\varrho} \mathbf{R}^T \boldsymbol{\sigma} \mathbf{R}, \quad \boldsymbol{\Gamma} = \frac{\varrho_{\text{ref}}}{\varrho} \mathbf{R}^T \boldsymbol{\gamma} \mathbf{R}, \quad \mathbf{M} = \mathbf{R}^T \mathbf{m}.$$

The equilibrium equations have then the alternative material form

$$(3.7) \quad \varrho_{\text{ref}} \ddot{\mathbf{u}} = \operatorname{Div} \mathbf{R} \boldsymbol{\Sigma} \mathbf{U}^{-T} + \mathbf{f},$$

$$(3.8) \quad \varrho_{\text{ref}} (\boldsymbol{\Theta} \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{\Theta} \boldsymbol{\omega}) = -\boldsymbol{\epsilon} : \boldsymbol{\Sigma} + \mathbf{R}^T \operatorname{Div} \mathbf{R} \boldsymbol{\Gamma} \mathbf{U}^{-1} + \mathbf{M},$$

where  $\operatorname{Div}$  means the divergence operator at the reference configuration and we have  $\boldsymbol{\Theta} = \mathbf{R}^T \hat{\boldsymbol{\Theta}} \mathbf{R}$ . In deriving the left-hand side of (3.8), use is made of the relation  $\hat{\boldsymbol{\Theta}} \hat{\boldsymbol{\omega}} = \mathbf{R} \boldsymbol{\Theta} \boldsymbol{\omega}$ . Note further that  $\boldsymbol{\Theta}$  is assumed to be constant.

The boundary conditions hold

$$(3.9) \quad \mathbf{R} \boldsymbol{\Sigma} \mathbf{U}^{T-1} \mathbf{G}^i \mu_i = \mathbf{f}_s, \quad \boldsymbol{\Gamma} \mathbf{U}^{-1} \mathbf{G}^i \mu_i = \mathbf{M}_s \quad \text{on } \partial \mathcal{B}_s,$$

where  $\mu_i$  are the components of the normal vector at the reference configuration. The validity of (3.7) and (3.8) is proved in Sec. 3.2 by deriving them from a variational statement.

Dealing with dissipation later on, we need the following expression for the mechanical power  $\mathcal{P}$  which is derived straightforwardly under the assumption that the equilibrium equations hold

$$(3.10) \quad \mathcal{P} = \int_{\mathcal{B}_t} \left[ \boldsymbol{\sigma} : \mathbf{l} + \boldsymbol{\gamma} : (\hat{\boldsymbol{\omega}}_{,i} \otimes \mathbf{g}^i) \right] dv.$$

Making use of the material rates as given in (2.27) or (2.31) as well as of (2.6), (2.12), (2.43), and (3.6), the above relation can be rewritten in the form

$$(3.11) \quad \mathcal{P} = \int_{\mathcal{B}} (\boldsymbol{\Sigma} : \hat{\mathbf{L}} + \boldsymbol{\Gamma} \mathbf{U}^{-1} : \dot{\mathbf{k}}_i \otimes \mathbf{G}^i) dV,$$

or alternatively in the form

$$(3.12) \quad \mathcal{P} = \int_B (\Xi : \mathbf{L} + \Gamma \mathbf{U}^{-1} : \dot{\mathbf{k}}_i \otimes \mathbf{G}^i) dV.$$

Here we have

$$(3.13) \quad \Xi = \frac{\rho_{\text{ref}}}{\rho} \mathbf{F}^{-1} \boldsymbol{\sigma} \mathbf{F}$$

which is nothing but the mixed variant pull-back of the Kirchhoff stress tensor.

### 3.2. Multiplicative-additive split of the strain measures

A first step in formulating elasto-viscoplasticity will be an adequate split of the strain measures in elastic and inelastic parts. Starting with  $\mathbf{U}$  we note that this strain measure is not symmetric. From its physical meaning it may itself be understood as an element of the group  $GL^+(3)$  for which multiplicative products are a natural operation defining the group action. Accordingly two possible splits can be considered. First

$$(3.14) \quad \mathbf{U} = \mathbf{U}^p \mathbf{U}^e,$$

where  $\mathbf{U}^p$  stands for the inelastic part of the stretch tensor and, correspondingly,  $\mathbf{U}^e$  stands for the elastic part. Alternatively, the decomposition

$$(3.15) \quad \mathbf{U} = \bar{\mathbf{U}}^e \bar{\mathbf{U}}^p$$

may be considered as well. By the use of a "bar", the different decompositions are distinguished.

As a next step we consider the decomposition of the second Cosserat deformation tensor  $\mathbf{K}$ . Two observations are helpful. First, the deformation gradient  $\mathbf{F}$  or the stretch tensor  $\mathbf{U}$  can be understood as elements of a matrix group acting on the tangent space at the identity. This is reflected also in their physical meaning by stretching (for  $\mathbf{F}$  also rotating) the tangent space. For such an action, a multiplicative decomposition is a natural choice. Such a mathematical or physical meaning is not assigned to the tensor  $\mathbf{K}$ . Second, from its very definition, (Eq. (2.13)), the tensor  $\mathbf{K}$  is equivalent to the three vectors  $\mathbf{k}_i$  for which an additive decomposition is, due to lack of any motivation for a multiplicative decomposition, an appropriate operation. Accordingly we consider the following decomposition

$$(3.16) \quad \mathbf{K} = \mathbf{K}^e + \mathbf{K}^p.$$

Moreover, the couple stresses are assumed to be small in comparison with the macroscopic stress tensor  $\boldsymbol{\Sigma}$  which gives a further justification for the additive decomposition of  $\mathbf{K}$ .

We consider now the rates of the above decompositions, specifically the rates of  $\mathbf{U}$  and  $\mathbf{U}^p$ . From (2.31) we conclude that the material rate corresponding to  $\dot{\mathbf{U}}\mathbf{U}^{-1}$  is of special interest. By the first decomposition one has

$$(3.17) \quad \dot{\mathbf{U}} = \dot{\mathbf{U}}^p \mathbf{U}^e + \mathbf{U}^p \dot{\mathbf{U}}^e$$

and

$$(3.18) \quad \dot{\mathbf{U}}\mathbf{U}^{-1} = \dot{\mathbf{U}}^p \mathbf{U}^{p-1} + \mathbf{U}^p \dot{\mathbf{U}}^e \mathbf{U}^{e-1} \mathbf{U}^{p-1}.$$

From this equation we infer that it is useful to use a left rate for  $\mathbf{U}^p$  and a right one for  $\mathbf{U}^e$  according to

$$(3.19) \quad \dot{\mathbf{U}}^p = \mathbf{L}^p \mathbf{U}^p, \quad \dot{\mathbf{U}}^e = \mathbf{U}^e \mathbf{L}^e,$$

which inserted in (3.18) give

$$(3.20) \quad \dot{\mathbf{U}}\mathbf{U}^{-1} = \mathbf{L}^p + \mathbf{U} \mathbf{L}^e \mathbf{U}^{-1}.$$

In the case of the second decomposition (3.15) one has

$$(3.21) \quad \dot{\mathbf{U}}\mathbf{U}^{-1} = \bar{\mathbf{U}}^e \dot{\bar{\mathbf{U}}}^p \bar{\mathbf{U}}^{p-1} \bar{\mathbf{U}}^{e-1} + \dot{\bar{\mathbf{U}}}^e \bar{\mathbf{U}}^{e-1}.$$

From this relation it follows that it is more appropriate to choose a right rate for  $\bar{\mathbf{U}}^p$  and a left rate for  $\bar{\mathbf{U}}^e$  according to

$$(3.22) \quad \dot{\bar{\mathbf{U}}}^p = \bar{\mathbf{U}}^p \bar{\mathbf{L}}^p, \quad \dot{\bar{\mathbf{U}}}^e = \bar{\mathbf{L}}^e \bar{\mathbf{U}}^e.$$

Correspondingly we have

$$(3.23) \quad \dot{\mathbf{U}}\mathbf{U}^{-1} = \bar{\mathbf{U}} \bar{\mathbf{L}}^p \bar{\mathbf{U}}^{-1} + \bar{\mathbf{L}}^e$$

as the counterpart of (3.20).

The rate of the second Cosserat deformation tensor is directly given by

$$(3.24) \quad \dot{\mathbf{K}} = \dot{\mathbf{K}}^e + \dot{\mathbf{K}}^p.$$

With the rates of deformation at hand we can proceed to discuss the frame of the theory.

### 3.3. The weak form of the equilibrium equations

The theory is completely determined by two functions: the internal energy function and the flow rule. The internal energy function is assumed to depend on the elastic strain tensors as well as on further internal variables which we

collect in the vector  $\mathbf{q}$ . Hence we define  $\psi(\mathbf{U}^e, \mathbf{K}^e, \mathbf{q})$  as the free energy under consideration.

Under these assumptions, the equilibrium equations (3.7) and (3.8) can be derived as Euler-Lagrange equations of an appropriate action. Note that even in a purely elastic response, a Hamiltonian can not be formulated. The last statement is due to the fact that, even in the case of constant external moments, an external potential does not exist since the variation of the rotation vector itself does not constitute the work conjugate of an external moment. These issues are discussed in detail in SANSOUR and BEDNARCZYK [7] to which the reader is referred. The statements stand in contradistinction to the formulation of a variational principle for the Cosserat continuum (see SACZUK [20]).

Pointwise the kinetic energy is defined by

$$(3.25) \quad T = \frac{1}{2} \rho_{\text{ref}} (\dot{\mathbf{u}} \cdot \dot{\mathbf{u}} + \Theta \boldsymbol{\omega} \cdot \boldsymbol{\omega}).$$

Note that the last term can be rewritten by means of spatial quantities as  $1/2 \hat{\Theta} \hat{\boldsymbol{\omega}} \cdot \hat{\boldsymbol{\omega}}$ .

Now we consider the following functional:

$$(3.26) \quad \delta \int_{t_0}^{t_1} \int_B T dV dt - \int_{t_0}^{t_1} \left[ \int_B (\mathbf{n} : \delta \mathbf{U} + \mathbf{m} : \delta \mathbf{K}) dV - \int_B \mathbf{f} \cdot \delta \mathbf{u} dV - \int_{\partial B} \mathbf{f}_s \cdot \delta \mathbf{u} dA - \int_B \mathbf{M} \cdot \boldsymbol{\omega} dV - \int_{\partial B} \mathbf{M}_s \cdot \boldsymbol{\omega} dA \right] dt = 0.$$

The assumption of the existence of a free energy function is equivalent to the assumption that the relations hold

$$\mathbf{n} = \rho_{\text{ref}} \frac{\partial \psi}{\partial \mathbf{U}}, \quad \mathbf{m} = \rho_{\text{ref}} \frac{\partial \psi}{\partial \mathbf{K}}.$$

Since  $\psi$  is a function of  $\mathbf{U}^e$ ,  $\mathbf{K}^e$  and via the latter a function of  $\mathbf{U}$ ,  $\mathbf{K}$ , we conclude

$$(3.27) \quad \int_B (\mathbf{n} : \delta \mathbf{U} + \mathbf{m} : \delta \mathbf{K}) dV = \int_B \rho_{\text{ref}} \left( \frac{\partial \psi_{\text{int}}(\mathbf{U}^e, \mathbf{K}^e)}{\partial \mathbf{U}^e} : \mathbf{U}^{p-1} \delta \mathbf{U} + \frac{\partial \psi_{\text{int}}(\mathbf{U}^e, \mathbf{K}^e)}{\partial \mathbf{K}^e} : \delta \mathbf{K} \right) dV.$$

Using (2.12), (2.13), and (2.42) we have

$$(3.28) \quad \delta \mathbf{U} = \delta \mathbf{R}^T \mathbf{F} + \mathbf{R}^T \delta \mathbf{F} = \delta \mathbf{R}^T \mathbf{R} \mathbf{U} + \mathbf{R}^T \delta \mathbf{F},$$

$$(3.29) \quad \delta \mathbf{K} = \delta \mathbf{k}_i \otimes \mathbf{G}^i = (\mathbf{w}_{,i} + \mathbf{k}_i \times \mathbf{w}) \otimes \mathbf{G}^i.$$

Correspondingly, we use (2.41), (3.25) to get

$$(3.30) \quad \begin{aligned} \delta T &= \varrho_{\text{ref}} (\dot{\mathbf{u}} \cdot \delta \dot{\mathbf{u}} + \Theta \boldsymbol{\omega} \cdot \delta \boldsymbol{\omega}), \\ &= \varrho_{\text{ref}} [\dot{\mathbf{u}} \cdot \delta \dot{\mathbf{u}} + \Theta \boldsymbol{\omega} \cdot (\dot{\mathbf{w}} + \boldsymbol{\omega} \times \mathbf{w})]. \end{aligned}$$

From the latter relation the essence of Sec. 4 and its great value, especially of relation (2.41), becomes apparent.

By introducing (3.27)–(3.30) in (3.26) and after some manipulations we arrive at

$$(3.31) \quad \int_{t_0}^{t_1} \left[ \int_{\mathcal{B}} \varrho_{\text{ref}} [\dot{\mathbf{u}} \cdot \delta \dot{\mathbf{u}} + \Theta \boldsymbol{\omega} \cdot (\dot{\mathbf{w}} + \boldsymbol{\omega} \times \mathbf{w})] dV \right. \\ \left. - \int_{\mathcal{B}} \varrho_{\text{ref}} \mathbf{R} \mathbf{U}^{p-T} \frac{\partial \psi_{\text{int}}(\mathbf{U}^e, \mathbf{K}^e)}{\partial \mathbf{U}^e} \mathbf{G}^i \cdot \delta \mathbf{u}_{,i} dV \right. \\ \left. - \int_{\mathcal{B}} \varrho_{\text{ref}} \left( -\mathbf{U}^{p-T} \frac{\partial \psi_{\text{int}}(\mathbf{U}^e, \mathbf{K}^e)}{\partial \mathbf{U}^e} \mathbf{U}^T : \mathbf{W} + \frac{\partial \psi_{\text{int}}(\mathbf{U}^e, \mathbf{K}^e)}{\partial \mathbf{K}^e} \mathbf{G}^i \cdot (\mathbf{w}_{,i} + \mathbf{k}_i \times \mathbf{w}) \right) dV \right. \\ \left. + \int_{\mathcal{B}} \mathbf{f} \cdot \delta \mathbf{u} dV + \int_{\partial \mathcal{B}} \mathbf{f}_s \cdot \delta \mathbf{u} dA + \int_{\mathcal{B}} \mathbf{M} \cdot \mathbf{w} dV + \int_{\partial \mathcal{B}} \mathbf{M}_s \cdot \mathbf{w} dA \right] dt = 0.$$

Note that (2.36) has been used and that  $\mathbf{w}$  is the axial vector of the skew-symmetric tensor  $\mathbf{W}$ . Standard regularity assumptions with respect to time together with the fact that the variations vanish at  $t = t_0$  and  $t = t_1$ , lead to

$$(3.32) \quad \int_{\mathcal{B}} \varrho_{\text{ref}} [\ddot{\mathbf{u}} \cdot \delta \mathbf{u} + (\Theta \dot{\boldsymbol{\omega}} - \Theta \boldsymbol{\omega} \times \boldsymbol{\omega}) \cdot \mathbf{w}] dV \\ + \int_{\mathcal{B}} \varrho_{\text{ref}} \mathbf{R} \mathbf{U}^{p-T} \frac{\partial \psi_{\text{int}}(\mathbf{U}^e, \mathbf{K}^e)}{\partial \mathbf{U}^e} \mathbf{G}^i \cdot \delta \mathbf{u}_{,i} dV \\ + \int_{\mathcal{B}} \varrho_{\text{ref}} \left( -\mathbf{U}^{p-T} \frac{\partial \psi_{\text{int}}(\mathbf{U}^e, \mathbf{K}^e)}{\partial \mathbf{U}^e} \mathbf{U}^T : \mathbf{W} + \frac{\partial \psi_{\text{int}}(\mathbf{U}^e, \mathbf{K}^e)}{\partial \mathbf{K}^e} \mathbf{G}^i \cdot (\mathbf{w}_{,i} + \mathbf{k}_i \times \mathbf{w}) \right) dV \\ - \int_{\mathcal{B}} \mathbf{f} \cdot \delta \mathbf{u} dV - \int_{\partial \mathcal{B}} \mathbf{f}_s \cdot \delta \mathbf{u} dA - \int_{\mathcal{B}} \mathbf{M} \cdot \mathbf{w} dV - \int_{\partial \mathcal{B}} \mathbf{M}_s \cdot \mathbf{w} dA = 0.$$

The last equation splits into the two equations

$$(3.33) \quad \int_{\mathcal{B}} \varrho_{\text{ref}} \ddot{\mathbf{u}} \cdot \delta \mathbf{u} = \int_{\mathcal{B}} \mathbf{f} \cdot \delta \mathbf{u} dV + \int_{\partial \mathcal{B}} \mathbf{f}_s \cdot \delta \mathbf{u} dA \\ - \int_{\mathcal{B}} \varrho_{\text{ref}} \mathbf{R} \mathbf{U}^{p-T} \frac{\partial \psi_{\text{int}}(\mathbf{U}^e, \mathbf{K}^e)}{\partial \mathbf{U}^e} \mathbf{G}^i \cdot \delta \mathbf{u}_{,i},$$

$$(3.34) \quad \int_B \varrho_{\text{ref}} \left( \Theta \dot{\omega} + \omega \times \Theta \omega \right) \cdot \mathbf{w} \, dV = \int_B \mathbf{M} \cdot \mathbf{w} \, dV + \int_{\partial B} \mathbf{M}_s \cdot \mathbf{w} \, dA$$

$$- \int_B \varrho_{\text{ref}} \left( -\mathbf{U}^{p-T} \frac{\partial \psi_{\text{int}}(\mathbf{U}^e, \mathbf{K}^e)}{\partial \mathbf{U}^e} \mathbf{U}^T : \mathbf{W} + \frac{\partial \psi_{\text{int}}(\mathbf{U}^e, \mathbf{K}^e)}{\partial \mathbf{K}^e} \mathbf{G}^i \cdot (\mathbf{w}_{,i} + \mathbf{k}_i \times \mathbf{w}) \right) dV,$$

which can be recognized as the weak form of the field equations (3.7), (3.8) if the identifications are made

$$(3.35) \quad \Sigma = \varrho_{\text{ref}} \frac{\partial \psi(\mathbf{U}^e, \mathbf{K}^e)}{\partial \mathbf{U}} \mathbf{U}^T = \varrho_{\text{ref}} \mathbf{U}^{p-T} \frac{\partial \psi(\mathbf{U}^e, \mathbf{K}^e)}{\partial \mathbf{U}^e} \mathbf{U}^T,$$

$$(3.36) \quad \Gamma = \varrho_{\text{ref}} \frac{\partial \psi(\mathbf{U}^e, \mathbf{K}^e)}{\partial \mathbf{K}} \mathbf{U}^T = \varrho_{\text{ref}} \frac{\partial \psi(\mathbf{U}^e, \mathbf{K}^e)}{\partial \mathbf{K}^e} \mathbf{U}^T.$$

These identifications are justified on the basis of the principle of positive dissipation to be discussed in the next section.

Clearly, the recovery of the field equations (3.7) and (3.8) as Euler-Lagrange equations necessitates some algebraic operations which we have omitted for the sake of brevity. Note also that standard regularity assumptions of the involved fields are assumed to hold.

### 3.4. Positive dissipation and flow rules

We proceed further and formulate the principle of positive dissipation

$$(3.37) \quad \int_B \mathcal{D} \, dV = \mathcal{W} - \int_B \varrho_{\text{ref}} \dot{\psi}_{\text{int}} \, dV > 0.$$

In the above expression  $\mathcal{D}$  means the dissipation function and  $\mathcal{W}$  is the difference between the mechanical power and the rate of the kinetic energy. It is defined by

$$(3.38) \quad \mathcal{W} = \mathcal{P} - \int_B \dot{T} \, dV = \int_B (\mathbf{f} \cdot \dot{\mathbf{u}} + \mathbf{M} \cdot \dot{\boldsymbol{\omega}}) \, dV + \int_{\partial B} (\mathbf{f}_s \cdot \dot{\mathbf{u}} + \mathbf{M}_s \cdot \dot{\boldsymbol{\omega}}) \, dA - \int_B \dot{T} \, dV.$$

Assuming that the equilibrium equations hold, it is straightforward to show that the mechanical power reduces to the forms given in (3.10), (3.11) or (3.12).

Making use of (3.11) as well as of

$$(3.39) \quad \int_B \varrho_{\text{ref}} \dot{\psi}(\mathbf{U}^e, \mathbf{K}^e, \mathbf{q}) \, dV = \int_B \varrho_{\text{ref}} \left( \frac{\partial \psi}{\partial \mathbf{U}^e} : \dot{\mathbf{U}}^e + \varrho_{\text{ref}} \frac{\partial \psi}{\partial \mathbf{K}^e} : \dot{\mathbf{K}}^e + \varrho_{\text{ref}} \frac{\partial \psi}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}} \right) dV$$

$$= \int_B \varrho_{\text{ref}} \left[ \mathbf{U}^{p-T} \frac{\partial \psi}{\partial \mathbf{U}^e} \mathbf{U}^T : (\dot{\mathbf{U}} \mathbf{U}^{-1} - \dot{\mathbf{U}}^p \mathbf{U}^{p-1}) \right.$$

$$\left. + \frac{\partial \psi}{\partial \mathbf{K}^e} : (\dot{\mathbf{K}} - \dot{\mathbf{K}}^p) + \frac{\partial \psi}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}} \right] dV,$$

where the decomposition (3.14) is underlined, and assuming that the equilibrium equations hold along with the standard thermomechanical arguments, the principle of positive dissipation results in the constitutive relations

$$(3.40) \quad \underline{\Sigma} = \varrho_{\text{ref}} \mathbf{U}^{p-T} \frac{\partial \psi(\mathbf{U}^e, \mathbf{K}^e, \mathbf{q})}{\partial \mathbf{U}^e} \mathbf{U}^T,$$

$$(3.41) \quad \underline{\Xi} = \varrho_{\text{ref}} \mathbf{U}^{eT} \frac{\partial \psi(\mathbf{U}^e, \mathbf{K}^e, \mathbf{q})}{\partial \mathbf{U}^e},$$

$$(3.42) \quad \underline{\Gamma} = \varrho_{\text{ref}} \frac{\partial \psi(\mathbf{U}^e, \mathbf{K}^e, \mathbf{q})}{\partial \mathbf{K}^e} \mathbf{U}^T,$$

$$(3.43) \quad \mathbf{y} = -\varrho_{\text{ref}} \frac{\partial \psi(\mathbf{U}^e, \mathbf{K}^e, \mathbf{q})}{\partial \mathbf{q}},$$

as well as in the statement

$$(3.44) \quad \mathcal{D} = \underline{\Sigma} : \mathbf{L}^p + \underline{\Gamma} \mathbf{U}^{-1} : \dot{\mathbf{K}}^e + \mathbf{y} \cdot \dot{\mathbf{q}} > 0,$$

which specifies the dissipation function. In (3.43), (3.44),  $\mathbf{y}$  has been introduced as the conjugate variable of  $\mathbf{q}$ . Note further that  $\mathbf{L}^p$  is defined by (3.19)<sub>1</sub>.

It should be mentioned that the above constitutive relations are formulated in a general form. Possible representations of these relations must allow for the fulfillment of the corresponding field equations. The discussion of such general representations is out of the scope of the paper.

Until now we have dealt with the first function of the theory, namely the free energy function  $\psi$ . The second function is the flow rule. We assume the existence of an elastic range defined by a function  $\phi$  formulated in the stress space in terms of the real stress tensors. The material moment tensor to enter the formulation is evidently  $\underline{\Gamma}$ . Contrasting this, the stress tensor can be chosen either as  $\underline{\Sigma}$  or as  $\underline{\Xi}$ . Both of them have the physical meaning of the Kirchhoff stress tensor (e.g. have the same invariants as the Kirchhoff stress tensor). Taking a look at the dissipation function it is evident that in the case of the decomposition (3.14), the tensor  $\underline{\Sigma}$  is the appropriate one. Accordingly we formulate the function  $\phi$  as

$$(3.45) \quad \phi(\underline{\Sigma}, \underline{\Gamma}, \mathbf{y}) : \mathbb{R}^9 \times \mathbb{R}^9 \times \mathbb{R}^n \rightarrow \mathbb{R}^+.$$

Elastic behaviour is given for  $\phi(\underline{\Sigma}, \underline{\Gamma}, \mathbf{y}) < 0$ .

The elastic-viscoplastic theory is complete when evolution equations for the internal variables are specified. Strictly speaking this is an experimental task. In elasto-plasticity, the so-called normal flow rule has been frequently used and proved to work well especially within metal plasticity. Classically, this flow rule is equivalent and can be derived by means of Hill's postulate of maximum dissipation. It is well known that the postulate does not constitute a physical law. It should be understood as a useful instrument to derive flow rules valid for a specific class of materials. In view of the fact that sufficient experimental data

concerning the behavior of the elastic-viscoplastic Cosserat continuum is not available, we appeal to the postulate of maximum dissipation in order to derive the necessary flow rules. It is clear that these rules can and should be modified, once experimental data is available giving reason for such a modification. According to the mentioned postulate we have

$$(3.46) \quad -\mathcal{D} + \frac{1}{\eta} \phi^+(\Sigma, \Gamma, \mathbf{y}) = \text{maximum},$$

where  $1/\eta$  can be understood as a penalty term which is physically interpreted as the viscosity. The elaboration of the postulate leads with (3.44) to the following flow rules

$$(3.47) \quad \mathbf{L}^p = \frac{1}{\eta} \frac{\partial \phi^+(\Sigma, \Gamma, \mathbf{y})}{\partial \Sigma},$$

$$(3.48) \quad \dot{\mathbf{K}}^p = \frac{1}{\eta} \frac{\partial \phi^+(\Sigma, \Gamma, \mathbf{y})}{\partial \Gamma} \mathbf{U}^T,$$

$$(3.49) \quad \dot{\mathbf{q}} = \frac{1}{\eta} \frac{\partial \phi^+(\Sigma, \Gamma, \mathbf{y})}{\partial \mathbf{y}}.$$

In the case we are adopting the second decomposition (3.15), the dissipation function reads

$$(3.50) \quad \mathcal{D} = \Xi : \bar{\mathbf{L}}^p + \Gamma \mathbf{U}^{-1} : \dot{\mathbf{K}}^e + \mathbf{y} \cdot \dot{\mathbf{q}} > 0,$$

where  $\bar{\mathbf{L}}^p$  is defined by (3.22)<sub>1</sub>. Accordingly, in the case of decomposition (3.15), the formulation of  $\phi$  is appropriately carried out in terms of  $\Xi$ . That is, we define the flow rule by the function

$$(3.51) \quad \phi^+(\Xi, \Gamma, \mathbf{y}) : \mathbb{R}^9 \times \mathbb{R}^9 \times \mathbb{R}^n \rightarrow \mathbb{R}^+.$$

The transformation of (3.46) leads now to the alternative flow rules

$$(3.52) \quad \bar{\mathbf{L}}^p = \frac{1}{\eta} \frac{\partial \phi^+(\Xi, \Gamma, \mathbf{y})}{\partial \Xi},$$

$$(3.53) \quad \dot{\mathbf{K}}^p = \frac{1}{\eta} \frac{\partial \phi^+(\Xi, \Gamma, \mathbf{y})}{\partial \Gamma} \mathbf{U}^T,$$

$$(3.54) \quad \dot{\mathbf{q}} = \frac{1}{\eta} \frac{\partial \phi^+(\Xi, \Gamma, \mathbf{y})}{\partial \mathbf{y}}.$$

Note that in (3.47)  $\mathbf{L}^p$  is a left rate. The updating of  $\mathbf{U}^p$  must be carried out according to the product  $\exp(t\mathbf{L}^p)\mathbf{U}^p$ . In (3.52) the rate is a right one which means that the updating of  $\mathbf{U}^p$  is carried out according to the product  $\mathbf{U}^p \exp(t\bar{\mathbf{L}}^p)$ .

With the specific formulation of the functions  $\psi$  and  $\phi$  the theory is completed. The following simple generalization of the von Mises theory can be adopted which we include for completeness:

$$(3.55) \quad \phi = h^\epsilon, \quad h = J + \beta I - y,$$



where we have

$$(3.56) \quad J = \text{dev } \Sigma : \text{dev } \Sigma, \quad I = \Gamma : \Gamma, \quad y = y_0 + Hq.$$

In the above equations,  $\epsilon$  is a material parameter and only isotropic hardening is considered where  $H$  is the hardening parameter.  $\beta$  is a material parameter related directly to the influence of the moments in the flow rule. In the following two boxes the complete set of equations in either case of decomposition is summarized.

$$(3.57) \quad \mathbf{U} = \mathbf{U}^p \mathbf{U}^e, \quad \mathbf{K} = \mathbf{K}^e + \mathbf{K}^p, \quad \dot{\mathbf{U}}^p = \mathbf{L}^p \mathbf{U}^p.$$

Internal dissipation

$$(3.58) \quad \mathcal{D} = \Sigma : \mathbf{L}^p + \Gamma \mathbf{U}^{-T} : \dot{\mathbf{K}}^p + \mathbf{y} \cdot \dot{\mathbf{q}}.$$

Constitutive relations

$$(3.59) \quad \Sigma = \rho_{\text{ref}} \frac{\partial \psi(\mathbf{U}^e, \mathbf{K}^e, \mathbf{q})}{\partial \mathbf{U}} \mathbf{U}^T = \rho_{\text{ref}} \mathbf{U}^{p-T} \frac{\partial \psi(\mathbf{U}^e, \mathbf{K}^e, \mathbf{q})}{\partial \mathbf{U}^e} \mathbf{U}^T,$$

$$(3.60) \quad \Gamma = \rho_{\text{ref}} \frac{\partial \psi(\mathbf{U}^e, \mathbf{K}^e, \mathbf{q})}{\partial \mathbf{K}^e} \mathbf{U}^T, \quad \mathbf{y} = -\rho_{\text{ref}} \frac{\partial \psi(\mathbf{U}^e, \mathbf{K}^e, \mathbf{q})}{\partial \mathbf{q}}.$$

Evolution equations

$$(3.61) \quad \mathbf{L}^p = \frac{1}{\eta} \frac{\partial \phi^+(\Sigma, \Gamma, \mathbf{y})}{\partial \Sigma}, \quad \dot{\mathbf{K}}^p = \frac{1}{\eta} \frac{\partial \phi^+(\Sigma, \Gamma, \mathbf{y})}{\partial \Gamma} \mathbf{U}, \quad \dot{\mathbf{q}} = \frac{1}{\eta} \frac{\partial \phi^+(\Sigma, \Gamma, \mathbf{y})}{\partial \mathbf{y}}.$$

$$(3.62) \quad \mathbf{U} = \bar{\mathbf{U}}^e \bar{\mathbf{U}}^p, \quad \mathbf{K} = \mathbf{K}^e + \mathbf{K}^p, \quad \dot{\bar{\mathbf{U}}}^p = \bar{\mathbf{U}}^p \bar{\mathbf{L}}^p,$$

Internal dissipation

$$(3.63) \quad \mathcal{D} = \Xi : \bar{\mathbf{L}}^p + \Gamma \mathbf{U}^{-T} : \dot{\mathbf{K}}^p + \mathbf{y} \cdot \dot{\mathbf{q}}.$$

Constitutive relations

$$(3.64) \quad \Xi = \rho_{\text{ref}} \mathbf{U}^T \frac{\partial \psi(\bar{\mathbf{U}}^e, \mathbf{K}^e, \mathbf{q})}{\partial \mathbf{U}} = \rho_{\text{ref}} \mathbf{U}^T \frac{\partial \psi(\mathbf{U}^e, \mathbf{K}^e, \mathbf{q})}{\partial \bar{\mathbf{U}}^e} \bar{\mathbf{U}}^{p-T},$$

$$(3.65) \quad \Gamma = \rho_{\text{ref}} \frac{\partial \psi(\bar{\mathbf{U}}^e, \mathbf{K}^e, \mathbf{q})}{\partial \mathbf{K}^e} \mathbf{U}^T, \quad \mathbf{y} = -\rho_{\text{ref}} \frac{\partial \psi(\mathbf{U}^e, \mathbf{K}^e, \mathbf{q})}{\partial \mathbf{q}}.$$

Evolution equations

$$(3.66) \quad \bar{\mathbf{L}}^p = \frac{1}{\eta} \frac{\partial \phi^+(\Xi, \Gamma, \mathbf{y})}{\partial \Xi}, \quad \dot{\mathbf{K}}^p = \frac{1}{\eta} \frac{\partial \phi^+(\Xi, \Gamma, \mathbf{y})}{\partial \Gamma} \mathbf{U}, \quad \dot{\mathbf{q}} = \frac{1}{\eta} \frac{\partial \phi^+(\Xi, \Gamma, \mathbf{y})}{\partial \mathbf{y}}.$$

The question of linearization of formulations as given by (3.33) and (3.34) is of special theoretical and practical interest. This issue is discussed briefly in the Appendix.

#### 4. Concluding remarks

A theory of the elastic-viscoplastic Cosserat continuum has been presented. Basic features of the formulation are: i) It is based on the multiplicative decomposition of the stretch tensor and an additive one for the second Cosserat deformation tensor. ii) The rotation field is treated as a kinematical variable and it was emphasized that no decomposition could be adopted for such a field. iii) Two functions determine the structure of the theory completely, the free energy function and the flow rule. As a first step, the evolution equations has been derived by means of Hill's postulate of maximum dissipation.

It was shown that two possible multiplicative decompositions can be adopted. Depending on the decomposition underlined, the flow rule has to be formulated in terms of two different stress tensors corresponding to a dissipation function. Special emphasis has been given for the definition of rates and variations. Relying on the basic concepts of Lie groups, the formulation of such rates has been systematically derived, a method which clarifies many concepts seem otherwise to be arbitrary.

#### 5. Appendix. Second derivatives and the geometric structure of the configuration space

The question of linearization of formulations as given by (3.33) and (3.34) is of special theoretical and practical interest. At least, when computations have to be carried out. It turns out that the geometry of the configuration space as a Lie group is of crucial importance. In the context of shell computations at finite deformations, these aspects have been presented in SANSOUR and BEDNARCZYK [7] and MAKOWSKI and STUMPF [21] to which the reader is referred. In brief we sketch in the following basic aspects of the linearization process to be carried out with respect to the geometrical nonlinearities. These aspects are not restricted to the Cosserat continuum but are carried over to any configuration space defined as a Lie group; e.g. within a micromorphic continuum.

The importance of the linearization concepts becomes obvious from the following observation. The second variation of a relation as (2.36) reads

$$(5.1) \quad \delta^2 \mathbf{R} = (\delta \mathbf{R}) \mathbf{W} = \mathbf{R} \mathbf{Y} \mathbf{W}.$$

Evidently the expression is not symmetric with respect to the variations  $\mathbf{W}$  and  $\mathbf{Y}$ . Based on these formulas, the linearization of (3.33) and (3.34) with respect to the geometric nonlinearities results necessarily in a non-symmetric tangent

operator. Here we touch a crucial difference between linear vector spaces and Lie groups which are naturally defined as a nonlinear manifold. A fundamental observation is that the linearization can be carried out differently if the configuration space is equipped with the structure of a Killing metric. In that case and since the first variation  $\delta\mathbf{R}$  is to be understood as a tangent vector, the second variation can be carried out as a covariant derivation of that tangent vector. A related formulation which treats the first variation of the energy function as a co-vector which then is derived covariantly, is due to SIMO [22]. Our formulation has the basic feature of operating at the level of the configuration space itself.

Let us first rewrite the tangent vectors by making use of the following notation (DUBROVIN *et al.* [17])  $L_{\mathbf{W}}(\mathbf{R}) = \mathbf{R}\mathbf{W}$ . At an arbitrary point on  $SO(3)$ , say at  $\mathbf{R}$ , the Killing metric is defined by the scalar product of the tangent vectors defined by

$$(5.2) \quad \langle L_{\mathbf{W}}(\mathbf{R}), L_{\mathbf{Y}}(\mathbf{R}) \rangle = \text{tr} | (\mathbf{R}\mathbf{W}(\mathbf{R}\mathbf{Y})^T) = \text{tr} (\mathbf{W}\mathbf{Y}^T).$$

The Lie bracket  $[\mathbf{W}, \mathbf{Y}]$  is given by

$$(5.3) \quad [\mathbf{W}, \mathbf{Y}] = \mathbf{W}\mathbf{Y} - \mathbf{Y}\mathbf{W}, \quad \mathbf{W}, \mathbf{Y} \in so(3).$$

On the space of tangent vectors there exists a connection given by the relation (BRICKELL and CLARK [23], CHOQUET *et al.* [16], DUBROVIN *et al.* [17]).

$$(5.4) \quad 2 \langle \nabla_{\mathbf{U}}\mathbf{V}, \mathbf{W} \rangle = \mathbf{U}(\langle \mathbf{V}, \mathbf{W} \rangle) + \mathbf{V}(\langle \mathbf{W}, \mathbf{U} \rangle) - \mathbf{W}(\langle \mathbf{U}, \mathbf{V} \rangle) \\ - \langle \mathbf{U}, [\mathbf{V}, \mathbf{W}] \rangle + \langle \mathbf{V}, [\mathbf{W}, \mathbf{U}] \rangle + \langle \mathbf{W}, [\mathbf{U}, \mathbf{V}] \rangle \\ - \langle \mathbf{U}, \mathbf{T}(\mathbf{V}, \mathbf{W}) \rangle + \langle \mathbf{V}, \mathbf{T}(\mathbf{W}, \mathbf{U}) \rangle + \langle \mathbf{W}, \mathbf{T}(\mathbf{U}, \mathbf{V}) \rangle$$

where the torsion tensor  $\mathbf{T}$  is defined as

$$\mathbf{T}(\mathbf{U}, \mathbf{V}) = \nabla_{\mathbf{U}}\mathbf{V} - \nabla_{\mathbf{V}}\mathbf{U} - [\mathbf{U}, \mathbf{V}].$$

For a symmetric connection the torsion  $\mathbf{T}$  vanishes and we can obtain from (5.4) the expression

$$(5.5) \quad \nabla_{L_{\mathbf{Y}}}L_{\mathbf{W}} = \frac{1}{2}L_{[\mathbf{Y}, \mathbf{W}]}$$

Hereby one has to make use of the metric as defined in (5.2) as well as of the idea that tangent vectors can be understood as derivatives operating on function spaces. By the latter fact, the first line in (5.4) vanishes identically since the metric is independent of the particular point on the curve in  $\mathcal{C}$ .

Dealing with vector fields, it makes sense to define with the help of this connection a covariant derivative (or covariant variation). In fact such a derivative is given on the right invariant vector fields as

$$(5.6) \quad \nabla_{\mathbf{Y}}L_{\mathbf{W}}(\mathbf{R}) = L_{\mathbf{W}\mathbf{Y}}(\mathbf{R}) + \nabla_{L_{\mathbf{Y}}}L_{\mathbf{W}}(\mathbf{R}) = \mathbf{R}\mathbf{W}\mathbf{Y} + \frac{1}{2}\mathbf{R}[\mathbf{Y}, \mathbf{W}] \\ = \frac{1}{2}\mathbf{R}(\mathbf{Y}\mathbf{W} + \mathbf{W}\mathbf{Y}).$$

Evidently the relation is symmetric with respect to the variations  $\mathbf{W}$ ,  $\mathbf{Y}$ . Accordingly, the last result and not (5.1) should be used to accomplish possible linearizations. For more details and applications to practical computations within the shell theory, the reader is referred to SANSOUR and BEDNARCZYK [7].

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