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FIFTY YEARS OF THE ARCHIVES OF MECHANICS

Objectivity and frame indifference, revisited

Dedicated to Prof. Henryk Zorski
on the occasion of his 70-th birthday

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BECAUSE ONE HAS to distinguish between changing the observer and changing the state of motion of materials, objectivity and material frame indifference are redefined: Objectivity denotes a special tensor property in case of changing the observer, whereas material frame indifference is characterized by quantities being independent of different states of motion of the material. To describe these different states of motion of the material, an arbitrary standard frame of reference and a Constitutive Family is introduced. We prove that the constitutive map is isotropic in the state variables, if these and the material properties are objective.

1. Introduction

OBJECTIVITY AND/OR FRAME-INDIFFERENCE is one of the five "principles" governing constitutive equations in Rational Continuum Theories [1]. Although there is a flood of publications in the field [2–11] (only some of them can be mentioned here) and nearly all aspects of these two items are discussed in detail, one can see that during a meeting, experienced scientists suddenly begin to discourse controversially on "objectivity". This only can mean that questions around objectivity are not answered so satisfactorily that a common agreement could exist. The aim of the paper is to demonstrate that no vagueness remains, if we properly distinguish between two different situations. In the first situation, one material is described by various observers; in the second one, two differently moving materials are described by the same observer. If the two differently moving materials have additionally the property that they are identical, if resting with respect to each other, then the question arises, whether they are also identical when they are moving with respect to each other. As we will see, in general the answer is "no", provided they belong to the same Constitutive Family.

2. Changing frames

As it is well known [12], in non-relativistic theory the change of frame $\mathsf{B} \to \mathsf{B}^*$

is achieved locally by an Euclidean transformation of the position coordinates [1]

$$\mathbf{x}^* = \mathbf{Q}\cdot(\mathbf{x}-\mathbf{c}). \tag{2.1}$$

Here we denote an arbitrary, but fixed B^* as a *standard frame of reference*, whereas B is the chosen frame. This Euclidean change of frame is described by the orthogonal time-dependent transformation $\mathbf{Q}(\mathrm{t})$ [2]

$$\widetilde{\mathbf{Q}}\cdot\mathbf{Q} = \mathbf{Q}\cdot\widetilde{\mathbf{Q}} = \mathbf{1}, \qquad \det\mathbf{Q} = 1, \tag{2.2}$$

and by the components of the difference vector between the origins of the frames $\mathbf{c}(t)$.

The material velocity transforms by changing the frame

$$\mathbf{v}^* = \mathbf{Q}\cdot\mathbf{V} = \mathbf{Q}\cdot(\mathbf{v}+\mathbf{v}^{\mathrm{rel}}) =: \mathbf{Q}\cdot(\mathbf{v}-\dot{\mathbf{c}}+\boldsymbol{\Omega}\cdot(\mathbf{x}-\mathbf{c})). \tag{2.3}$$

Here $\mathbf{v}^{\mathrm{rel}}$ is the relative velocitiy between the two frames [3] which decomposes into the translational part $-\dot{\mathbf{c}}$ and the rotational part $\boldsymbol{\Omega}\cdot(\mathbf{x}-\mathbf{c})$ by use of the skew-symmetric spin matrix:

$$\boldsymbol{\Omega} := \widetilde{\mathbf{Q}}\cdot\dot{\mathbf{Q}} = -\widetilde{\boldsymbol{\Omega}}. \tag{2.4}$$

The relative velocity $\mathbf{v}^{\mathrm{rel}}$ depends only on quantities characterizing the relative motion of the two frames considered, whereas the material velocity $\mathbf{v}$ is independent of the relative motion of both the frames. In non-relativistic theories time and also the mass density are *frame-independent*

$$t^* = t, \qquad \varrho^* = \varrho, \tag{2.5}$$

whereas according to (2.3) the material velocity is *frame-dependent.*

A balance equation in an arbitrary frame B has the form

$$\partial_t(\varrho\Psi) + \nabla\cdot(\varrho\mathbf{v}\Psi + \boldsymbol{\Phi}) + \Sigma = 0. \tag{2.6}$$

Because no frame (observer) is distinguished, balance equations are *frame-invariant*, that means, they have the same shape in all frames. Thus we have according to (2.6) in B^*

$$\partial_t^*(\varrho^*\Psi^*) + \nabla^*\cdot(\varrho^*\mathbf{v}^*\Psi^* + \boldsymbol{\Phi}^*) + \Sigma^* = 0, \tag{2.7}$$

and the question arises, how the field quantities Ψ, $\boldsymbol{\Phi}$, and Σ transform. In general they transform as the material velocity (2.3) does,

$$\Psi^* = \Psi + \Psi^{\mathrm{rel}}, \qquad \boldsymbol{\Phi}^* = \mathbf{Q}\cdot(\boldsymbol{\Phi}+\boldsymbol{\Phi}^{\mathrm{rel}}), \qquad \Sigma^* = \Sigma + \Sigma^{\mathrm{rel}}, \tag{2.8}$$

(1) We write the equations for the coordinates in a symbolic way: $\mathbf{x}$ is the column of the position coordinates.

(2) Note: $\mathbf{Q}(t)$ is a proper orthogonal matrix.

(3) The notion "relative velocity" is often used in another context. Here it denotes the relative velocity between the standard frame of reference B^* and the chosen frame B.

or if the balance (2.6) has its range in R^3, we have the transformations

$$\mathbf{\Psi}^* = \mathbf{Q}\cdot[\mathbf{\Psi} + \mathbf{\Psi}^{\mathrm{rel}}], \qquad \mathbf{\Phi}^* = \mathbf{Q}\cdot[\mathbf{\Phi} + \mathbf{\Phi}^{\mathrm{rel}}]\cdot\widetilde{\mathbf{Q}}, \qquad \mathbf{\Sigma}^* = \mathbf{Q}\cdot[\mathbf{\Sigma} + \mathbf{\Sigma}^{\mathrm{rel}}]. \tag{2.9}$$

In the standard frame of reference, the relative part of all quantities is identically zero by definition

$$\Psi^{\mathrm{rel}*} \equiv 0, \qquad \mathbf{\Phi}^{\mathrm{rel}*} \equiv \mathbf{0}, \qquad \Sigma^{\mathrm{rel}*} \equiv 0. \tag{2.10}$$

Quantities whose relative parts vanish in all frames transform according to (2.8) or (2.9) as tensor components. This gives rise to the

DEFINITION. *Quantities whose relative parts vanish in all frames are called objective.*

3. Derivatives

We consider a function of arbitrary range

$$f(\mathbf{x}^*, t) = f(\mathbf{Q}\cdot(\mathbf{x} - \mathbf{c}), t), \tag{3.1}$$

taking (2.1) into account. Its gradient is

$$\nabla f(\mathbf{x}^*, t) = \nabla^* f(\mathbf{x}^*, t)\cdot\mathbf{Q}. \tag{3.2}$$

Thus we have proved the

□ PROPOSITION.

$$\nabla = \widetilde{\mathbf{Q}}\cdot\nabla^* = \nabla^*\cdot\mathbf{Q}. \tag{3.3}$$

□

There are two time derivatives belonging to different frames

$$\text{in } \mathsf{B}^*: \quad \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}^*} \equiv \partial_t{}^*, \tag{3.4}$$

$$\text{in } \mathsf{B}: \quad \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} \equiv \partial_t. \tag{3.5}$$

We now can prove the

□ PROPOSITION. The transformation equation of the partial time derivative is

$$\partial_t{}^* = \partial_t - \mathbf{v}^{\mathrm{rel}}\cdot\nabla \tag{3.6}$$

for arbitrary components in its domain.

For proving this proposition we obtain by the chain rule

$$\partial_t f(\mathbf{x}^*, t) = [\partial_t{}^* + \partial_t\mathbf{x}^*\cdot\nabla^*]\, f(\mathbf{Q}\cdot(\mathbf{x} - \mathbf{c}), t). \tag{3.7}$$

According to (2.1), relation

$$\partial_t \mathbf{x}^* = \dot{\mathbf{Q}}\cdot(\mathbf{x}-\mathbf{c}) - \mathbf{Q}\cdot\dot{\mathbf{c}} \tag{3.8}$$

is valid. By use of (2.4) and (2.3) we obtain

$$\partial_t \mathbf{x}^* = \mathbf{Q}\cdot\mathbf{v}^{\mathrm{rel}} \tag{3.9}$$

by which Eq. (3.7) results in the transformation equation (3.6) we are looking for, if Eq. (3.3) was taken into account. □

According to (3.6) the partial time derivative is non-objective, because its relative part is not identically equal zero

$$\partial_t^{\mathrm{rel}} \equiv -\mathbf{v}^{\mathrm{rel}}\cdot\nabla. \tag{3.10}$$

The balance equations (2.7) and (2.6) are often written down by means of the material time derivative

DEFINITION. *The frame-invariantly defined material time derivative is*

$$\frac{d^*}{dt} := \partial_t^* + \mathbf{v}^*\cdot\nabla^*, \qquad \frac{d}{dt} := \partial_t + \mathbf{v}\cdot\nabla. \tag{3.11}$$

We now prove the

□ PROPOSITION. The material time derivative is frame-independent, and therefore objective.

By (3.6) we obtain from $(3.11)_1$

$$\frac{d^*}{dt} = \partial_t - \mathbf{v}^{\mathrm{rel}}\cdot\nabla + \mathbf{v}^*\cdot\nabla^*. \tag{3.12}$$

Taking (2.3) and (3.3) into account, Eq. (3.11) yields the proposition

$$\frac{d^*}{dt} = \partial_t + \mathbf{v}\cdot\nabla = \frac{d}{dt}. \tag{3.13}$$

□

The frame independence of the material time derivative needs an interpretation. For that purpose we introduce the so-called local rest frame for which we have locally (at point $\mathbf{x}^0$) $\mathbf{v}^0(\mathbf{x}^0(t),t) = \mathbf{0}$. In this rest frame B^0 we obtain for the material time derivative $(3.11)_2$

$$\frac{d^0}{dt} = \partial_t^{\,0}, \tag{3.14}$$

what means, that in this frame the material time derivative describes the explicit temporal change in the rest frame. Because of (3.13), the material time derivative in an arbitrary frame B describes this explicit time rate in the rest frame, too.

If we introduce the frame-invariant *material spin matrix*

$$\boldsymbol{\omega}(\mathbf{x},t) := \frac{1}{2}[\nabla\mathbf{v} - (\nabla\mathbf{v})^{\sim}], \tag{3.15}$$

we can prove the following well-known

□ PROPOSITION.

$$\boldsymbol{\omega}^*(\mathbf{x}^*,t) = \mathbf{Q}\cdot[\boldsymbol{\omega}(\mathbf{x},t) + \boldsymbol{\Omega}]\cdot\widetilde{\mathbf{Q}}, \tag{3.16}$$

$$\frac{d}{dt}^* a^* = \frac{d}{dt}(a + a^{\mathrm{rel}}), \tag{3.17}$$

$$\frac{d}{dt}^* \mathbf{a}^* - \boldsymbol{\omega}^*\cdot\mathbf{a}^* = \mathbf{Q}\cdot\left[\frac{d}{dt}(\mathbf{a}+\mathbf{a}^{\mathrm{rel}}) - \boldsymbol{\omega}\cdot(\mathbf{a}+\mathbf{a}^{\mathrm{rel}})\right], \tag{3.18}$$

$$\frac{d}{dt}^* \mathbf{a}^* - \boldsymbol{\omega}^*\cdot\mathbf{a}^* + \mathbf{a}^*\cdot\boldsymbol{\omega}^* = \mathbf{Q}\cdot\left[\frac{d}{dt}(\mathbf{a}+\mathbf{a}^{\mathrm{rel}}) - \boldsymbol{\omega}\cdot(\mathbf{a}+\mathbf{a}^{\mathrm{rel}}) + (\mathbf{a}+\mathbf{a}^{\mathrm{rel}})\cdot\boldsymbol{\omega}\right]\cdot\widetilde{\mathbf{Q}}. \tag{3.19}$$

□

Here we can see from (3.18) that also for an objective **a** its material time derivative is not objective in general, although the material time derivative itself is an objective operator according to (3.13). Now we have to take constitutive equations into account.

4. Material frame indifference

We now consider three frames: the standard frame of reference B^*, an arbitrary frame B, and the *local co-rotational rest frame* B^0 which is fixed at a material point of position $\mathbf{x}^0(t)$

$$\text{in } \mathsf{B}^0: \quad \mathbf{v}^0(\mathbf{x}^0(t),t) = \mathbf{0}, \qquad \boldsymbol{\omega}^0(\mathbf{x}^0(t),t) = \mathbf{0}. \tag{4.1}$$

We need B^0 for describing the motion of the material with respect to the standard frame of reference. If we introduce an abstract state space spanned by a set of variables z, these variables will transform by an abstract linear mapping B (because it operates on tensor components of different orders included in z) depending on the relative motion of both the frames with respect to each other

$$z^* = B(z + z^{\mathrm{rel}}) = B^0(z^0 + z^{0\mathrm{rel}}). \tag{4.2}$$

Here B describes the change of frames $\mathsf{B} \to \mathsf{B}^*$, and B^0 that of $\mathsf{B}^0 \to \mathsf{B}^*$.

There is no doubt about the validity of the following statement:

I. *Constitutive properties do not depend on the relative motion of frames.*

Two observers in different frames, B and B*, investigating the same material observe the same constitutive properties. This trite statement should not be confused with the second statement.

II. *Constitutive properties do not depend on the motion of the material with respect to the standard frame of reference.*

Experience shows that this statement is wrong: Materials perform their motion with respect to the standard frame of reference B*, and therefore the constitutive properties depend on B^0. Hence we introduce the *Constitutive Family*

$$M(\mathbf{x},t) = \mathcal{M}(z(\mathbf{x},t);\square^*), \qquad \square^* \equiv \mathcal{F}[\mathbf{Q}^0(t),\mathbf{c}^0(t)]. \tag{4.3}$$

The material properties M are generated by the constitutive map $\mathcal{M}$ which is defined on the state space spanned by the z. The family parameter is the functional $\mathcal{F}$ which describes the influence of motion of the material. If this functional is of differential type, we have

$$\square^* = F[\mathbf{Q}^0,\mathbf{c}^0,\mathbf{\Omega}^0,\dot{\mathbf{c}}^0,\dot{\mathbf{\Omega}}^0,\ddot{\mathbf{c}}^0,...]. \tag{4.4}$$

Thus different observers describe the same material in a special state of motion by

$$M(\mathbf{x},t) = \mathcal{M}(z(\mathbf{x},t);\square^*), \tag{4.5}$$

$$M^*(\mathbf{x}^*,t) = \mathcal{M}^*(z^*(\mathbf{x}^*,t);\square^*) \qquad M^0(\mathbf{x}^0,t) = \mathcal{M}^0(z^0(\mathbf{x}^0,t);\square^*). \tag{4.6}$$

Here it seems that to each frame belongs its own constitutive family, although the different observers see the same material. In this case we cannot define what identical materials are. Therefore we formulate the

Principle of material frame-indifference:

i. Because observers are not distinguished, the constitutive map $\mathcal{M}(\bullet;\sharp)$ describing one constitutive family is frame-invariant

$$\mathcal{M}(\bullet;\sharp) = \mathcal{M}^*(\bullet;\sharp) = \mathcal{M}^0(\bullet;\sharp) = \dots . \tag{4.7}$$

ii. Uniform motions of the material with respect to the standard frame of reference do not influence constitutive properties. Consequently the domain of (4.4) is

$$\square^* = F[\mathbf{\Omega}^0,\dot{\mathbf{\Omega}}^0,\ddot{\mathbf{c}}^0,...]. \tag{4.8}$$

We now prove the following

□ PROPOSITION. If the state variables z and the constitutive properties M are objective quantities, the constitutive map $\mathcal{M}$ is isotropic in the state variables.

From $(4.6)_1$ we obtain taking (4.7) into account

$$M^* = \mathcal{M}(z^*;\square^*), \tag{4.9}$$

and application of the transformation properties (4.2) yields

$$B(M + M^{\text{rel}}) = \mathcal{M}(B(z + z^{\text{rel}}); \square^*). \tag{4.10}$$

From this and (4.5) we obtain

$$M + M^{\text{rel}} = B^{-1}\mathcal{M}(B(z + z^{\text{rel}}); \square^*) = \mathcal{M}(z; \square^*) + M^{\text{rel}}. \tag{4.11}$$

If the state variables z and the constitutive properties M are objective quantities, we obtain

$$\mathcal{M}(z; \square^*) = B^{-1}\mathcal{M}(Bz; \square^*), \tag{4.12}$$

which is the maintained isotropy of $\mathcal{M}$. □

Consequently all representation theorems of isotropic functions are valid for the Constitutive Family.

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