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FIFTY YEARS OF THE ARCHIVES OF MECHANICS

Pseudomomentum in relativistic continuum mechanics

Dedicated to Prof. Henryk Zorski
on the occasion of his 70-th birthday

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IN CLASSICAL continuum mechanics the balance or unbalance equation of *pseudomomentum* reflects the material invariance of the system under study. It relates the time derivative of pseudo-momentum and the flux of the *Eshelby stress*. It is legitimate to inquire whether this structure is conserved in a *relativistic four-dimensional* background. We examine here the relativistic definition of *pseudo/material* momentum using simultaneously variational and direct approaches (the latter using the canonical projection of space-time onto the material manifold). It appears that the truly *material entities*, just as those in a proper frame, should be the basic ones, being independent of the relativity framework used.

1. Introduction

THE NOTION of *pseudomomentum* in a continuum is so much intriguing that Sir Rudolph Peierls, a sharp observer of the physical scene, recurrently came back to that subject matter [1–3]. In nondissipative continua described in the usual Newtonian background, pseudomomentum is none other than the *canonical momentum* of *analytical continuum mechanics* [4–6]. It has thus an ontologial status which equals that of *energy*, i.e. it is the *spatial* part of a four-dimensional vectorial object or, equivalently, the mixed space-time part of a four-dimensional second-order tensor known as the canonical *energy-momentum tensor* [7]. The remarkable facts about pseudomomentum are that (i) unless the considered body is rigid, it is different from *physical momentum* (the "quantity of motion" in classical mechanics), (ii) one part of it plays a fundamental role in *crystal physics* under the name of *crystal momentum* [8], and in *electromagnetic optics* and *wavelike* phenomena under the name of *wave momentum* whether in optics or acoustics [1, 9, 10], (iii) it does play a role in the discussion of the notion of *electromagnetic momentum* in the electrodynamics of magnetized and polarized bodies [9, 11–13], and (iv) in *global form* its conservation or *non*conservation plays both a theoretical and computational role in the *dynamics of fracture* in elastic [14, 15] or inelastic [16] solids and in the *dynamics of perturbed solitonic structures* [17]. Its role in elastic solids was also recognized by other authors [18, 19]. Syntheses emphasizing the last two aspects are given in book form [20] and in two more recent review papers [21, 22].

The balance or *unbalance of pseudomomentum* of the Newtonian mechanics of continua is related to the invariance with respect to *material coordinates*, i.e. it expresses the material homogeneity or inhomogeneity of the material, while the balance of energy relates to the invariance with respect to *time*, and the balance of *physical momentum* (momentum in the *current configuration*) relates to the homogeneity of *physical space* (and *not* of the material). As shown in previous papers, all material inhomogeneities, whether of inertial, elastic or inelastic origins, are captured by the balance of pseudomomentum. Although pseudomomentum is naturally defined in a Lagrangian - Hamiltonian variational context [6, 20], it can also be given an intrinsic differential-geometrical definition (it is the natural pull-back of physical momentum to the material manifold, up to a sign, or: it is the *material covector associated, via the deformed metric, with the inverse-motion velocity*). These definitions were first given by one of the authors [6, 11]. Because of the intimate relationship between the notions of pseudomomentum and invariance, it is a natural move to look at the notion of pseudomomentum in the *relativistic* framework. In doing so we will essentially build on the approach to relativistic continuum mechanics advocated by GROT and ERINGEN [23] and MAUGIN [24, 25] - see also Chapter 16 in Ref. [26] - while recognizing our debt to pioneering works by ROGULA and KURLANDZKI [27, 28].

2. Inverse-motion description

As we know in relativistic continuum mechanics, the kinematics is best described in terms of the *inverse motion*, that is: if x^α, $\alpha = 1, 2, 3, 4$, x^4 timelike, is the actual placement of a material point X in the Riemannian physical spacetime $\mathcal{V}^4$ with metric $g_{\alpha\beta}$ of Minkowskian signature $+2$, then the matter deformation is described by the (here supposedly) regular mapping [23 - 26]

$$X^K = \overline{X}^K(x^\alpha), \qquad K = 1, 2, 3, \tag{2.1}$$

where X^K designate the local coordinates of the material point X on the *material manifold* $\mathcal{M}^3$, the set of *material points*. The latter has a geometry which in general is part of the solution, i.e., it is induced by the space-time metric. World lines $\mathcal{C}_X$ of material particles X in $\mathcal{V}^4$ are given by the parametrization

$$\mathcal{C}_X : x^\alpha = \overline{x}^\alpha\left(X^K, \tau\right), \tag{2.2}$$

where τ is the so-called *proper time* of X. Local spatial sections of $\mathcal{V}^4$ at $\mathbf{x} \in \mathcal{C}_X$ are defined by means of the *projector* or spatial metric

$$P_{\alpha\beta} = g_{\alpha\beta} + c^{-2} u_\alpha u_\beta = P_{\beta\alpha}, \tag{2.3}$$

a symmetric idempotent operator, where u^α is the four-velocity, a field tangent to $\mathcal{C}_X$ and normalized in such a way that $g_{\alpha\beta}u^\alpha u^\beta + c^2 = 0$, where c is the velocity of light in vacuum. Obviously, $P_{\alpha\beta}$ and u^α satisfy the orthogonality condition $P_{\alpha\beta}u^\beta = 0$. Any space-time geometric object which admits $\mathbf{u}$ as a null vector is said to be "*essentially spatial*". The main ingredient of deformation theory is the *inverse motion gradient* $\mathbf{F}^{-1}$ defined from (2.1) by

$$\mathbf{F}^{-1} := \{\nabla_\mu \mathbf{X}\} = \left\{X^K_\mu \equiv X^K_{,\mu};\ \ K = 1,2,3;\ \ \mu = 1,2,3,4\right\} \tag{2.4}$$

which is such that

$$\mathbf{u}\cdot\mathbf{F}^{-1} = D_u\mathbf{X} = 0, \qquad D_u := u^\alpha\nabla_\alpha = \frac{D}{D\tau}. \tag{2.5}$$

In Eqs. (2.5) D_u denotes the invariant directional derivative or gradient in the u^α-direction.

In essence $(2.5)_1$ means that τ and the X^K are good independent time and space coordinates in the parametrization (2.2). From $\mathbf{F}^{-1}$ one constructs the following space-time invariant which acts as reciprocal deformed metric on $\mathcal{M}^3$ (T – transpose):

$$\mathbf{C}^{-1} := \mathbf{F}^{-1}.\left(\mathbf{F}^{-1}\right)^T, \qquad \text{i.e.} \qquad \mathbf{C}^{-1KL} = X^K_\alpha X^L_\beta g^{\alpha\beta} = X^K_\alpha X^L_\beta P^{\alpha\beta}. \tag{2.6}$$

This establishes the *canonical projection* of $\mathcal{V}^4$ onto $\mathcal{M}^3$. General relativistic elastic materials were first described by means of this procedure in Ref. [24].

3. Balance equations for elastic materials

In a *nondissipative* relativistic background these balance laws consist *a priori* of the law of conservation of *mass* and *energy-momentum*. Let $\varrho_0(\mathbf{X})$ be the mass density at X on $\mathcal{M}^3$ and $\varrho(x^\alpha)$ the matter density at x^α in $\mathcal{V}^4$, where $\mathbf{X}$ and x^α are related by (2.1). These two densities are related by [26]

$$\varrho\,(x^\alpha) = \varrho_0(\mathbf{X})\ \left(\det \mathbf{C}^{-1}\right)^{1/2}. \tag{3.1}$$

For a *purely elastic* material body the other balance laws can be derived from a Hamiltonian – Lagrangian variational principle. To that purpose we consider the following Lagrangian density per unit volume of $\mathcal{V}^4$ at x^α [27]

$$\mathcal{L} = \overline{\mathcal{L}}\,(\mathbf{X}, \nabla_\mu\mathbf{X}), \tag{3.2}$$

where, for the sake of simplicity, we do not envisage metric-dependent effects. The Lagrangian (3.2) describes the response of elastic materials, irrespectively of their anisotropy and material homogeneity, as an explicit dependence on the "particle" X through $\mathbf{X}$ obviously indicates *material inhomogeneity*. The only restriction present in (3.2) is that elasticity manifests only through the *first* gradient of $\mathbf{X}$, and this materializes interactions of a *local* type involving no dispersion (i.e. no characteristic length). In (3.2) according to (2.1), the X^K are the fields and the x^α are the parameters. Thus the *field equations* are given by the following evident Euler–Lagrange equations:

$$\frac{\delta\mathcal{L}}{\delta\mathbf{X}} := \left(\frac{\partial\mathcal{L}}{\partial\mathbf{X}}\right)_{\mathrm{expl}} - \nabla_\mu \frac{\partial\mathcal{L}}{\partial(\nabla_\mu\mathbf{X})} = 0, \tag{3.3}$$

or

$$\nabla_\mu \underline{\Im}^\mu = \mathbf{f}^{\,\mathrm{inh}}, \qquad \text{where} \quad \underline{\Im}^\mu := \frac{\partial\mathcal{L}}{\partial(\nabla_\mu\mathbf{X})}, \quad \mathbf{f}^{\,\mathrm{inh}} := \left(\frac{\partial\mathcal{L}}{\partial\mathbf{X}}\right)_{\mathrm{expl}} \tag{3.4}$$

For all practical purposes $\underline{\Im}^\mu$ is a four-vector in space-time $\mathcal{V}^4$ and $\mathbf{f}^{\,\mathrm{inh}}$ is a co-vector on $\mathcal{M}^3$ (whose components in $\mathcal{V}^4$ are pure scalars!); but $\underline{\Im}^\mu$, just like $\nabla_\mu\mathbf{X}$ but with opposite variance, is a good example of a two-point tensor field, so that Eq. $(3.4)_1$ is indeed a co-vector equation on $\mathcal{M}^3$.

Through Noether's celebrated theorem, the variation of the *parameters* x^α of the description (3.2), yields the *conservation law of energy-momentum* as

$$\nabla_\mu T^\mu_{.\nu} = 0, \tag{3.5}$$

with a *canonical* stress-energy-momentum tensor classically defined by (compare to [7], Sec. 32)

$$T^\mu_{.\nu} = -\left(\mathcal{L}\delta^\mu_\nu - \nabla_\nu\mathbf{X}\cdot\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\mathbf{X})}\right), \tag{3.6}$$

where the dot indicates summation over the K's of $\mathbf{X}$.

On account of (3.1) that is already in integrated form, and the fact that $\mathcal{L}$ must be at least Lorentz-invariant, there follows that

$$P_{\beta[\mu}\nabla_{\nu]}\mathbf{X}\cdot\frac{\partial\mathcal{L}}{\partial(\nabla_\beta\mathbf{X})} = 0, \tag{3.7}$$

where square brackets indicate skew-symmetrization. Equations $(3.4)_1$, and (3.5) exhaust the list of available balance equations. A natural question is whether these last two equations are independent. The answer is negative. Indeed, multiplying $(3.4)_1$ scalarly on $\mathcal{M}^3$ by $\nabla_\nu\mathbf{X}$, integrating by parts and noting that $X^K_{.,\mu\nu} = X^K_{.,\nu\mu}$ (remember that the X^K are just scalars in so far as space-time transformations are concerned), we obtain that

$$(\nabla_\nu\mathbf{X})\cdot\left(\nabla_\mu\underline{\Im}^\mu - \mathbf{f}^{\,\mathrm{inh}}\right) + (\nabla_\mu T^\mu_{.\nu})_\perp = 0, \tag{3.8}$$

where the symbol $(..)_\perp$ means the space-like part obtained by full projection, i.e. in the present case

$$(\nabla_\mu T^\mu_{.\nu})_\perp \equiv P^{.\beta}_\nu \nabla_\mu T^\mu_{.\beta}, \qquad u^\nu(\nabla_\mu T^\mu_{.\nu})_\perp \equiv 0. \tag{3.9}$$

Equation (3.8) means that $(3.4)_1$ entails the spatial part of (3.5). The reciprocal statement is true although its proof is more tedious. Equation (3.8) is a relativistic dynamical statement that generalizes the Ericksen identity known for classical finite-strain elastostatics [29]. The timelike complement of (3.8) is none other than the *energy equation* which obviously reads [23]

$$u^{\nu}\nabla_{\mu}T^{\mu}_{.\nu} = 0, \tag{3.10}$$

so that, instead of (3.8) and (3.10) we could as well write the generalized Ericksen identity:

$$(\nabla_{\nu}\mathbf{X}) \cdot \left(\nabla_{\mu}\underline{\Im}^{\mu} - \mathbf{f}^{\text{inh}}\right) + c^{-2}u_{\nu}\left(u^{\alpha}\nabla_{\mu}T^{\mu}_{.\alpha}\right) + \nabla_{\mu}T^{\mu}_{.\nu} = 0. \tag{3.11}$$

While Eq. (3.5) is properly written in covariant form with space-time parameters and operations, we notice that the same is not true of Eq. $(3.4)_1$ on the material manifold because $\underline{\Im}^{\mu}$ still is a two-point tensor field whereas one would certainly prefer to have at hand an entirely material equation, i.e. a truly *canonical* equation fully independent of the space-time representation. That equation in classical continuum mechanics is the balance of *pseudomomentum* or *canonical material momentum* [11].

4. Balance of pseudomomentum

This equation should involve a Lagrangian density per unit volume in material space $\mathcal{M}^3$ at X and *material time* and space differentiations. Let

$$\nabla^{\perp}_{\alpha} \equiv P^{.\beta}_{\alpha}\nabla_{\beta} = \nabla_{\alpha} + c^{-2}u_{\alpha}D_u\,, \qquad J := \left(\det \mathbf{C}^{-1}\right)^{-1/2}. \tag{4.1}$$

Multiplying $(3.4)_1$ by J we obtain

$$J\left(\nabla^{\perp}_{\mu} + c^{-2}u_{\mu}D_u\right) \cdot \underline{\Im}^{\mu} = \mathbf{f}^{\text{inh}}_0, \tag{4.2}$$

where

$$\mathcal{L}_0 = J\mathcal{L}, \qquad \mathbf{f}^{\text{inh}}_0 = \left(\frac{\partial \mathcal{L}_0}{\partial \mathbf{X}}\right)_{\text{expl}}. \tag{4.3}$$

This should be the equation looked for. This goal is reached by integrating by parts and rearranging terms. We note that

$$\nabla^{\perp}_{\mu} = X^{L}_{\mu}\nabla_L\,, \qquad \nabla_L \equiv \partial/\partial X^L, \qquad \nabla_L\left(JX^{L}_{\mu}\right) \equiv 0. \tag{4.4}$$

On account of $(4.3)_1$ and the fact that J depends on $\mathbf{F}^{-1}$ through $\mathbf{C}^{-1}$, we let the reader prove that

$$J\nabla^{\perp}_{\mu} \cdot \underline{\Im}^{\mu} = \nabla_K b^{K}_{.L}, \qquad b^{K}_{.L} \equiv X^{K}_{\mu}\frac{\partial \mathcal{L}_0}{\partial X^{L}_{\mu}} - \mathcal{L}_0\delta^{K}_{L}\,. \tag{4.5}$$

Notice that the material object **b** defined by $(4.5)_2$ is formally the material analogue of $T^{\mu}_{.\nu}$ (but the latter is not essentially spatial). This is indeed the *material energy-momentum tensor* called *Eshelby stress.* Furthermore, with the obvious condition that $u_\mu \dfrac{\partial \mathcal{L}_0}{\partial(\nabla_\mu \mathbf{X})} = 0$, we show that

$$c^{-2} J u_\mu D_u \left(\frac{\partial \mathcal{L}}{\partial \nabla_\mu X^K} \right) = -c^{-2} \left(\frac{\partial \mathcal{L}_0}{\partial \nabla_\mu X^K} - \mathcal{L}_0 x^\mu_K \right) D_u u_\mu , \tag{4.6}$$

where x^μ_K is such that

$$x^\mu_K = P^\mu_{.\nu} \frac{\partial x^\nu}{\partial X^K}, \qquad x^\mu_K u_\mu = 0, \qquad x^\mu_K X^K_\nu = P^\mu_{.\nu}, \qquad X^K_\nu x^\nu_L = \delta^K_L . \tag{4.7}$$

But

$$\frac{\partial \mathcal{L}_0}{\partial \nabla_\mu X^K} - \mathcal{L}_0 x^\mu_K = \frac{\partial \mathcal{L}_0}{\partial \nabla_\nu X^K} P^\mu_{.\nu} - \mathcal{L}_0 \delta^L_K x^\mu_L = b^L_{.K} x^\mu_L \tag{4.8}$$

as a result of $(4.7)_3$. Thus Eq. (4.2) reads in components

$$\nabla_K b^K_{.L} - c^{-2} b^K_{.L} x^\mu_K D_u u_\mu = \left(f_0^{\text{inh}} \right)_L . \tag{4.9}$$

Because of $(4.7)_2$ this can also be written as

$$\nabla_K b^K_{.L} + c^{-2} b^K_{.L} u_\mu D_u x^\mu_K = \left(f_0^{\text{inh}} \right)_L . \tag{4.10}$$

This is the fully material equation of linear momentum looked for, in which we identify the materially co-variant *inhomogeneity force* $\mathbf{f}_0^{\text{inh}}$ and the *Eshelby material stress* **b**. The material or *pseudo*-momentum, being of inertial origin, is *not* obviously present in this formulation without expliciting the Lagrangian density.

5. Explicit forms

The Lagrangian density should account for the rest and internal energies since kinetic energy is not "apparent" in the relativistic framework. For a generally anisotropic, materially inhomogeneous elastic solid, the internal energy per unit proper mass reads $\epsilon = \overline{\epsilon}\,(\mathbf{X}; \nabla_\mu \mathbf{X})$ or, in *objective form*, i.e. as a form-invariant expression in space-time $\mathcal{V}^4$,

$$\epsilon = \overline{\epsilon} \left(\mathbf{X}; \mathbf{C}^{-1} \right) . \tag{5.1}$$

As a matter of fact, this is an integral of the first-order system provided by the Lorentz-invariance condition (3.7). The result is purely *material*, and is thus invariant by all means in so far as transformations of physical space-time are concerned, that is. whether the latter is Galilean, Minkowskian or Einsteinian (i.e., accounting for general relativistic effects). Such a general invariance was perceived by pioneers of "good" relativistic continuum mechanics such as OLDROYD [30].

The Lagrangian densities $\mathcal{L}_0$ and $\mathcal{L}$ are given by

$$\begin{aligned} \mathcal{L}_0 &= -\varrho_0(\mathbf{X})c^2\left(1+\frac{\in}{c^2}\right), \\ \mathcal{L} &= -\varrho c^2\left(1+\frac{\in}{c^2}\right) = -\varrho_0(\mathbf{X})c^2\left(\det \mathbf{C}^{-1}\right)^{1/2}\left\{1+\frac{1}{c^2}\overline{\in}\left(\mathbf{X};\mathbf{C}^{-1}\right)\right\}. \end{aligned} \tag{5.2}$$

In an *inertial* frame (noted by the equality sign $\stackrel{*}{=}$) at the nonrelativistic limit, expression $(5.2)_1$ yields

$$\mathcal{L}_0 = \varrho_0(\mathbf{X})g_{\alpha\beta}u^\alpha u^\beta\left(1+\frac{\in}{c^2}\right) \stackrel{*}{=} -\varrho_0(\mathbf{X})\left(1+\frac{\in}{c^2}\right)\left(1-\beta^2\right)^{1/2}, \tag{5.3}$$

where $\beta^2 = \mathbf{v}^2/c^2$ if $\mathbf{v}$ is the physical velocity of matter. For small β this yields

$$\mathcal{L}_0 = \varrho_0(\mathbf{X})\frac{\mathbf{v}^2}{2} - \varrho_0 \in\left(\mathbf{X};\mathbf{C}^{-1}\right) = \varrho_0(\mathbf{X})\frac{\mathbf{v}^2}{2} - W\left(\mathbf{X};\mathbf{C}^{-1}\right), \tag{5.4}$$

which is a possible Lagrangian density per unit volume in the reference configuration in classical finite-strain elasticity [14, 20]. In the same approximation where

$$\begin{aligned} g_{\alpha\beta} &\stackrel{*}{=} \mathrm{diag}(+1,+1,+1,-1), \\ u^\alpha &\stackrel{*}{=} (\gamma\mathbf{v},\gamma c), \qquad i=1,2,3, \quad \gamma \equiv \left(1-\beta^2\right)^{-1/2} \end{aligned} \tag{5.5}$$

as β goes to zero we find that Eq. (2.5) reduces to

$$\mathbf{v} + \mathbf{F}_*\cdot\mathbf{V} \stackrel{*}{=} 0, \tag{5.6}$$

where $\mathbf{F}_*$ is the nonrelativistic direct-motion gradient and $\mathbf{V}$ is the so-called *material velocity*. With a general motion now described by either $\mathbf{x} = \chi(\mathbf{X},t)$ or $\mathbf{X} = \chi^{-1}(\mathbf{x},t)$, respectively in the direct and inverse-motion descriptions, $\mathbf{v}$, $\mathbf{F}_*$, $\mathbf{V}$, $\mathbf{F}_*^{-1}$ and $\mathbf{C}_*^{-1}$ are given by (compare [14, 20])

$$\begin{aligned} &\mathbf{v} = \left.\frac{\partial\chi}{\partial t}\right|_{\mathbf{X}}, \qquad \mathbf{F}_* = \left.\frac{\partial\chi}{\partial\mathbf{X}}\right|_t, \qquad \mathbf{V} = \left.\frac{\partial\chi^{-1}}{\partial t}\right|_{\mathbf{x}}, \\ &\mathbf{F}_* = \left.\frac{\partial\chi^{-1}}{\partial\mathbf{x}}\right|_t, \qquad \mathbf{C}_*^{-1} = \mathbf{F}_*^{-1}\left(\mathbf{F}_*^{-1}\right)^T. \end{aligned} \tag{5.7}$$

We readily check that

$$\mathbf{C}^{-1} = \mathbf{C}_*^{-1} - c^{-2}\mathbf{V}\otimes\mathbf{V}, \qquad \det\mathbf{C}^{-1} \stackrel{*}{=} \left(1-\beta^2\right)\det\left(\mathbf{F}_*^{-1}\right)^2 \geq 0. \tag{5.8}$$

The nonrelativistic pseudomomentum, a *material co-vector*, is usually defined by

$$\boldsymbol{\mathcal{P}} = -\varrho_0(\mathbf{X})\mathbf{F}_*^T\cdot\mathbf{v} = \varrho_0(\mathbf{X})\mathbf{C}\cdot\mathbf{V}, \tag{5.9}$$

where $\mathbf{V}$ is such that (cf. Eq. (5.6)) $\mathbf{V} = -\mathbf{F}_*^{-1}\cdot\mathbf{v}$.

We easily check that (cf. Eq. (5.8))

$$\mathbf{C} \stackrel{*}{=} \mathbf{C}_* + \frac{1}{\varrho_0^2 c^2}\boldsymbol{\mathcal{P}}\otimes\boldsymbol{\mathcal{P}}. \tag{5.10}$$

The closest we can come to the definition of a relativistic (material) pseudomomentum is

$$\boldsymbol{\mathcal{P}} \stackrel{\text{def}}{=} -\frac{1}{c^2}\mathcal{L}_0 u_\alpha \frac{\partial x^\alpha}{\partial\mathbf{X}}, \tag{5.11}$$

where x^α and $\mathcal{L}_0$ are given by $x^\alpha = \overline{x}^\alpha(\mathbf{X},\tau)$ and Eq. $(5.2)_1$. This, indeed, reduces to the classical definition $(5.9)_1$ in the nonrelativistic limit. This can be complemented by a fourth – timelike – component $\mathcal{P}_4$ such that $\mathcal{P}_4 = \varrho_0\left(c^2 + \in\right)$, i.e., the total energy density and then both this and (5.11) enter a unique four-dimensional definition ($\Delta = 1,2,3,4$)

$$\mathcal{P}_\Delta = -\frac{1}{c^2}\mathcal{L}_0 u_\alpha \frac{\partial x^\alpha}{\partial X^\Delta}, \qquad X^\Delta = \left(X^K, \tau\right), \qquad u^\alpha \equiv \frac{\partial x^\alpha}{\partial\tau}. \tag{5.12}$$

Returning now to the relativistic expression (5.2) we immediately show that

$$\frac{\partial\mathcal{L}}{\partial X^K_\mu} = -\varrho\frac{\partial\in}{\partial\mathbf{C}^{-1}} : \frac{\partial\mathbf{C}^{-1}}{\partial X^K_\mu} + \frac{1}{2}\varrho_0\left(\det\mathbf{C}^{-1}\right)^{-3/2}\frac{\partial\left(\det\mathbf{C}^{-1}\right)}{\partial X^K_\mu}. \tag{5.13}$$

But

$$\begin{aligned} \frac{\partial\left(\mathbf{C}^{-1}\right)^{MN}}{\partial X^K_\mu} &= \left(\delta^M_K X^N_\alpha + \delta^N_K X^M_\alpha\right)P^{\alpha\mu}, \\ \frac{\partial\left(\det\mathbf{C}^{-1}\right)}{\partial X^K_\mu} &= 2\left(\det\mathbf{C}^{-1}\right)x^\mu_K, \end{aligned} \tag{5.14}$$

from which it follows on account of $\delta^\alpha_\beta - P^\alpha_{.\beta} = -c^{-2}u^\alpha u_\beta$ that

$$T^\mu_{.\nu} = \varrho\left(1 + \frac{\in}{c^2}\right)u^\mu u_\nu - t^\mu_{.\nu}, \tag{5.15}$$

wherein

$$t_{\mu\nu} = -2\varrho \frac{\partial \overline{\in}}{\partial (\mathbf{C}^{-1})^{KL}} X^K_\mu X^L_\nu = t_{\nu\mu}, \quad t_{\mu\nu} u^\nu = 0. \tag{5.16}$$

The latter quantity is the relativistic stress *per se* (an essentially spatial tensor) for a description based on the inverse motion [23, 24, 31], while the energy-momentum tensor (5.15) admits the standard space-time decomposition which, in the absence of heat flow, microstructure, and electromagnetic fields, presents no mixed space-time and time-space elements. From (5.16) we see that the spatial relativistic stress is none other than the "*push forward*" of the *covariant material stress* T_{KL} defined thermodynamically by (Note: this is *not* the second Piola – Kirchhoff stress, which is materially contravariant) $T_{KL} = -2\varrho \dfrac{\partial \overline{\in}}{\partial (\mathbf{C}^{-1})^{KL}}$.

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