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FIFTY YEARS OF THE ARCHIVES OF MECHANICS

Some properties of connections on iterated tangent bundles

Dedicated to Prof. Henryk Zorski
on the occasion of his 70-th birthday

M. KUREŠ (BRNO)

POSSIBILITIES of a generalization of the original Grifone's approach to connections are studied. Semisprays associated to connections and torsions on the iterated tangent bundle TTM are described.

1. Introduction

MANY PAPERS dealing with the theory of connections, semisprays and mutual relations are motivated by mechanics, especially by Lagrangian dynamics. GRIFONE [5], gave a new definition of the connection on a manifold M as a certain special $(1,1)$-tensor field on TM. There is a semispray with the same paths associated with each such connection. The converse is also proved; with each semispray we can associate a connection and hence we are provided with connections attached to regular Lagrangians. DE ANDRÉS, DE LEÓN and RODRIGUES [2, 3], generalized Grifone's results to tangent bundles of higher order T^rM. The applicability of their construction in higher order mechanics is known.

We study the case of iterated tangent bundle TTM, especially the possibilities of a further generalization of Grifone's results. A nice discussion of the structure of the bundle TTM is presented in the monograph of ABRAHAM and MARSDEN [1], where it is also possible to find mechanical interpretations. Furthermore, we present torsions of connections on TTM defined by the Fröhlicher-Nijenhuis bracket of the associated horizontal projection and natural affinors on this bundle. The torsion is also a subject of interest for mathematical physics. It plays an important role in the Einstein-Cartan theory of gravitation, in modern development of field theories, etc. Our approach makes use of the theory of natural operations in differential geometry. The techniques of finding the natural objects are studied in the monograph of KOLÁŘ, MICHOR and SLOVÁK [7].

2. Natural affinors and natural vector fields

In general, by an *affinor* A on a manifold M we mean a (1,1)-tensor field, i.e. a linear morphism $A : TM \to TM$ over id_M. A *natural affinor* on a natural bundle F over m-dimensional manifolds is a system of (1,1)-tensor fields $A_M : TFM \to TFM$ for every m-dimensional manifold M satisfying $TFf \circ A_M = A_N \circ TFf$

for every local diffeomorphism $f : M \to N$. Analogously, a *natural vector field* on a natural bundle F over m-dimensional manifolds is a system of vector fields $\xi_M : FM \to TFM$ for every m-dimensional manifold M satisfying $TFf \circ \xi_M = \xi_N \circ TFf$ for every local diffeomorphism $f : M \to N$. The natural vector fields on F can be interpreted as the so-called absolute natural operators $C^\infty TM \to C^\infty TFM$ transforming vector fields on M into vector fields on FM. Now, let G be another natural bundle such that $\pi : FM \to GM$ is a fibered manifold (it is possible that $GM = M$). We denote by $V^\pi FM \subset TFM$ the vertical bundle with respect to the tangent projection $T\pi$. If A is an affinor such that $\operatorname{im} A \in V^\pi FM$ and $\operatorname{rank} A = \dim V^\pi FM$, then we call it the *$\pi$-affinor*. Analogously, if ξ is a vector field such that $\operatorname{im}\xi \in V^\pi FM$ and $\operatorname{rank}\operatorname{im}\xi = \dim V^\pi FM$, then we call it the *$\pi$-vector field.*

Weil bundles, [13], which we can see as generalizations of bundles T_k^r of k-dimensional velocities of order r, play an important role in the theory of natural operations, cf. [5]. We call reader's attention to Kolář's papers [6, 8], in which all natural affinors and all natural vector fields, respectively, on an arbitrary Weil bundle are described. In the case of TTM, we obtain these natural objects in a geometrical way here. Because the iterated tangent bundle is the Weil bundle, our results are included in Kolář's classifications for $F = TT$ and represent their geometrical interpretations.

We denote by $\pi^1 : TM \to M$ the tangent bundle of a smooth m-dimensional manifold M. The *(second) iterated tangent bundle* $TTM = T(TM)$ obtained by the additional application of the functor T disposes of the following bundle structures: $\pi^2 : TTM \to M$, $\pi_1^1 := \pi_{TM}^1 : T(TM) \to (TM)$, ${}_1\pi^1 := T\pi_M^1 : T(TM) \to T(M)$ (for more details see e.g. [14]). Given some local coordinates x^i on M, let us denote by x^i, y^i the induced coordinates on TM and by x^i, y^i, X^i, Y^i the induced coordinates on TTM. Then $\pi^2 : (x^i, y^i, X^i, Y^i) \mapsto (x^i)$, $\pi_1^1 : (x^i, y^i, X^i, Y^i) \mapsto (x^i, y^i)$, ${}_1\pi^1 : (x^i, y^i, X^i, Y^i) \mapsto (x^i, X^i)$.

Firstly, we shall construct canonical affinors on TTM. We have the following exact sequence of vector bundles over TTM:

$$0 \longrightarrow V^{\pi^2}TTM \xrightarrow{\ i\ } TTTM \xrightarrow{\ s\ } TM \times_M TTM \longrightarrow 0.$$

Because there exists the canonical isomorphism

$$h : \pi^{2*}TTM = TM \times_M TTM \longrightarrow V^{\pi^2}TTM,$$

the canonical affinor is defined $A^{\pi^2} = i \circ h \circ s$. Similarly, if we take the exact sequences of vector bundles

$$0 \longrightarrow V^{\pi_1^1}TTM \longrightarrow TTTM \longrightarrow TTM \times_{TM} TTM \longrightarrow 0,$$
$$0 \longrightarrow V^{{}_1\pi^1}TTM \longrightarrow TTTM \longrightarrow TTM \times_{TM} TTM \longrightarrow 0,$$

and isomorphisms

$$\pi_1^{1*}TTM = TTM \times_{TM} TTM \longrightarrow V^{\pi_1^1}TTM,$$
$$_1\pi^{1*}TTM = TTM \times_{TM} TTM \longrightarrow V^{_1\pi^1}TTM,$$

we obtain the canonical π_1^1-affinor $A^{\pi_1^1}$ and the canonical $_1\pi^1$-affinor $A^{_1\pi^1}$, respectively. Furthermore, let A_0 denote the identical affinor. All natural affinors on TTM constitute a 4-parameter family linearly generated by A_0, $A^{\pi_1^1}$, $A^{_1\pi^1}$, A^{π^2}.

Secondly, we shall construct canonical vector fields on TTM. For the projection π^2 let $\alpha := \pi_1^1 \times_M \mathrm{id}_{TTM}$, $\beta := {}_1\pi^1 \times_M \mathrm{id}_{TTM}$, be two canonical sections of the vector bundle $TM \times_M TTM$; the canonical vector fields are defined $\xi_1^{\pi^2} = i \circ h \circ \alpha$, $\xi_2^{\pi^2} = i \circ h \circ \beta$. Similarly, for the projection π_1^1 ($_1\pi^1$, respectively) we take the canonical section $\gamma := \mathrm{id}_{TTM} \times_{TM} \mathrm{id}_{TTM}$, and by the analogous composition we obtain the π_1^1-vector field $\xi^{\pi_1^1}$ ($_1\pi^1$-vector field $\xi^{_1\pi^1}$, respectively). All natural vector fields on TTM constitute a 4–parameter family linearly generated by $\xi^{\pi_1^1}$, $\xi^{_1\pi^1}$, $\xi_1^{\pi^2}$, $\xi_2^{\pi^2}$.

REMARK 1. The Weil algebra for TT is $D_2 = D \otimes D$ (D is the algebra of dual numbers) with generators 1, δ_1, δ_2 and with relations $\delta_1^2 = \delta_2^2 = 0$.

REMARK 2. It is easy to verify identities $A^{\pi_1^1} \circ A^{_1\pi^1} = A^{_1\pi^1} \circ A^{\pi_1^1} = A^{\pi^2}$, $A^{\pi_1^1} \circ \xi^{_1\pi^1} = \xi_1^{\pi^2}$, $A^{_1\pi^1} \circ \xi^{\pi_1^1} = \xi_2^{\pi^2}$, etc.

3. Grifone's connections

Trying to follow the original Grifone's procedure, we must put necessary general questions. A *Grifone's semiconnection* on an arbitrary fibered manifold $Y \to M$ means any $(1,1)$-tensor field $\widehat{\Gamma}$ on Y satisfying

$$J_A\widehat{\Gamma} = J_A\,,$$
$$\widehat{\Gamma} J_B = -J_B\,,$$

for any chosen natural affinors J_A, J_B on Y. A Grifone's semiconnection $\widehat{\Gamma}$ is said to be the *Grifone's connection*, if the $(1,1)$-tensor field $\widehat{\gamma} = \frac{1}{2}(\mathrm{id} + \widehat{\Gamma})$ corresponds to the horizontal lifting $\gamma : Y \times_M TM \to TY$ of the general connection Γ which is defined as a section $\Gamma : Y \to J^1Y$ of the first jet prolongation of $J^1Y \to Y$ of Y, see [12]. In this case J_A and J_B are called *A-affinor* and *B-affinor*, respectively. Let us study conditions of their existence.

PROPOSITION 1. An A-affinor exists on an arbitrary Weil bundle.

P r o o f. We denote by x^i the local coordinates on M, $i = 1, \ldots, m$, and by y^p, the fiber coordinates on Y, $p = 1, \ldots, n$, and $z = 1, \ldots, m, m+1, \ldots, m+n$.

The corresponding horizontal lifting of a general connection Γ has the coordinate form

$$dy^p = \Gamma_i^p dx^i.$$

By a direct application of the definition of the A-affinor we obtain the coordinate form of J_A as

$$A_j^i \frac{\partial}{\partial x^i} \otimes dx^j + A_i^p \frac{\partial}{\partial y^p} \otimes dx^i.$$

Moreover, the rank A_i^z must be maximal, i.e. it equals m. Really, if rank $A_i^z < m$, than there exists at least one zero column $A_{i_0}^z$ or it is possible to obtain it after linear transformations. But it means $dx^{i_0} = \Gamma_j^{i_0} dx^j + \Gamma_p^{i_0} dy^p$, where we can take $\Gamma_j^{i_0}$, $\Gamma_p^{i_0}$ arbitrarily.

Further, elements of Weil algebra are sums of monomials of the form

$$a_{p_1 \dots p_k} \delta_1^{p_1} \dots \delta_k^{p_k},$$

where a are real numbers and δ are generators. There exist so-called maximal monomials, i.e. non-zero monomials, which vanish after multiplication by an arbitrary δ_i. For any such maximal monomial

$$a_{\bar{p}_1 \dots \bar{p}_k} \delta_1^{\bar{p}_1} \dots \delta_k^{\bar{p}_k},$$

we put $a_{\bar{p}_1 \dots \bar{p}_k} = 1$ and all other numbers $a = 0$. Natural affinors correspond to the multiplication by the elements of Weil algebra. Multiplication by our special element provides an affinor satisfying conditions for the coordinate form and for the rank, too. □

PROPOSITION 2. *B-affinors are just natural π-affinors (with respect to the investigated projection π).*

P r o o f. As in the proof of the previous proposition, we make sure that the coordinate form of J_B is

$$B_i^p \frac{\partial}{\partial y^p} \otimes dx^i + B_q^p \frac{\partial}{\partial y^p} \otimes dy^q,$$

and rank B_z^p must be maximal, i.e. it equals n. □

REMARK 3. A-affinors and B-affinors also exist on T^*M, TT^*M, higher order cotangent bundles $T^{r*}M$ and higher order frame bundles P^rM, cf. [8, 4, 9, 10]. The A-affinor exists on TTM, it is the affinor A^{π^2}. But we immediately see, that B-affinor does not exist on TTM.

We can state that using of the definition of the general connection Γ as a section $\Gamma : Y \to J^1Y$ of the first jet prolongation of $J^1Y \to Y$ of Y, removes the problem of the incidental non-existence of Grifone's connection.

4. Connections and associated semisprays

We denote by Γ^{π} a connection with respect to the projection π. So, we have connections Γ^{π^2}, $\Gamma^{\pi_1^1}$, $\Gamma^{1\pi^1}$ on TTM. But we confine ourselves to connections with respect to the projection π^2 it this chapter, because the cases $TTM \to TM$ represent, roughly speaking, the case $TM \to M$ described by Grifone. We shall call *tensions* of Γ^{π^2} the $(1,1)$-tensor fields $H_1^{\pi^2}$, $H_2^{\pi^2}$ on TTM given by $H_1^{\pi^2} = [\xi^{1\pi^1}, \Gamma^{\pi^2}]$, $H_2^{\pi^2} = [\xi^{\pi_1^1}, \Gamma^{\pi^2}]$.

Vector fields $\sigma_1^{\pi^2}$, $\sigma_2^{\pi^2}$ on TTM are said to be *semisprays* if $A^{\pi^2} \circ \xi_1^{\pi^2} = \sigma_1^{\pi^2}$, $A^{\pi^2} \circ \xi_2^{\pi^2} = \sigma_2^{\pi^2}$, respectively. We shall call *deviation* of $\sigma_1^{\pi^2}$ ($\sigma_2^{\pi^2}$) the vector field $\sigma_1^{\pi^2*}$ ($\sigma_2^{\pi^2*}$) defined by $\sigma_1^{\pi^2*} = [\xi^{1\pi^1}, \sigma_1^{\pi^2}] - \sigma_1^{\pi^2}$ ($\sigma_2^{\pi^2*} = [\xi^{\pi_1^1}, \sigma_2^{\pi^2}] - \sigma_2^{\pi^2}$, respectively).

A $(1,l)$-tensor field L on TTM, with $l \geq 1$, is said to be *semibasic*, if

(1) $L(\xi_1, \ldots, \xi_l) \in V^{\pi^2}TTM$, for every vector fields $\xi_1, \ldots, \xi_l$ on TTM, and

(2) $L(\xi_1, \ldots, \xi_l) = 0$, if ξ_1 belongs to $V^{\pi^2}TTM$.

Let L be a semibasic $(1,l)$-tensor field. We call *potentials* of L the semibasic $(1, l-1)$-tensor fields L_1^0, L_2^0 given by $L_1^0 = \mathrm{i}_{\sigma_1^{\pi^2}} L$, $L_2^0 = \mathrm{i}_{\sigma_2^{\pi^2}} L$.

Let $\sigma_1^{\pi^2}$, $\sigma_2^{\pi^2}$ be arbitrary semisprays. We denote the horizontal projector of Γ^{π^2} by the same symbol. Let us consider the semisprays $\underline{\sigma_1}$, $\underline{\sigma_2}$ given by $\underline{\sigma_1} = \Gamma^{\pi^2} \circ \sigma_1^{\pi^2}$, $\underline{\sigma_2} = \Gamma^{\pi^2} \circ \sigma_2^{\pi^2}$, respectively. These semisprays are said the *first* and the *second associated semisprays* to Γ^{π^2}.

PROPOSITION 3. For any connection Γ^{π^2} on $TTM \to M$ and their associated semisprays $\underline{\sigma_1}$, $\underline{\sigma_2}$, the identities

$$\underline{\sigma_1}^* = (H_1^{\pi^2})_1^0,$$
$$\underline{\sigma_2}^* = (H_2^{\pi^2})_2^0,$$

are satisfied.

P r o o f. If

$$dy^i = F_j^i dx^j,$$
$$dX^i = G_j^i dx^j,$$
$$dY^i = H_j^i dx^j,$$

are the local equations of Γ^{π^2}, then a direct evaluation gives the following coordinate expression of $(H_1^{\pi^2})_1^0$:

$$dy^i = y^j y^k \frac{\partial F_k^i}{\partial y^j} + Y^j y^k \frac{\partial F_k^i}{\partial Y^j} - y^k F_k^i,$$

$$dX^i = y^j y^k \frac{\partial G^i_k}{\partial y^j} + Y^j y^k \frac{\partial G^i_k}{\partial Y^j},$$
$$dY^i = y^j y^k \frac{\partial H^i_k}{\partial y^j} + Y^j y^k \frac{\partial H^i_k}{\partial Y^j} - y^k H^i_k.$$

If we evaluate the deviation of the first associated semispray

$$\sigma_1^{\pi^2} \equiv y^i \otimes \frac{\partial}{\partial x^i} + F^i_j y^j \otimes \frac{\partial}{\partial y^i} + G^i_j y^j \otimes \frac{\partial}{\partial X^i} + H^i_j y^j \otimes \frac{\partial}{\partial Y^i},$$

we come directly to the same expression. The procedure is the same for the second associated semispray. Application of local coordinates is not necessary, see the proof of an analogous assertion for T^2M in [11]. □

A parametric curve $c : I \to M$ is called a *path* of a connection Γ, if $j^1(j^1c)$ is a horizontal curve in TTM. The connection Γ is said to be *homogeneous* if its tension vanishes. Paths of a homogeneous connection are called *geodesics*.

A *path* of a semispray σ is a parametric curve $c : I \to M$ such that $j^1(j^1c)$ is an integral curve of σ. A semispray σ is called *spray*, if σ has zero deviation. If σ is a spray then their paths are called *geodesics*.

PROPOSITION 4. The paths of a connection Γ are the same as the paths of the first and the second associated semisprays.

P r o o f. The paths of a connection Γ^{π^2} satisfy the system of ordinary differential equations

$$\frac{d^2x^i}{dt^2} = F^i_j \frac{dx^j}{dt},$$
$$\frac{d^2x^i}{dt^2} = G^i_j \frac{dx^j}{dt},$$
$$\frac{d^3x^i}{dt^3} = H^i_j \frac{dx^j}{dt}.$$

If we evaluate the paths of the first (of the second) associated semispray, we come to the same equations. □

5. Torsions

We recall that the Fröhlicher - Nijenhuis bracket $[\Gamma, A]$ of Γ and an arbitrary natural affinor is called the *(general) torsion* of Γ, see [8, 10]. The geometrical interpretation of such general torsions may be complicated and their applicability may be very questionable. That is why we study only *weak torsions* as a special case here: they represent brackets of a type $[\Gamma^\pi, A^\pi]$. Thus, we have three weak torsions on TTM: $t^{\pi^1_1} = [\Gamma^{\pi^1_1}, A^{\pi^1_1}]$, $t^{1\pi^1} = [\Gamma^{1\pi^1}, A^{1\pi^1}]$, $t^{\pi^2} = [\Gamma^{\pi^2}, A^{\pi^2}]$.

REMARK 4. We differ from original Grifone's notations of torsions purposely, because we prefer the definition from [8].

PROPOSITION 5. The weak torsion t^{π^2} has the coordinate expression

$$\frac{\partial F^i_k}{\partial Y^j} dx^j \wedge dx^k \otimes \frac{\partial}{\partial y^i} + \frac{\partial G^i_k}{\partial Y^j} dx^j \wedge dx^k \otimes \frac{\partial}{\partial X^i} + \frac{\partial H^i_k}{\partial Y^j} dx^j \wedge dx^k \otimes \frac{\partial}{\partial Y^i} .$$

Geometrically, we can characterize t^{π^2} by a bracket expression

$$t^{\pi^2}(\eta, \theta) = [\Gamma^{\pi^2}\eta, \pi^{2*}\theta] - [\Gamma^{\pi^2}\theta, \pi^{2*}\eta] - \pi^{2*}[\eta, \theta]$$

for every vector fields η, θ on M.

P r o o f. We obtained the formula by a direct evaluation of the Fröhlicher–Nijenhuis bracket in local coordinates. The idea of geometrization is from [8]. □

Because analogous calculations for $t^{\pi^1_1}$ and $t^{1\pi^1}$ are equally technical, we do not go into details here.

The *strong torsion* of Γ is the (1,1)-tensor field T given by $H + t^0$, where H is the tension of Γ and t^0 is the potential of the weak torsion of Γ. So we obtain $T^{\pi^2}_1 = H^{\pi^2}_1 + (t^{\pi^2})^0_1$, $T^{\pi^2}_2 = H^{\pi^2}_2 + (t^{\pi^2})^0_2$ for Γ^{π^2}. Closing this paper, we view the coordinate expression of $T^{\pi^2}_1$

$$\begin{aligned} &\left(y^j\left(\frac{\partial F^i_k}{\partial y^j} + \frac{\partial F^i_k}{\partial Y^j} - \frac{\partial F^i_j}{\partial Y^k}\right) + Y^j \frac{\partial F^i_k}{\partial Y^j} - F^i_k\right) dx^k \otimes \frac{\partial}{\partial y^i} \\ &+ \left(y^j\left(\frac{\partial G^i_k}{\partial y^j} + \frac{\partial G^i_k}{\partial Y^j} - \frac{\partial G^i_j}{\partial Y^k}\right) + Y^j \frac{\partial G^i_k}{\partial Y^j}\right) dx^k \otimes \frac{\partial}{\partial X^i} \\ &+ \left(y^j\left(\frac{\partial H^i_k}{\partial y^j} + \frac{\partial H^i_k}{\partial Y^j} - \frac{\partial H^i_j}{\partial Y^k}\right) + Y^j \frac{\partial H^i_k}{\partial Y^j} - H^i_k\right) dx^k \otimes \frac{\partial}{\partial Y^i} \end{aligned}$$

as a show-piece, but we recall that some bundle projections provide to associate a connection to a given semispray and a strong torsion, and it is the way how to construct a connection having the generalized Euler–Lagrange vector field as an associated semispray, [3].

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DEPARTMENT OF MATHEMATICS,
TECHNICAL UNIVERSITY OF BRNO,
Technická 2, 616 69 Brno, Czech Republic
e-mail: kures@mat.fme.vutbr.cz

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