

Hyperbolic framework for thermoelastic materials

*Dedicated to Prof. Henryk Zorski
on the occasion of his 70-th birthday*

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A THERMODYNAMIC framework of a deformable continuum is developed in which the conservative state variable vector is enlarged by adding the spatial gradient of a scalar thermal internal state variable responsible for the description of thermal history effects. The theory leads to a modified model of thermoelasticity with internal state variables and with a wave-type heat conduction governed by a system of quasi-linear hyperbolic equations. In a general non-deformable case, the observed material properties such as specific heat, quasi-equilibrium thermal conductivity and speed of thermal (the so-called second sound) waves, all regarded as functions of ϑ , lead to a particular specification of all material functions and the evolution equation for the scalar internal state variable. The short review of the heat pulse experiment is made. Main assumptions of the present approach are formulated and some arguments referred to the rate-type description are presented. A set of remarks and comparisons with another modification of the Fourier law characterizes the model. An analysis of hyperbolicity of thermoelasticity by propagation conditions of weak discontinuity waves is performed.

1. Introduction

MANY physical experiments at low temperatures and technological situations at moderate and high temperatures show the necessity of taking into account the wave structure of the heat transport. Those experiments performed at low temperatures dealt with

- second sound in dielectrics, e.g. [29, 30, 52, 61, 63], and
- second sound in solid helium, e.g. [2].

On the other hand, those technological situations at moderate and high temperatures are related to

- the distribution of temperature around propagating⁽¹⁾ cracks [68, 69, 73], and
- temperature distribution in solid materials due to laser pulse train of a very short duration (in the picosecond range), e.g. [26].

The aim of this paper is to build a modified framework of thermomechanical theory of heat conducting and deformable materials in which the structure of the set of governing equations is of hyperbolic type; it means that both disturbances:

⁽¹⁾ For continuum damage mechanics bibliography and its trends, compare [10].

mechanical and thermal ones propagate with finite speeds. Moreover, our further aim is to model quantitatively the observed wave phenomena in solids and fluids.

The organization of the paper is as follows. In Sec. 2 a typical experiment performed at low temperature for solid crystals will be shortly presented together with its results. In Sec. 3 the rate-type approaches most often used in the literature of thermodynamic theories with hyperbolic-type heat conduction equation will be discussed. Then in Sec. 4 some arguments supporting the internal state variable approach in the case of a non-deformable continuum will be given. From them general forms of evolution equation for the scalar internal state variable and heat flux constitutive equation, both based on the thermal history effects, will be more evident. Some particular forms of the evolution equation are presented. Then the developed model is characterized by a set of remarks and comparison with the extended thermodynamic approach of Morro and Ruggeri. In Sec. 5 the thermomechanical framework will be outlined. Then in Sec. 6 the hyperbolicity of the thermoelasticity is analysed with the help of weak discontinuity wave. Propagation conditions are formulated together with the general form of the system of governing equations suitable for the analysis of strong discontinuity waves.

2. Heat pulse experiment

According to the authors of [18], the heat pulse experiment was first introduced into crystal physics by GUTFELD and NETHERCOT [25]. The experiment was intended to detect second sound. The possibility that under certain conditions, a temperature wave might propagate in a good thermal conductor was first considered in detail by WARD and WILKS [71], following an earlier suggestion by PESHKOV [64]. The authors of [25], however, failed to see the evidence of the second sound.

Finite speed thermal waves, known collectively as second sound, distinguishing them from generally faster propagating mechanical waves, were first detected in ^3He , ([2]), and then in high purity dielectric crystals of sodium fluoride, NaF, [29], and bismuth, Bi, [61].

For heat pulse experiments the NaF and Bi single crystals of high chemical and isotopic purity were used (cf. [30, 52, 61, 66]). Since both crystals have a particular symmetry, the authors performed experiments for several different samples geometries with path lengths varying between 3.6 and 13 mm, and for pulse propagation direction along $\langle 100 \rangle$, $\langle 110 \rangle$ and $\langle 111 \rangle$.

The heat pulse experiments were performed in a simple metal cryostat [66]. The sample was mounted in a vacuum on stainless-steel post anchored to the helium bath which could be pumped to 1.8 K. A heater on the post allowed the sample temperature to be raised above the bath temperature, a carbon resistance thermometer mounted on the crystal provided a measure of its temperature.

Two opposite surfaces of the crystal were covered by an evaporation technique with thin metallic films, which serve as heater and bolometer, respectively. The distance between those surfaces varied from 3.5 to 13 mm.

Pure Pb was chosen for the bolometer film. The lead detector was biased with a constant current so that resistance changes gave rise to signal voltage across the film. The change of the resistance of the metallic film at the other side of the crystal was taken as a measure for the temperature. A current pulse generator supplied the impulse to the crystal heater; its duration varied between 0.2 and 0.5 μ s. The signals from the lead detector were usually of a sufficient amplitude to allow their direct display on an oscilloscope after some wide-band preamplification. The temperature rise of the bolometer never exceeded 1 K [52]. Signals were recorded graphically. The arrival time of the leading edges of the pulses were measured. At low temperatures (i.e. at 12 K for NaF and 1.38 K for Bi) the fastest peak, denoted by L , travelled with the speed of a longitudinal sound wave. The peak which arrives next, denoted by T , travelled with the speed of a transversal sound wave (cf. Fig. 1).

This behaviour is interpreted theoretically as the ballistic phonon effect: phonons⁽²⁾ representing eigenvibrations of the atom of a crystal lattice, travel through the crystal without any interaction. On the other hand, in the framework of thermoelasticity this behaviour can be viewed as a thermomechanical coupling in acoustic wave propagation: due to the plane strain condition, two sound waves: longitudinal and transversal, may propagate under such condition. They were initiated by the temperature rise at one end (side) of the crystal, while the temperature measurement on the other end (side) reported the arrival signal of thermomechanical wave, propagating with those sound speeds. No mechanical variables, like strain, stress or particle velocity were measured in the heat pulse experiments.

The experiments were repeated at different bath temperatures, and no particular dependence of those sound speeds were reported. However, in some range of temperature, an extra peak following those two already observed peaks were detected (cf. Fig. 1). That third peak has been interpreted as partially developed second sound. It is interesting that the second sound peaks in Fig. 1 are followed by long tails; they could represent the diffusive part of the heat pulse ([18, 61]).

In the case of the NaF crystals it seen from the plots of heat-pulse arrival time *vs* temperature, that the so-called ballistic longitudinal and transverse pulses denoted by L and T , respectively, occur at the lowest temperatures, and that the second-sound pulse begins to appear at 10 K.

⁽²⁾ In the phonon model the actual crystal is replaced by a box containing a gas of phonons: i.e. a gas of quasi-particles bearing energy and momentum. Transport processes in the crystal are treated analogously to transport processes in the gas. Phonons may interact among themselves as well as with lattice imperfections and with boundaries of the crystal. In the so-called N(normal)-processes phonon momentum is conserved, in contrast to the so-called R(resistive)-processes, that do not conserve the momentum.

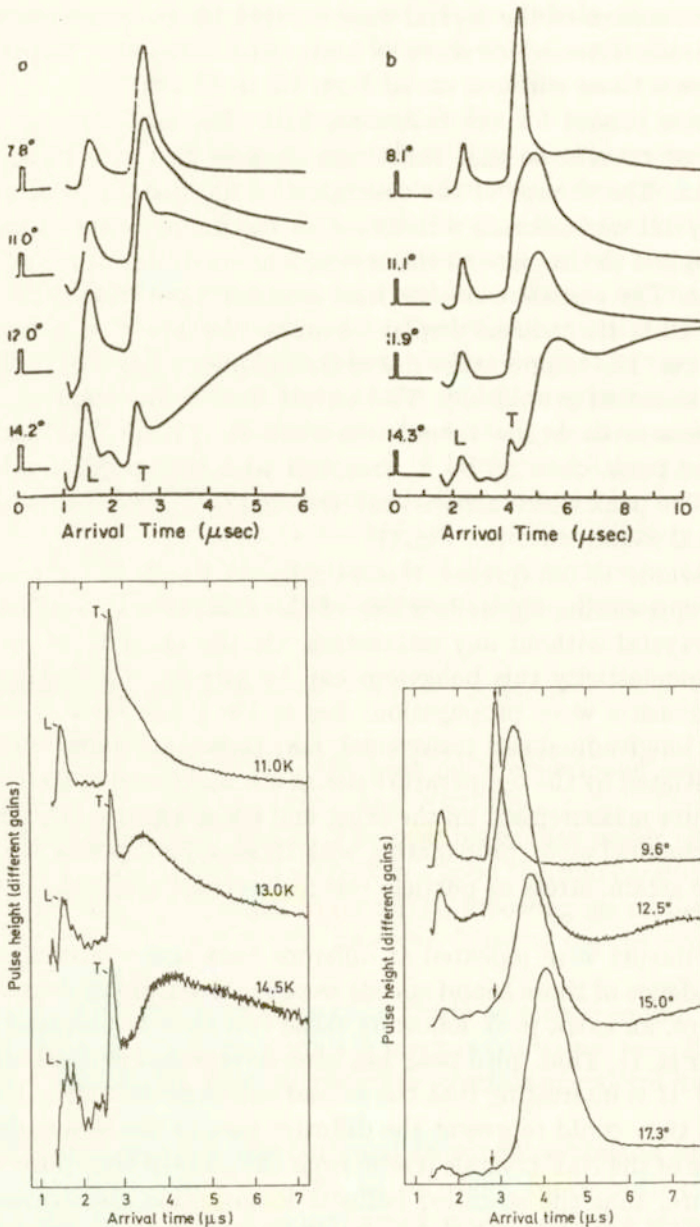


FIG. 1. Heat pulses in a pure NaF sample (a - $l = 7.9$ mm, b - $l = 12.7$ mm), after [29] and [52]. *L* and *T* mark the peaks of the longitudinal and transverse ballistic pulses, respectively. The third distinct pulse (second sound) appears at higher temperatures.

It was there observed (cf. Fig. 2, and figures from [16] and the review article [18]), that in the range of low temperature at which experiments have been performed, the time of arrival of the third heat pulse is approximately a linear

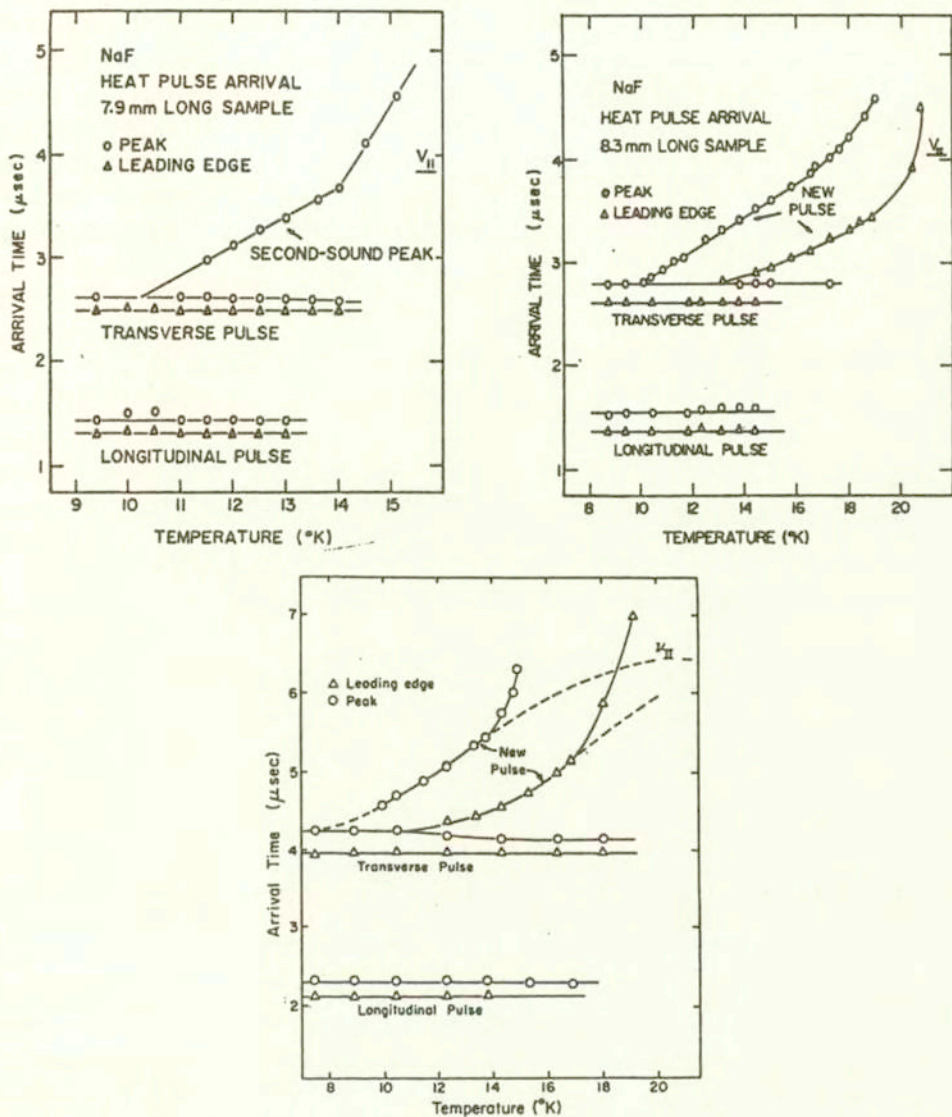


FIG. 2. Arrival times versus temperatures for leading edges and peaks of longitudinal ballistic pulse, the transverse ballistic pulse, and the second-sound pulse for the sample of Fig. 1, after [29] and [52] ($l = 12.7 \text{ mm}$, $\nu_{II} = 2 \times 10^5 \text{ cm s}^{-1}$).

function of the reference (bath) temperature. However, near the upper limit of the measured bath temperature values, the arrival time of the leading edge of heat pulses rises rapidly with increasing temperature. The latter corresponds to a very fast decay (with respect to temperature) of the second sound speed. We can repeat after the authors of [52]: "Eventually a temperature is reached at which this pulse disappears into the diffusive signals".

Experimental investigations with the second sound reported by the authors of [29, 30, 66] have been preceded with steady-state thermal-conductivity measurements. Figure 3 shows thermal conductivity versus temperature data for several samples of different purity. For the heat conductivity a very strong temperature dependence has been observed. Even more: for both crystals NaF and Bi, there is a particular temperature at which the conductivity of the material reaches a peak. The thermal conductivity is highest for purest NaF fourth-regrowth sample A (cf. Fig. 3.)

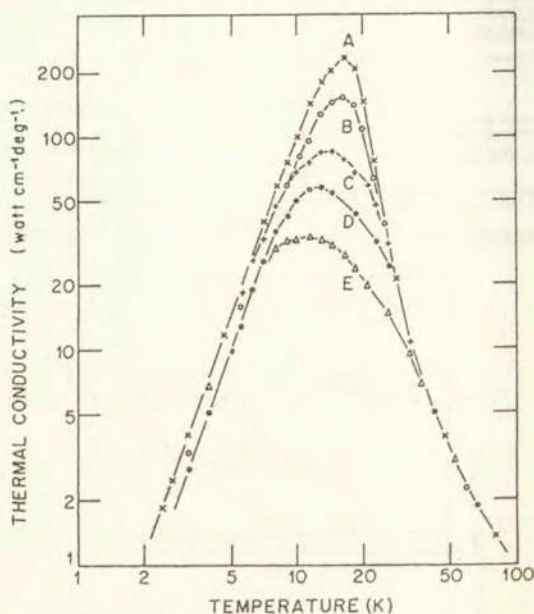


FIG. 3. Steady-state heat conductivity of NaF crystals of different purity, after [30].

Comparing those experimental results on the second sound and heat conductivity, one could derive a possible conclusion, that the second sound phenomenon appears more strongly close to a particular temperature at which the conductivity of the material reaches a peak [30]. In our model developed further in the paper, motivated by the experimental data, and trying to fit experiments for the second sound speed in NaF, we make in [43, 67] the hypothesis that the temperature of maximum heat conductivity coincides with that below which second sound appears. Above this temperature value the heat conduction becomes purely diffusive, obeying a general nonlinear Fourier law. We have called this *critical temperature* ϑ_{λ} .

There is an additional possible explanation why in the first range of low temperatures, the thermal pulse of the second sound were not observed. Namely, at that range of temperature, the speed of the second sound is very close to the speed of the transversal wave, and both the peaks are reported at the same

time as a rather big peak (cf. Figs. 2, 3 for NaF for temperature values 8.1 K, 11.1 K, 11.9 K).

3. Rate-type approach to thermal waves

The above cited experimental facts contradict the results of phenomenological approach to the heat transfer based on a proportionality law between the heat flux and the temperature gradient, called the Fourier law. That law incorporated with the balance of energy leads to a parabolic equation for the temperature field, in solutions of which heat pulses are transmitted with infinite speed. Even including mechanical fields leading to thermoelasticity, the final system of equations is hyperbolic only in a mechanical variable – the displacement vector, being still parabolic in terms of the temperature field. Consequently, only ballistic thermo-mechanical wave propagation could be described in such a model. No possibility exists for describing the second sound, i.e. thermal wave (i.e. the third peak in experimental results) then exists.

There are several phenomenological approaches aimed at the new heat conduction equation, that could describe that phenomena. Some of them are reviewed in the papers [31, 32].

In order to give more reasons for the present approach, we start with referring to the rate-type methods leading to a kind of telegraph equation for the temperature field, which is the second order hyperbolic equation.

Let us start our short presentation with the following

OBSERVATION.

Assume that a non-deformable material is characterized by its energy function e that depends on the thermodynamic temperature ϑ . If for the heat flux \mathbf{q} one incorporates a classical *Fourier proportionality law*

$$(3.1) \quad \mathbf{q} = -k(\vartheta)\nabla\vartheta$$

then, as a consequence of the *balance of energy*

$$(3.2) \quad \rho\dot{e}(\vartheta) + \operatorname{div} \mathbf{q} = \rho r,$$

where ρ represents the mass density and r – the body heat supply, per unit time, one gets a *parabolic nonlinear heat conduction equation*

$$(3.3) \quad \rho c_v(\vartheta)\dot{\vartheta} = \operatorname{div} (k(\vartheta)\nabla\vartheta) + \rho r \quad \text{with} \quad c_v(\vartheta) = \frac{d\varepsilon(\vartheta)}{d\vartheta},$$

which turns out to be a *diffusion equation*

$$(3.4) \quad \rho c_v \dot{\vartheta} = k\Delta\vartheta + \rho r,$$

for constant coefficients, i.e. thermal disturbances propagate with infinite speed. The same is true if additionally one introduces the dependence of the free energy function $\psi = \varepsilon - \vartheta\eta$, where η is the specific entropy, on ϑ and on its gradient $\nabla\vartheta$. This is due to the *second law of thermodynamics*.

Hence, to incorporate a *wavelike phenomenon* (hyperbolicity) in the heat transfer, one needs to enlarge the set of thermal state variables $(\vartheta, \nabla\vartheta)$ in the energy constitutive function by additional quantities. Such an enlargement will cause of course an additional source of dissipation.

There is the well-known method which can lead to a wavelike heat conduction, namely a rate-type approach, in which the Fourier law (3.1) is modified by adding to (3.1) a term in which the time derivative of the heat flux is multiplied by a coefficient τ ; the latter is called a *thermal relaxation time*. Then the governing equation of this approach called the Maxwell–Cattaneo–Vernotte–Kaliski equation, is

$$(3.5) \quad \tau \dot{\mathbf{q}} + \mathbf{q} = -k(\vartheta)\nabla\vartheta,$$

where τ is a *thermal relaxation time*.

If τ is constant then the 2-nd order dissipative wave equation follows for the thermodynamic temperature

$$(3.6) \quad \tau \frac{\partial}{\partial t}(\rho c_v(\vartheta)\dot{\vartheta}) + \rho c_v(\vartheta)\dot{\vartheta} = \operatorname{div}(k(\vartheta)\nabla\vartheta),$$

which occurs to be a *telegraph equation*

$$(3.7) \quad \tau \rho c_v \ddot{\vartheta} + \rho c_v \dot{\vartheta} = k \Delta \vartheta,$$

for constant coefficients, and is *hyperbolic* if $\tau c_v \kappa$ is positive. The prototypes of this modification one can find in [1, 8, 9, 11, 33, 53, 70] for non-deformable (rigid) conductors (cf. also [54]). The thermoelastic developments of this modification in the earlier stages of development one can find in [5, 19, 51]. For more recent development we refer to the review paper of IGNACZAK [28].

Another method had been proposed by Bogy and Naghdi, and the McCarthy approach: a dependence of the specific entropy η on ϑ , $\nabla\vartheta$ and temperature rate $\dot{\vartheta}$:

$$(3.8) \quad \eta = \eta(\vartheta, \nabla\vartheta, \dot{\vartheta}).$$

They arrived at an equation similar to (3.7) with additional term $\nabla\dot{\vartheta}$. In [6] and [55] thermodynamic derivations of the heat conduction equations can be found with the temperature-rate dependence of free energy and entropy. Thermomechanical developments of that approaches in the earlier stages one can find in [19, 23, 51].

In both the above approaches based on the rate-type method, the final equation is a kind of telegraph equation for the temperature field, which is the second order hyperbolic equation. Such a formulation has the main drawback, namely

the order of the equation, for which the initial conditions in terms of the time derivative of the temperature field have to be given additionally to the temperature itself. Moreover, thermal shocks in the form of temperature pulses cannot be described. In the approach we are going to present, this inconvenience will be omitted.

Almost parallel to the rate-type methods, a history approach was developed by GURTIN and PIPKIN [24] and NUNZIATO [62]. In their papers the heat flux vector was assumed to depend on the summed history of the temperature gradient $\bar{\mathbf{g}}$

$$(3.9) \quad \mathbf{q}(t) = \int_0^{\infty} \frac{d}{ds} \mathbf{A}(s) \bar{\mathbf{g}}(t-s) ds,$$

where

$$\bar{\mathbf{g}}(t-s) = \int_0^s \text{grad } \vartheta(t-\lambda) d\lambda, \quad \text{with } \mathbf{g} = \text{grad } \vartheta.$$

NUNZIATO in [62] assumed a more general dependence on the actual value of the temperature gradient. That assumption changes drastically the type of the final heat conduction equation. Under the isotropy assumption concerning the tensor function \mathbf{A} , the derivative of which appears in (3.9), took the form $\mathbf{A}(s) = a(s)\mathbf{1}$, with a differentiable scalar-valued function $a(\cdot)$ defined on $[0, \infty)$ with $a(\infty) = 0$; the authors of [24] wrote the heat flux constitutive equation (3.9) in the form

$$(3.10) \quad \mathbf{q}(t) = - \int_0^{\infty} a(s) \mathbf{g}(t-s) ds.$$

It is interesting to notice that the Maxwell-Cattaneo-Vernotte-Kaliski equation (3.5) can be obtained from (3.10) under rather weak smoothness assumptions about the function $\mathbf{g}(t-s)$ and the following kernel representation $a(s) = \exp(-t/\tau) k/\tau$.

There are some drawbacks in applying the equation (3.5) to the thermodynamic theory of thermomechanics, especially because of the internal dissipation inequality. It was pointed out by MAZILU in [54], MORRO and RUGGERI in [58, 59], by the authors of review papers [18, 31, 32] (cf. also [7, 50, 56, 60]) and by others.

Due to the lack of space we only mention two other approaches which make use of the phonon gas for the wave phenomena in heat transport, namely those of LARECKI and PIEKARSKI [48], and the authors of [18] that make use of many-moments model of extended thermodynamics.

4. Internal variable as a thermal history of material

The present author with PERZYNA in [40] (and latter in [34] in a more general setup) calculated for the first time finite speeds of thermal (and thermomechan-

ical) waves in the framework of a thermodynamic theory with internal state variables. The results of those derivations were applied to 3D thermoelasticity in [57]. Further generalizations for more complex materials (cf. [47], and [72]) have been partially reported in the monograph [35]. Later on COLEMAN *et al.* [15] repeated the first results of KOSIŃSKI and PERZYNA from 1972 in a slightly different set-up, without referring to the internal state variable approach and to the original paper [40].

In [36], the material gradient of an internal, scalar state variable was introduced as a fundamental state variable in the response functions of thermoelastic materials. In the course of obtaining the consequences for the laws of thermodynamics, a modified Fourier-type law was found leading to finite speeds of propagation of thermal and thermomechanical waves. This model differed from an earlier one, ([40]), in the form of the evolution and constitutive equations.

In the model proposed by the author in [36], a different procedure for the derivation of a wavelike heat propagation equation was proposed. In that model a scalar internal state variable β has been introduced to represent at the same time a history of the thermodynamic temperature and the potential for its gradient. That gradient appears in the heat flux constitutive equation. The model has been mostly applied to heat conductors in 1D and 3D cases and to thermoelastic solids (cf. [3, 12, 13, 14, 17, 41, 42, 37, 38, 39, 43, 67]). Its numerical implementations are discussed in [4, 20, 21, 22].

In the present section we are going to give additional arguments for the model initiated in [36] and developed further in a dozen of publications.

Let us look once more at the Maxwell – Cattaneo – Vernotte – Kaliski equation (3.5). At each material point X this equation is regarded as an ODE for the heat flux vector \mathbf{q} . For any time instant $t \geq 0$ its solution under homogeneous initial conditions for \mathbf{q} and $\nabla\vartheta$ at $t = 0$ will be

$$(4.1) \quad \mathbf{q}(t) = - \int_0^t \frac{k}{\tau} \exp\left(\frac{s-t}{\tau}\right) \nabla\vartheta(s) ds.$$

In the linear case one assumes that the heat conductivity coefficient k is constant as well as the relaxation time τ . Then one can write the gradient operator in front of the integral to get

$$(4.2) \quad \mathbf{q}(t) = -k\nabla \int_0^t \frac{1}{\tau} \exp\left(\frac{s-t}{\tau}\right) \vartheta(s) ds.$$

If we denote the whole integral by β

$$(4.3) \quad \beta(t) = \int_0^t \frac{1}{\tau} \exp\left(\frac{s-t}{\tau}\right) \vartheta(s) ds,$$

then the last equation can be written in the form

$$(4.4) \quad \mathbf{q}(t) = -k\nabla\beta(t).$$

On the other hand, let us notice that the variable β satisfies the following ODE

$$(4.5) \quad \dot{\beta} = \frac{1}{\tau}(\vartheta - \beta).$$

From this point it is not difficult to arrive at the more general *evolution equation*

$$(4.6) \quad \dot{\beta} = F(\vartheta, \beta),$$

which could be accompanied by the more general heat conduction constitutive equation

$$(4.7) \quad \mathbf{q} = -\alpha^*(\vartheta)\nabla\beta$$

in a particular quasi-linear case, instead of the previous (4.4).

Further on, this extra thermal variable β (called in [12] a new semi-empirical temperature) will be regarded in this *more general set-up* (instead of the previous way (4.3)), as a solution of the evolution equation (4.6) with a thermodynamic temperature nonlinear, in general, dependence on the right-hand side (RHS).

This short derivation has been performed to present the role of the variable β , and its interpretation as an internal state variable. The derivation shows that this variable has been introduced in order to represent a *thermal history* of the material. The fact that a scalar internal state variable can represent a history of the temperature (while its gradient can represent the history of the temperature gradient) and in this way, influence the response of the material at hand, is compatible with the general framework of the internal state variable approach. As it was shown in several publications (cf. for example [44]), the internal state variable approach and the theory using constitutive functionals with a past history dependence have some common points. Those two approaches are to some extent equivalent as far as the description of memory effects of the material is concerned. The difference appears in the very far past: in the internal state variable approach one has to give an initial value of the internal state variable at time t tending to minus infinity, on the other hand in the constitutive functional description one has to assume some value of the history at that far past.

Coming back to the main stream of the paper, we would like to point out that in some applications analyzing the propagation of weak discontinuity waves (cf. [41, 42]) we have used the linear evolution equation (4.5). However, in more advanced problems related to the temperature-dependence of the speed of thermal waves (i.e. the speed of the second sound), such linear equation is not suitable and we had to look for a nonlinear one. It was the case discussed in the recent papers [43, 67] where, based on low temperature experiment data in solid dielectric crystals NaF, the present author together with the Saxtons used in the

evolution equation (4.6) the function F of the form $F = F_1(\vartheta) + F_2(\beta)$, with

$$(4.8) \quad F_1(z) = a(|z|^{p-1}z)_-, \quad 1 < p < 2,$$

where a is a positive constant, and the subscript “ $-$ ” means that when $z \geq 0$, F_1 is taken to be zero. For the second function F_2 , they put

$$(4.9) \quad F_2(z) = -b|z|^{h-1}z, \quad h \geq \frac{p}{2-p},$$

where b is another positive constant. In the first case z represents $\vartheta - \vartheta_\lambda$ where ϑ_λ denotes the *critical* temperature at which the heat conductivity of the material reaches a peak. The model developed in [43, 67] based on the general framework developed in [45], is intended to admit wavelike propagation of heat below – and diffusive conduction above – a particular temperature value ϑ_λ . This is mainly due to the fact that the range of temperature for which the second sound (thermal waves) is detectable is in fact quite small, and normal diffusive propagation takes place above it.

5. Foundation of thermodynamics with modified Fourier law

The aim of this section is to list fundamental assumptions concerning the thermodynamics with modified Fourier law as well as its main features. The present consideration will be restricted, for simplicity of the presentation, to a non-deformable continuum, while in the next section a thermodynamics of deformable, elastic continuum will be shortly presented.

To this end we formulate an alternative, gradient model with internal state variables to describe thermal wave phenomena, with a possibility to pass to the Fourier model of heat conduction. Then we propose, in the next section, the full thermomechanical coupling with wave phenomenon.

Based on that as well as on other physical observations, the main assumption of the model developed by the author in [36] can be summarized in the following assumptions (cf. [13]):

ASSUMPTION 1

Memory effect is introduced by a scalar-valued function β that plays the role of a potential in the heat conduction law, i.e. β is a solution of an initial value problem

$$(5.1) \quad \dot{\beta} = F(\vartheta, \beta), \quad \beta(t_0) = \beta_0.$$

In one of our previous papers [12] we have called β a *semi-empirical temperature*. However, this term is not essential for the development.

ASSUMPTION 2

Gradient $\nabla\beta$ appears in constitutive equations instead of the gradient of the thermodynamic (absolute) temperature ϑ ,

$$(5.2) \quad \begin{aligned} \psi &= \psi^*(\vartheta, \nabla\beta), \\ \eta &= \eta^*(\vartheta, \nabla\beta), \\ \mathbf{q} &= \mathbf{q}^*(\vartheta, \nabla\beta). \end{aligned}$$

ASSUMPTION 3

Fourier law is necessary to obtain in the limit transition, called a quasi-equilibrated (or steady-state) case, when the left-hand side of (5.1) tends to zero. In other words, we suppose there exists an invertible *quasi-equilibrated* relation between ϑ and β , i.e. a differentiable homeomorphism $B(\vartheta)$ such that when the left-hand side of (5.1) vanishes then $F(\vartheta, \beta) = 0$, and under the assumption of the local monotonicity of F one gets β as a function of ϑ . Hence in terms of the function B , we get

$$(5.3) \quad \text{if } \dot{\beta} \rightarrow 0 \text{ then } \beta = B(\vartheta), \text{ with } F(\vartheta, B(\vartheta)) \equiv 0.$$

This function will appear in the representation of the steady-state thermal conductivity coefficient. Let us present this for the case of the quasi-linear proportionality constitutive law (4.7). Namely, admitting that in general dynamic case, the heat flux vector \mathbf{q} is proportional to the gradient of β (cf. (4.7), then under the quasi-equilibrated condition (or steady-state) (5.3), the heat flux turns to be proportional to $\nabla\vartheta$ itself, as it is in the Fourier law (3.1), however, with another heat conduction coefficient \mathbf{k}^* , called now *steady-state thermal conductivity coefficient*, and given by

$$(5.4) \quad \mathbf{q} = -\mathbf{k}^*\nabla\vartheta \quad \text{with} \quad \mathbf{k}^* := \alpha^*(\vartheta)\frac{dB(\vartheta)}{d\vartheta}.$$

Now, let us make the derivation of the thermodynamic consequences for the general case of the constitutive equations (5.2) if we employ the 1-st and 2-nd laws of thermodynamics in the form:

$$(5.5) \quad \rho\dot{e} + \text{div } \mathbf{q} = \rho r,$$

$$(5.6) \quad \rho\dot{\eta} + \text{div } (\mathbf{q}/\vartheta) \geq \rho r/\vartheta,$$

where the specific internal energy $e = \psi + \eta\vartheta$. If we require that each Lipschitz continuous solution of (5.1), (5.2), (5.5) satisfies the thermodynamic inequality (5.6) then we get

REMARK 0

The necessary and sufficient conditions for satisfying the second law of thermodynamics by each Lipschitz continuous solution to (5.1), (5.2) and (5.5), are:

$$(5.7) \quad \begin{aligned} \mathbf{q} &= -\varrho\vartheta(\partial F/\partial\vartheta)(\partial\psi^*/\partial\nabla\beta), \\ \eta &= -\partial\psi^*/\partial\vartheta, \end{aligned}$$

and the residual internal dissipation inequality

$$(5.8) \quad (\partial F/\partial\beta)/(\vartheta\partial F/\partial\vartheta)\mathbf{q}\cdot\nabla\beta \geq 0.$$

Moreover, one gets the next consequence

$$(5.9) \quad F(\vartheta, \beta) = F_1(\vartheta) + F_2(\beta).$$

In the present paper we will follow the internal state variable approach based on the evolution equation (4.6). Now we list some most fundamental observations concerning the developed model.

REMARK 1

If Eq. (5.7) for \mathbf{q} is linear in $\nabla\beta$, i.e.

$$(5.10) \quad \mathbf{q} = -\alpha^*(\vartheta)\nabla\beta,$$

then, two functions ψ_1^* and ψ_2^* should exist such that

$$(5.11) \quad \psi = \psi_1^*(\vartheta) + 0.5\psi_2^*(\vartheta)|\nabla\beta|^2,$$

where

$$(5.12) \quad \alpha^*(\vartheta) = \varrho\vartheta F_1'(\vartheta)\psi_2^*(\vartheta).$$

Notice that the assumption on the linearity in the heat flux constitutive equation (5.7)₂ is equivalent to the splitting of the free energy function ψ into two terms, with quadratic dependence on $\nabla\beta$. Moreover, the first term $\psi_1^*(\vartheta)$ is rather classical. In [37] 3D thermal weak discontinuity waves were investigated. It was shown there, for example, that in the rigid case with (4.7) and (5.11) the governing system of equations for determining β and ϑ is composed of (5.1) with F given by (5.9), and

$$(5.13) \quad \begin{aligned} \varrho c_v(\vartheta)\ddot{\beta} - \alpha^{*'}(\vartheta)\nabla\dot{\beta}\cdot\nabla\beta - \alpha^*(\vartheta)F_1'(\vartheta)\Delta\beta \\ - \varrho c_v(\vartheta)F_2'(\beta)\dot{\beta} + \alpha^{*'}(\vartheta)F_2'(\beta)\nabla\beta\cdot\nabla\beta = \varrho rF_1'(\vartheta). \end{aligned}$$

Since ϑ can be expressed in terms of β and $\dot{\beta}$ from (5.1), the equation (5.13) can be solved independently for β . The speed λ of a pure thermal wave is given by

$$(5.14) \quad \varrho c_v(\vartheta)\lambda^2 - \alpha^{*'}(\vartheta)\nabla\beta\cdot\mathbf{n}\lambda - \alpha^*(\vartheta)F_1'(\vartheta) = 0,$$

with \mathbf{n} as a unit normal to the wave front. We can see that the wave propagation is not symmetric if $\nabla\beta$ differs from zero, and the hyperbolicity condition is

$$(5.15) \quad c_v(\vartheta)\alpha^*(\vartheta)F_1'(\vartheta) \geq 0 \quad \text{or} \quad c_v(\vartheta)\vartheta(F_1'(\vartheta))^2\psi_2^*(\vartheta) \geq 0,$$

where the second equivalent form is due to the representation (5.12).

Let us check the consequences of the residual dissipation inequality (5.8) in the present form of Eq. (5.10) for \mathbf{q} . Substituting it into (5.8) we get

$$(5.16) \quad -\varrho F_2'(\beta)\psi_2^*(\vartheta) \geq 0.$$

Let us make the additional requirement on the asymptotic stability of solutions β of the evolution equation

$$(5.17) \quad \dot{\beta} = F_1(\vartheta) + F_2(\beta).$$

Then due to the classical Lapunov stability result we can require

$$(5.18) \quad F_2'(\beta) \leq 0.$$

This implies

$$\psi_2^*(\vartheta) \geq 0.$$

Hence in view of the (5.15) we may formulate the proposition:

REMARK 2

Under the assumption of Remark 1 and the asymptotic stability requirements (5.18), the *hyperbolicity condition* (5.15) is satisfied if $c_v(\vartheta) \geq 0$.

REMARK 3

Under the assumption of Remark 1, if the internal energy $\varepsilon = \psi + \eta\vartheta$ is a function of ϑ only, then for the constitutive functions ψ_2^* and α^* we obtain

$$(5.19) \quad \psi_2^*(\vartheta) = \psi_{20}^*\vartheta \quad \text{and} \quad \alpha^*(\vartheta) = \varrho\psi_{20}^*\vartheta^2 F_1'(\vartheta),$$

with a positive constant ψ_{20}^* , that could be written as

$$\psi_{20}^* = \frac{\tau_0 k_0}{\varrho\vartheta_0^2},$$

where the dimensions of material constants $k_0, \tau_0, \vartheta_0^2$ are obvious and follow from the dimensional analysis (cf. [36]). Then the specific free energy, entropy and heat flux are given by

$$(5.20) \quad \begin{aligned} \psi &= \psi_1^*(\vartheta) + 0.5\psi_{20}^*\vartheta|\nabla\beta|^2, \\ \eta^*(\vartheta, \nabla\beta) &= \eta_E(\vartheta) - 0.5\psi_{20}^*|\nabla\beta|^2, \quad \text{with} \quad \eta_E(\vartheta) = -\psi_1'^*(\vartheta), \\ \mathbf{q} &= -\varrho\psi_{20}^*\vartheta^2 F_1'(\vartheta)\nabla\beta, \end{aligned}$$

and the principle of maximum entropy at equilibrium ($\mathbf{q} = \mathbf{0}$) holds, for $\psi_{20} \geq 0$. Let us notice that the first term $\eta_E(\vartheta)$ is rather classical and can be called the equilibrium entropy.

REMARK 4

If in the thermodynamic inequality (5.6) one assumes a more general form of the entropy flux, namely

$$\frac{\mathbf{q}}{\vartheta} + \mathbf{h}$$

then in the derivation of the consequences one gets the potential relations (5.7) with the same residual dissipation inequality together with the requirement $\mathbf{h} = \mathbf{0}$.

REMARK 5

With the representations (4.8) and (4.9), it can be shown (cf. [67]) that in that model, an expression for the second sound speed, U_E , (the speed of temperature rate waves propagating in an undisturbed region) is given by

$$(5.21) \quad U_E^2 = \frac{\alpha^*(\vartheta) F_1'(\vartheta)}{\rho c_v(\vartheta)} = \frac{\psi_{20}}{c_0 \vartheta} a^2 p^2 (\vartheta - \vartheta_\lambda)_-^{2(p-1)}.$$

At the same time, in the *steady state* case, for which $F_1(\vartheta) + F_2(\beta) = 0$, (i.e. β is a function of ϑ) the heat flux (4.7) now reduces to (cf. (5.4)) and

$$(5.22) \quad \mathbf{q} = -(c\psi_{20}\vartheta^2 |(\vartheta - \vartheta_\lambda)_-|^{p(1+1/h)-2}) \nabla \vartheta,$$

where c depends on a, b, p and h .

We note that the obtained expression for the second sound propagation speed in undisturbed region described very well the experimental results reported in Sec. 2. Moreover, the solid curve in Fig. 1 in [43] fits very well the experimental curve of Coleman and Newman. On the other hand, the expression (5.22) predicts a peak in heat conductivity as ϑ tends to ϑ_λ from below, followed by a sharp drop. At the same time, in particular if p is close to 1, Eq. (5.21) delivers a sudden drop to zero of the wave speed U_E , where it could be expected that we were entering the range of temperature of purely diffusive heat conductivity in which parabolic heat conduction equation holds.

If we look once more at the general expression (5.21) for the second sound speed $U_E(\vartheta)$, then using (5.12) we get

$$(5.23) \quad U_E^2(\vartheta) = \frac{(\vartheta F_1'(\vartheta))^2 \psi_2^*(\vartheta)}{c_v(\vartheta)}.$$

From this expression and the form (5.4) of the steady-state heat conductivity coefficient \mathbf{k}^* , we can get the following conclusion:

REMARK 6

The model is determined, i.e. all its constitutive functions can be determined, if the following quantities: specific heat $c_v(\vartheta)$, steady-state heat conductivity coefficient \mathbf{k}^* and speed of propagation of thermal second sound wave U_E , all in terms of the absolute temperature ϑ , are known.

REMARK 7

One can get the model of extended thermodynamics developed by MORRO and RUGGERI [59] with the following evolution equation for the heat flux:

$$(5.24) \quad \frac{\partial}{\partial t} \left(\frac{\mathbf{q}}{\alpha(\vartheta)} \right) + \text{grad } \nu(\vartheta) = - \frac{\nu'(\vartheta)}{k(\vartheta)} \mathbf{q},$$

under the identities

$$\alpha(\vartheta) F_1'(\vartheta) \equiv 1/\alpha^*(\vartheta) \quad \text{and} \quad \nu \equiv F_1(\vartheta),$$

and $F_2(\beta)$ being linear in β . Similarly one can get a particular model with internal state variable developed by KOSIŃSKI and PERZYNA [40].

REMARK 8

There is a parabolic thermodynamic development [45, 46] of the present approach in which the additional dependence on the actual value of the temperature gradient is allowed. In a particular isotropic, linearized case one gets for the heat flux the equation (cf. [43, 67])

$$(5.25) \quad \mathbf{q} = -k(\vartheta) \nabla \vartheta - \alpha^*(\vartheta) \nabla \beta, \quad \text{with} \quad \dot{\beta} = F(\vartheta, \beta).$$

That development can play two roles: the first role is a thermodynamically consistent parabolic regularization of the present hyperbolic model, the second one – a full model that can be used in order to describe the whole regime of the heat conduction: the wave-type and diffusive parts (cf. [43, 67]).

A new (non-hyperbolic) nonlinear model has been recently developed by HETNARSKI and IGNACZAK and used in [27] in the analysis of soliton-like waves in a non-deformable heat conductor. Their model could be regarded as a particular case of (5.25) in which they are not introducing (formally) the scalar internal state variable β , but its gradient, as a vector field $\boldsymbol{\beta}$. However, it could correspond to the present case with a scalar internal state case, however, the function F in the RHS of the evolution equation for β will not depend on β , having the form $F(\vartheta, \beta) = A \ln \vartheta$ with $A = \text{const}$ (cf. Eq. (11) in [27] and substitute their $\boldsymbol{\beta}$ with the present $\nabla \beta$).

In the recent papers of LARECKI [49] the analysis of the phenomenological model of a non-deformable conductor of heat with a vector-valued internal state variable has been performed. There its comparison with the present model as well as its relations to other models based on some generalizations of the Maxwell – Cattaneo – Vernotte – Kaliski equation have been broadly discussed.

6. Thermomechanics

To present shortly the case of thermomechanics we notice that at finite strains, the referential (Lagrangian) description is used. Then $\varrho_0(X)$ is the reference mass density, known for each X in the reference placement \mathcal{B} and related to the actual mass density ϱ by the mass continuity law $\varrho_0 = J\varrho$, with $J = \det \mathbf{F}$ and \mathbf{F} as the deformation gradient tensor. If \mathbf{T} is the symmetric Cauchy stress tensor then it is related to the first Piola–Kirchhoff stress tensor \mathbf{S} by the identity $\mathbf{S} = J \mathbf{T} \mathbf{F}^{-T}$. Here \mathbf{v} is the particle velocity, \mathbf{b} – the body force, \mathbf{q} – the heat flux vector related to the reference heat flux \mathbf{q}_κ by the identity $\mathbf{q}_\kappa = J\mathbf{q}\mathbf{F}^{-T}$, ε – the specific internal energy per unit mass, r – the body heat supply, η – the specific entropy, ϑ – the absolute temperature. By E we shall denote the sum of internal and kinetic energies. The dot \cdot denotes the scalar product operation.

In the recent paper of the author with ARCISZ [3], two formulations of a thermomechanics with a vector-valued internal state variable at finite strains have been presented incorporating a wave-type heat transport. The authors have made the fundamental assumption concerning the dependence of the Helmholtz free energy and the heat flux vector on thermal variables and concerning the limiting quasi-equilibrated case. In the case of finite strains the classical Fourier proportionality law can be written in two non-equivalent forms. Those two forms are crucial for the isotropic case mainly. Hence the question appears: is the proportionality isotropic law between the actual heat flux vector \mathbf{q} and the spatial gradient of temperature valid or rather the proportionality isotropic law between the referential heat flux vector \mathbf{q}_κ and the Lagrangean gradient of temperature $\text{Grad } \vartheta$? The authors of that paper answer “yes” to the first part of the question. For those and the present cases such an answer has implied that the free energy should depend on the spatial gradient of the scalar internal state variable β .

Now following [3, 38] we state the main assumption:

$$(6.1) \quad \psi = \psi^*(\mathbf{F}, \vartheta, \text{grad } \vartheta, \mathbf{h}),$$

with

$$(6.2) \quad \mathbf{h} = \text{grad } \beta,$$

where β is the solution of an initial value problem

$$(6.3) \quad \dot{\beta} = F(\vartheta, \beta), \quad \beta(t_0) = \beta_0,$$

where t_0 is an initial instant and β_0 is an initial distribution of β , assumed to be given at t_0 for each X of \mathcal{B} .

Incorporating (6.2) into (6.1) we obtain, as it was required above, the spatial gradient $\text{grad } \beta = \nabla \beta$ in the constitutive equation for the free energy

$$(6.4) \quad \psi = \psi^*(\mathbf{F}, \vartheta, \nabla \vartheta, \nabla \beta).$$

For the other constitutive quantities

$$(6.5) \quad \begin{aligned} \eta &= \eta^*(\mathbf{F}, \vartheta, \nabla\vartheta, \nabla\beta), \\ \mathbf{S} &= \mathbf{S}^*(\mathbf{F}, \vartheta, \nabla\vartheta, \nabla\beta), \\ \mathbf{q}_\kappa &= \mathbf{Q}^*(\mathbf{F}, \vartheta, \nabla\beta). \end{aligned}$$

Notice that the independence of the heat flux of the actual value of the temperature gradient is crucial for the development of the hyperbolic model.

Now the second law of thermodynamics

$$(6.6) \quad \rho_0(\dot{\eta}\vartheta - \dot{\varepsilon}) + \mathbf{S} \cdot \dot{\mathbf{F}} - \vartheta^{-1} \mathbf{q}_\kappa \cdot \text{Grad } \vartheta \geq \rho_0 r / \vartheta,$$

will be in use. To derive its consequences let us put the form of ψ into (6.6). Then if we perform the (material) time differentiation in (6.4), use the chain rule property and insert the result of these operations in (6.6) then we get, after grouping the terms standing in front of the appropriate time derivatives of the components the state variable vector $\mathbf{F}, \vartheta, \nabla\vartheta, \nabla\beta$,

$$(6.7) \quad \begin{aligned} (\rho_0 \partial\psi^* / \partial\mathbf{F} - \mathbf{S}) \cdot \dot{\mathbf{F}} + \rho_0 (\partial\psi^* / \partial\vartheta + \eta^*) \dot{\vartheta} + (\rho_0 \partial\psi^* / \partial\nabla\vartheta) \cdot \dot{\overline{\nabla\vartheta}} \\ + \rho_0 (\partial\psi^* / \partial\nabla\beta) \cdot \dot{\overline{\nabla\beta}} + \mathbf{q}_\kappa \cdot \text{Grad } \vartheta / \vartheta \leq 0. \end{aligned}$$

Now, taking the gradient of (6.3) calculated in the actual configuration and observing that the material time derivative does not commute with grad, we get

$$(6.8) \quad \dot{\overline{\nabla\beta}} = \text{grad } F(\vartheta, \beta) - \nabla\beta \dot{\mathbf{F}} \mathbf{F}^{-1}.$$

The latter enables us to express the product $\mathbf{q}_\kappa \cdot \text{Grad } \vartheta$ in the last inequality (6.7) as

$$\mathbf{q}_\kappa \cdot \text{Grad } \vartheta = (\partial F / \partial\vartheta)^{-1} \{ \mathbf{q}_\kappa \mathbf{F}^T \cdot \dot{\overline{\nabla\beta}} - (\partial F / \partial\beta) \mathbf{q}_\kappa \mathbf{F}^T \cdot \nabla\beta + (\nabla\beta \otimes \mathbf{q}_\kappa) \cdot \dot{\mathbf{F}} \}.$$

Hence, owing to this, the last inequality can be represented as

$$(6.9) \quad \begin{aligned} \{ \rho_0 \partial\psi^* / \partial\mathbf{F} - \mathbf{S} + (\vartheta \partial F / \partial\vartheta)^{-1} (\nabla\beta \otimes \mathbf{q}_\kappa) \} \cdot \dot{\mathbf{F}} \\ + \rho_0 (\partial\psi^* / \partial\vartheta + \eta^*) \dot{\vartheta} + (\rho_0 \partial\psi^* / \partial\nabla\vartheta) \cdot \dot{\overline{\nabla\vartheta}} \\ + \{ \rho_0 \partial\psi^* / \partial\nabla\beta + (\vartheta \partial F / \partial\vartheta)^{-1} \mathbf{q}_\kappa \mathbf{F}^T \} \cdot \dot{\overline{\nabla\beta}} \\ - (\vartheta \partial F / \partial\vartheta)^{-1} (\partial F / \partial\beta) \mathbf{q}_\kappa \mathbf{F}^T \cdot \nabla\beta \leq 0. \end{aligned}$$

Due to the independence of the time derivatives of the components of the (restricted) state variable vector $(\mathbf{F}, \vartheta, \nabla\vartheta, \nabla\beta)$, the inequality (6.9) leads to identities

$$(6.10) \quad \begin{aligned} \mathbf{0} &= \partial\psi^* / \partial\nabla\vartheta, \\ \eta &= -\partial\psi^* / \partial\vartheta, \\ \mathbf{S} &= \rho_0 \partial\psi^* / \partial\mathbf{F} + (\vartheta \partial F / \partial\vartheta)^{-1} (\nabla\beta \otimes \mathbf{q}_\kappa), \\ \mathbf{q}_\kappa &= -\rho_0 (\vartheta \partial F / \partial\vartheta) (\partial\psi^* / \partial\nabla\beta) \mathbf{F}^{-T}, \end{aligned}$$

and to a reduced inequality

$$(6.11) \quad (\vartheta \partial F / \partial \vartheta)^{-1} (\partial F / \partial \beta) \mathbf{q}_\kappa \mathbf{F}^T \cdot \nabla \beta \geq 0.$$

The independence and three other potential relations in (6.10) have particular meaning. The identities from the first three lines of (6.10) are well known, the last relation is rather not typical. The stress potential relation in (6.10)₃ contains two components: the first component is rather classical and appears in all stress potential relations, while the second one represents the direct coupling between mechanical and thermal fields in which extra stresses appear. The potential relation for the heat flux can be obtained in some models with internal state variables (cf. [34]). Eliminating \mathbf{q}_κ from (6.10)₃, (6.10)₄ and (6.11) we get a new potential relation for the Piola-Kirchhoff stress and a residual inequality

$$(6.12) \quad \begin{aligned} \mathbf{S} &= \varrho_0 \partial \psi^* / \partial \mathbf{F} - \varrho_0 (\nabla \beta \otimes \partial \psi^* / \partial \nabla \beta) \mathbf{F}^{-T}, \\ \varrho_0 (\partial F / \partial \beta) (\partial \psi^* / \partial \nabla \beta) \cdot \nabla \beta &\leq 0. \end{aligned}$$

Using (6.10) and (6.12) we can write the final form of the constitutive equations for the Cauchy stress and the actual heat flux vector in a general anisotropic case, as

$$(6.13) \quad \begin{aligned} \mathbf{T} &= \varrho \partial \psi^* / \partial \mathbf{F} \mathbf{F}^T - \varrho \nabla \beta \otimes \partial \psi^* / \partial \nabla \beta, \\ \mathbf{q} &= -\varrho \vartheta (\partial \psi^* / \partial \nabla \beta) (\partial F / \partial \vartheta), \end{aligned}$$

respectively.

The general constitutive equations (6.13) can be investigated for the particular case of a free energy with a general law between \mathbf{q} and $(\mathbf{F}, \vartheta, \nabla \beta)$

$$(6.14) \quad \mathbf{q} = -\boldsymbol{\alpha}^*(\mathbf{F}, \vartheta, \nabla \beta),$$

with a material tensor function (in a general, anisotropic case) $\boldsymbol{\alpha}^*$ given by

$$(6.15) \quad \boldsymbol{\alpha}^* = \varrho \vartheta \frac{\partial F}{\partial \vartheta} \frac{\partial \psi^*}{\partial \nabla \beta}.$$

The linearity and the isotropy of (6.14) in $\nabla \beta$, on the other hand, will be reached iff the free energy function is of the form (cf. Remark 1)

$$(6.16) \quad \psi = \psi_1^*(\mathbf{B}, \vartheta) + 0.5 \psi_2^*(\mathbf{B}, \vartheta) \nabla \beta \cdot \nabla \beta,$$

where the factor 0.5 has been assumed for convenience only, and the tensor $\boldsymbol{\alpha}^*$ is spherical, i.e. $\boldsymbol{\alpha}^* = \alpha^* \mathbf{1}$, with

$$(6.17) \quad \mathbf{q} = -\alpha^*(\mathbf{B}, \vartheta) \nabla \beta \quad \text{with} \quad \alpha^*(\mathbf{B}, \vartheta) = \varrho \vartheta \partial F / \partial \vartheta \psi_2^*(\mathbf{B}, \vartheta).$$

It is worthwhile to point out, that the above linearity assumption is compatible with the following observation. As discussed in [13], it becomes reasonable to make the following assumptions remaining consistent with classical thermo-statics, at the same time making it straightforward to use experimental results to identify the material functions needed:

- the free energy is independent of β and quadratic in $|\nabla\beta|$,
- the coefficient α^* may depend on \mathbf{B} and on ϑ .

REMARK 9

In the present thermomechanical case, the independence of the free energy function of β leads to splitting (compare (5.9) of the function $F(\vartheta, \beta)$ into two independent terms

$$F(\vartheta, \beta) = F_1(\vartheta) + F_2(\beta).$$

Moreover, in the linear and isotropic case of the heat flux constitutive relation (6.17) if the proportionality material coefficient α^* is independent of strain, i.e. $\alpha^*(\mathbf{B}, \vartheta) = \alpha^*(\vartheta)$, then

$$(6.18) \quad \mathbf{q} = -\alpha^*(\vartheta) \nabla\beta,$$

and due to (6.17)

$$(6.19) \quad \psi_2^*(\mathbf{B}, \vartheta) = \psi_{21}^*(\vartheta)J,$$

where, as before, $J = \det \mathbf{F} = \varrho_0/\varrho$, and ψ_{21}^* is a nonnegative material function⁽³⁾. In this case the final form of the Cauchy stress constitutive law (6.13)₁ due to (6.16) expression will take the form

$$(6.20) \quad \mathbf{T} = 2\varrho(\partial\psi_1^*/\partial\mathbf{B})\mathbf{B} + \varrho\psi_{21}^*(\vartheta)J(0.5|\nabla\beta|^2\mathbf{I} - \nabla\beta \otimes \nabla\beta),$$

or equivalently, by using (6.18),

$$(6.21) \quad \mathbf{T} = 2\varrho(\partial\psi_1^*/\partial\mathbf{B})\mathbf{B} + \varrho \frac{\psi_{21}^*(\vartheta)J}{\alpha^{*2}(\vartheta)}(0.5|\mathbf{q}|^2\mathbf{I} - \mathbf{q} \otimes \mathbf{q}).$$

Let us notice that even in this simplified case the stress-strain relation has an extra term due to the thermomechanical coupling; this term can be called an *extra thermal stress*. This extra contribution is similar to that already known in the Landau's superfluid model of liquid helium at low temperature [65]. It seems to us that this contribution can have a substantial meaning in describing the damage phenomena in deformable materials due thermomechanical coupling. The thermomechanics with damage effects introduced by the internal state variables has been developed in the last paper [39].

⁽³⁾ If the internal energy ε^* is independent of $\nabla\beta$ then the function ψ_{21}^* is linear in ϑ , cf. (5.19).

7. Hyperbolicity of thermoelasticity

In order to check the hyperbolicity of the model, let us discuss shortly the propagation of weak discontinuity waves. On the surface of discontinuity⁽⁴⁾ of the second derivatives of the displacement function \mathbf{u} and the function β , we have at our disposal the classical kinematical and geometrical compatibility conditions. A jump discontinuity in the field is denoted by a bracket $[[\cdot]]$. Here u_n is called its (intrinsic) *normal speed* of propagation and \mathbf{n} is a *unit normal* to the wave front, regarded as a 2D surface in 3D Euclidean space ([35]). Hence for derivatives of the fields \mathbf{v} , \mathbf{F} , ϑ and β , we have the following relations:

$$(7.1) \quad \begin{aligned} [[\text{grad } \mathbf{v}]] &= -Us \otimes \mathbf{n}, \\ [[\text{grad } (\partial\beta/\partial t)]] &= -u_n v \mathbf{n}, \\ [[\text{grad } \mathbf{F}]] &= \mathbf{s} \otimes \mathbf{F}^T \mathbf{n} \otimes \mathbf{n}, \\ [[\partial\mathbf{v}/\partial t + \text{grad } \mathbf{v}\mathbf{v}]] &= [[\dot{\mathbf{v}}]] = U^2 \mathbf{s}, \end{aligned}$$

where we have used the following notations for the relative speed U and the amplitudes of jumps

$$(7.2) \quad \begin{aligned} U &:= u_n - \mathbf{v} \cdot \mathbf{n}, & [[\text{grad } \mathbf{v} \mathbf{n}]] &= -Us, \\ & & [[\text{grad grad } \beta \mathbf{n}]] &= v \mathbf{n}, \end{aligned}$$

together with the continuity of the first order derivatives of β . On the other hand, for the second derivatives of the thermal variable β , in view of (6.3), and its prolongation (6.8), we get

$$(7.3) \quad [[\text{grad } \vartheta]] = -U\tau(\vartheta)(v + \nabla\beta \cdot \mathbf{s})\mathbf{n}, \quad [[\dot{\vartheta}]] = -U[[\nabla\vartheta]] \cdot \mathbf{n},$$

where we use the following short denotation

$$(7.4) \quad \tau(\vartheta) = 1/F_1'(\vartheta).$$

For further calculation we need the local form of the energy

$$(7.5) \quad \partial\rho(\varepsilon + 0.5\mathbf{v} \cdot \mathbf{v})/\partial t + \text{div}(E\mathbf{v} + \mathbf{q} - \mathbf{v} \mathbf{T}) = \rho r + \rho \mathbf{v} \cdot \mathbf{b},$$

and the motion equation

$$(7.6) \quad \partial\rho\mathbf{v}/\partial t + \text{div}(\rho\mathbf{v} \otimes \mathbf{v} - \mathbf{T}) = \rho\mathbf{b}$$

in the spatial description. Now performing the differentiation of the stress constitutive function \mathbf{T}^* in the equation of motion (7.6), and of ε^* in the reduced energy equation, written below as the consequence of (7.6) and (7.5),

$$(7.7) \quad \rho\dot{\varepsilon} + \text{div } \mathbf{q} - \mathbf{T} \cdot \text{grad } \mathbf{v} = \rho r,$$

⁽⁴⁾ Such a surface of discontinuity represents geometrically the thermo-acoustic wave front.

and taking the results across the wave front, we obtain

$$\begin{aligned}
 (\rho U^2 \mathbf{I} - \mathbf{Q} + \tau U \mathbf{P} \mathbf{n} \otimes \nabla \beta) \mathbf{s} &= (-\tau U \mathbf{P} + (\mathbf{G} \mathbf{n})) v \mathbf{n}, \\
 (7.8) \quad (\rho U^2 c_v \tau \nabla \beta + \tau U \alpha^{*'}(\vartheta) \nabla \beta \cdot \mathbf{n} \nabla \beta) \cdot \mathbf{s} \\
 &= (-\rho U^2 c_v \tau + U \mathbf{t} \cdot \mathbf{n} - \tau U \alpha^{*'}(\vartheta) \nabla \beta \cdot \mathbf{n} + \alpha^*) v,
 \end{aligned}$$

where we have assumed the following abbreviations:

$$\begin{aligned}
 \mathbf{Q} &:= \underset{(3,5)}{\text{tr}} \underset{(4,6)}{\text{tr}} \frac{\partial \mathbf{T}^*(\mathbf{F}, \vartheta, \nabla \beta)}{\partial \mathbf{F}} \otimes \mathbf{F}^T \mathbf{n} \otimes \mathbf{n}, \\
 \mathbf{P} &:= \partial \mathbf{T}^*(\mathbf{F}, \vartheta, \nabla \beta) / \partial \vartheta, \\
 \mathbf{G} &:= \partial \mathbf{T}^*(\mathbf{F}, \vartheta, \nabla \beta) / \partial \nabla \beta, \\
 (7.9) \quad \alpha^{*'} &:= \partial \alpha^*(\vartheta) / \partial \vartheta, \\
 \mathbf{t} &:= \partial \varepsilon^*(\mathbf{F}, \vartheta, \nabla \beta) / \partial \nabla \beta, \\
 c_v &:= \partial \varepsilon^*(\mathbf{F}, \vartheta, \nabla \beta) / \partial \vartheta.
 \end{aligned}$$

Here the symbol $\underset{(3,5)}{\text{tr}}$ denotes the operation of the simple saturation (composition) of the third index with the fifth one of the tensor following the symbol.

Let us notice that in 3D case we have a system of four homogeneous equations for two amplitudes: the vector-valued mechanical amplitude \mathbf{s} and the scalar thermal one v . To get a nontrivial solution, the determinant of the system has to vanish. This gives us the so-called characteristic (or dispersion) relation for the speed U in terms of \mathbf{n} and values of state variables at the wave front.

The hyperbolicity of the governing system of equations (6.3), (7.6), (7.7) will be guaranteed if for any normal vector \mathbf{n} eight real solutions for U and four linear independent amplitude vectors (\mathbf{s}, v) exist.

We are not going to write this relation, restricting ourselves to some particular cases. If the wave propagates in the state with the vanishing heat flux, i.e. $\nabla \beta = \mathbf{0}$, then the system (7.8) is simpler and the characteristic relation will take the form

$$(7.10) \quad \det(\rho U^2 \mathbf{I} - \mathbf{Q})(\rho U^2 c_v \tau - \alpha^*) = 0,$$

giving three symmetric mechanical wave speeds, provided the acoustic tensor \mathbf{Q} is positive definite, and also one symmetric thermal wave speed, provided the specific heat c_v is positive. In the general case, with nonvanishing $\nabla \beta$ we can have coupled thermomechanical wave speeds.

We will conclude this section with the governing system of equations of the modified model of thermoelasticity written in a 4D divergence form. Such a form is necessary in order to discuss strong discontinuity waves and weak solutions, that are piecewise continuous and Lipschitz continuous with jump discontinuities. The system should be written as a first order one for the following fields: $\mathbf{u} -$

displacement vector, \mathbf{F} – the deformation gradient tensor, β – internal state variable, \mathbf{v} – particle velocity vector, η – the specific entropy, \mathbf{p} – the spatial gradient of β , where for the E (total specific energy), the following relationship has been assumed: $E = \varepsilon^*(\mathbf{F}, \eta, \mathbf{p}) + 0.5\mathbf{v} \cdot \mathbf{v}$ with the thermodynamic (absolute) temperature ϑ , given by the potential relation $\vartheta = \partial\varepsilon^*/\partial\eta$.

$$\begin{aligned}
 (7.11) \quad & \dot{\mathbf{u}} = \mathbf{v}, \\
 & \dot{\mathbf{F}} - \text{Div}(\mathbf{v} \otimes \mathbf{1}) = \mathbf{0}, \\
 & \dot{\beta} = F_1(\vartheta) + F_2(\beta), \\
 & \varrho_0 \dot{\mathbf{v}} - \text{Div} \mathbf{S} = \varrho_0 \mathbf{b}, \\
 & \varrho_0 \dot{E} + \text{Div}(\mathbf{q}_\kappa - \mathbf{v} \mathbf{S}) = \varrho_0 r + \varrho_0 \mathbf{v} \cdot \mathbf{b}, \\
 & \overline{\mathbf{pF}} - \text{Div}(F_1(\vartheta)\mathbf{1}) = F_2'(\beta)\mathbf{pF}.
 \end{aligned}$$

Let us notice that the last equation in this system is the differential consequence of the evolution equation (6.3) (compare also (6.8)). Moreover, from this equation the jump condition follows

$$(7.12) \quad -\sigma[[\mathbf{pF}]] = [[F_1(\vartheta)]]\mathbf{N},$$

where σ denotes normal component of the strong discontinuity wave (shock speed) while the unit vector \mathbf{N} is normal to the wave front in the reference placement (configuration). This jump condition is compatible with the condition following from the evolution equation (6.3), namely $[[\dot{\beta}]] = [[F_1(\vartheta)]]$ and the differential compatibility conditions for the first and second derivatives of β , namely $\text{Grad} \beta = \mathbf{pF}$ and $\overline{\mathbf{pF}} = \text{Grad} \dot{\beta}$.

8. Conclusions

The extensive discussion of possible propagation conditions will be given in the next paper. Now we can refer to [41, 42], where some particular 1D cases were considered, while in [37] 3D thermal weak discontinuity waves were investigated. The reader is referred to the recent paper of FRISCHMUTH and CIMMELLI [22], where 1D thermomechanical case with linearized constitutive equations has been numerically treated. There two waves have been calculated: one corresponding to longitudinal ballistic phonon propagation and the other, the second sound wave. That paper is the first one in which the present model has been successfully applied in the linear case for numerical treatment of coupled thermomechanical and second sounds waves observed in the heat pulse experiments described shortly in Sec. 2 of the present paper. In the next paper we will be concerned with the numerical simulation in 1D nonlinear case.

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