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On generalized parallelisms

Dedicated to Prof. Henryk Zorski
on the occasion of his 70-th birthday

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M. EPSTEIN AND M. DE LEÓN defined the second order non-holonomic parallelism on a manifold and applied it to a geometric description of generalized Cosserat continua. We explain that the underlying geometric idea is the concept of generalized parallelism on an arbitrary principal fiber bundle. Some properties of generalized parallelisms are characterized in terms of induced connections or from the viewpoint of the theory of generalized G-structures.

1. Introduction

THE STARTING POINT of the present paper were some recent results by M. Epstein and M. de León [3, 4], which describe geometrically generalized Cosserat continua in terms of the second order non-holonomic frame bundle $\widetilde{P}^2M$ of a manifold M. In particular, they introduced a second order non-holonomic parallelism on M as a section $M \to \widetilde{P}^2M$ and deduced several geometric properties of this parallelism. Taking into account that $\widetilde{P}^2M$ is the first principal prolongation of the first order frame bundle P^1M of M, [5], we demonstrate that an interesting geometric situation appears even if one replaces the first order frame bundle by an arbitrary principal fiber bundle P over M. That's why we introduce a generalized parallelism on P as a section $M \to W^1P$ of the first principal prolongation W^1P of P.

In Sec. 3 we show that the functoriality of W^1 makes possible a simple construction of a prolongation $p(s, B) : M \to W^1P$ of every section $s : M \to P$ and every classical parallelism $B : M \to P^1M$. In Sec. 4 we deduce that every generalized parallelism on P induces two connections on P and one connection on P^1M with analogous properties to the special case of second order non-holonomic parallelisms by Epstein and de León. Then we apply our theory of generalized G-structures, [5]. In Sec. 5 we describe the local flatness of generalized parallelisms from such a point of view. In the last section we prove that the generalized parallelism can be characterized in terms of its structure function similarly to the case of classical parallelisms. All manifolds and maps are assumed to be infinitely differentiable. A connection means always a principal (i.e. right-invariant) connection in the terminology of the book [6].

2. Prolongation of principal fiber bundles

Consider an arbitrary principal fiber bundle $P(M,G)$ with projection $\pi : P \to M$, $\dim M = m$. The first principal prolongation W^1P of P is the space of all 1-jets at $(0,e) \in \mathbb{R}^m \times G$ of local principal bundle isomorphisms $\varphi : \mathbb{R}^m \times G \to P$, where e is the unit of G, [6]. It follows that W^1P is a principal bundle over M, the structure group of which is $W^1_mG = W^1_0(\mathbb{R}^m \times G)$ (= the fiber of $W^1(\mathbb{R}^m \times G)$ over $0 \in \mathbb{R}^m$), where both the multiplication in W^1_mG and the right action of W^1_mG on W^1P are defined by jet composition. Since φ is a principal bundle morphism, it is determined by the restriction $\widetilde{\varphi} = \varphi|\mathbb{R}^m \times \{e\} : \mathbb{R}^m \to P$. The composition $\varphi_0 = \pi \circ \widetilde{\varphi} : \mathbb{R}^m \to M$ is a local diffeomorphism, so that we can construct locally the inverse map $(\varphi_0)^{-1}$. Then $\widetilde{\varphi} \circ (\varphi_0)^{-1}$ is locally a section of P. Passing to 1-jets we find that W^1P coincides with the fiber product over M

$$W^1P = P^1M \times_M J^1P\,, \tag{2.1}$$

where P^1M is the first order (= linear) frame bundle of M and J^1P is the first jet prolongation of P, [6]. Every manifold M can be identified with a principal bundle $M(M,\{e\})$, whose projection is the identity of M and the structure group is the one-element group $\{e\}$. In this case we have $W^1M = P^1M$. Further, if P itself is the first order frame bundle of M, its first principal prolongation W^1P^1M coincides with the second order non-holonomic frame bundle $\widetilde{P}^2M$ of M, [3, 4, 5].

Consider another principal fiber bundle $\bar{P}(\bar{M},\bar{G})$ satisfying $\dim \bar{M} = \dim M$. Every principal bundle morphism $f : P \to \bar{P}$ such that its base map $f_0 : M \to \bar{M}$ is a local diffeomorphism is extended into a map $W^1f : W^1P \to W^1\bar{P}$ defined by

$$W^1f(j^1_0\widetilde{\varphi}) = j^1_0(f \circ \widetilde{\varphi})\,, \tag{2.2}$$

where $\widetilde{\varphi} : \mathbb{R}^m \to P$ is the above map generating an element of W^1P. Clearly, W^1 is a functor. In particular, every section $s : M \to P$ can be interpreted as a principal bundle morphism $M(M,\{e\}) \to P(M,G)$. Hence we have the induced map $W^1s : W^1M = P^1M \to W^1P$.

The adjoint bundle of any principal bundle $P(M,G)$ is the associated bundle

$$LP = P[\mathfrak{g}, Ad]\,, \tag{2.3}$$

whose standard fiber is the Lie algebra $\mathfrak{g}$ of G with the adjoint action. According to [6], p. 161, LP is identified with the vertical tangent bundle VP of P factorized by the induced action of G on VP, i.e.

$$LP = VP/G\,. \tag{2.4}$$

In general, the first jet prolongation J^1Y of any fibered manifold $Y \to M$ is an affine bundle over Y, whose associated vector bundle is $VY \otimes T^*M$ [6], p. 125.

Let $\pi_0^1 : J^1P \to P$ be the target jet prolongation. Using the identification (2.4), we obtain immediately

LEMMA 1. Let $s_1, s_2 : M \to J^1P$ be two sections satisfying $\pi_0^1 \circ s_1 = \pi_0^1 \circ s_2 : M \to P$. Then $s_1 - s_2$ is a section of $LP \otimes T^*M$. □

3. Generalized parallelisms

The classical parallelism on a manifold M means an m-tuple $(B_1, \ldots, B_m)$ of vector fields on M, which are linearly independent at every point. Clearly, this parallelism can be interpreted as a section $B : M \to P^1M$. We recall that B is said to be locally flat, if for every $x \in M$ there exists a neighbourhood $U \subset M$ and a local coordinate system x^i on U such that the restrictions $B_i|U$ are the coordinate vector fields $\dfrac{\partial}{\partial x^i}$, $i = 1, \ldots, m$.

As mentioned in the introduction, Epstein and de León defined a second order non-holonomic parallelism on M as a section $M \to \widetilde{P}^2M = W^1P^1M$. We are going to study the following general concept (the classical parallelism corresponds to the case $P = M(M, \{e\})$).

DEFINITION 1. *A generalized parallelism on a principal bundle $P(M,G)$ is a section $A : M \to W^1P$.*

Since $W^1P = P^1M \times_M J^1P$, generalized parallelisms on P are identified with pairs $A = (A_1, A_2)$ of sections $A_1 : M \to P^1M$ and $A_2 : M \to J^1P$. Applying the projection $\pi_0^1 : J^1P \to P$, we obtain an induced section $A_0 = \pi_0^1 \circ A_2 : M \to P$.

For every section $s : M \to P$, we have constructed $W^1s : P^1M \to W^1P$.

DEFINITION 2. *For every section $s : M \to P$ and every classical parallelism $B : M \to P^1M$, the generalized parallelism $p(s, B) := (W^1s) \circ B : M \to W^1P$ on P is called the prolongation of s with respect to B.*

This construction leads to the following important concept.

DEFINITION 3. *A generalized parallelism $A : M \to W^1P$ is said to be decomposable, if $A = p(A_0, A_1)$.*

By Lemma 1, the difference

$$D(A) = A - p(A_0, A_1) \tag{3.1}$$

is a section $M \to LP \otimes T^*M$. This is the obstruction for decomposability of A. We remark that in the case $P = P^1M$ we have $LP^1M = TM \otimes T^*M$.

PROPOSITION 1. Generalized parallelisms on P are in bijection with triples of sections $A_0 : M \to P$, $A_1 : M \to P^1M$ and $D : M \to LP \otimes T^*M$.

Proof. We set $(A_0, A_1, D) = p(A_0, A_1) + D$. □

Clearly, $A = (A_0, A_1, D)$ is decomposable, if and only if $D = D(A) = 0$.

We remark that a similar result for connections on W^1P was deduced in [8].

4. Induced connections

Every section of a principal bundle P induces an integrable connection Γ on P, which is determined by the tangent spaces of the right translations of the section. In the case of a classical parallelism $B : M \to P^1M$, a classical result reads that B is locally flat, if and only if the connection Γ is torsion-free [3, 4].

In the case of a generalized parallelism $A : M \to W^1P$, the underlying section $A_0 : M \to P$ or $A_1 : M \to P^1M$ induces an integrable connection Γ_0 on P or Γ_1 on P^1M, respectively. Another connection $\Gamma_2 : P \to J^1P$ on P is defined by prescribing its values along the section A_0 by

$$\Gamma_2(A_0(x)) = A_2(x) \tag{4.1}$$

and by using the right-invariance condition. The following assertion generalizes the results by Epstein and de León about the second order non-holonomic parallelism.

PROPOSITION 2. A generalized parallelism $A : M \to W^1P$ is decomposable, if and only if $\Gamma_0 = \Gamma_2$.

P r o o f. If we use the product formula (2.1), we have $p(A_0, A_1) = (A_1, j^1A_0)$, where $j^1A_0 : M \to J^1P$ is the first jet prolongation of A. Hence A is decomposable, if and only if $A_2 = j^1A_0$. By (4.1), $\Gamma_0 = \Gamma_2$ means $A_2(A_0(x)) = j^1A_0(x)$ for all $x \in M$. This is equivalent to $A_2 = j^1A_0$. □

5. Generalized G-structures

Let $G \subset GL(m, \mathbb{R})$ be a subgroup. We recall that a classical G-structure on a manifold M is a reduction Q of the frame bundle P^1M to G, [9]. For a classical parallelism $B : M \to P^1M$, $B(M)$ is an $\{e\}$-structure on M, where e is the unit of $GL(m, \mathbb{R})$. In [5] we introduced the following generalization (which was motivated by the theory of higher order G-structures). Let $H \subset W^1_mG$ be a subgroup.

DEFINITION 4. *An H-structure on principal bundle $P(M,G)$ is a reduction Q of W^1P to H.*

We also say that Q is a generalized G-structure.

We have $W^1(\mathbb{R}^m \times G) = \mathbb{R}^m \times W^1_mG$. The product $\mathbb{R}^m \times H$, which is an H-structure on $\mathbb{R}^m \times G$, is called the standard flat H-structure.

Write $P|U$ for the restriction of $P(M,G)$ over an open subset $U \subset M$.

DEFINITION 5. *An H-structure $Q \subset W^1P$ is said to be locally flat, if for every $x \in M$ there exists a neighbourhood $U \subset M$ and a principal bundle isomorphism $f : \mathbb{R}^m \times G \to P|U$ such that $W^1f(\mathbb{R}^m \times H) = Q|U$.*

For every generalized parallelism $A : M \to W^1P$, $A(M)$ is an $\{e\}$-structure on P, where e is the unit of W^1_mG.

DEFINITION 6. *A generalized parallelism is said to be locally flat, if it is locally flat as an $\{e\}$-structure.*

PROPOSITION 3. A generalized parallelism $A : M \to W^1P$ is locally flat, if and only if A is decomposable and the underlying classical parallelism A_1 on M is locally flat.

Proof. Assume that A is decomposable and A_1 is locally flat. Let $U \subset M$ be an open subset and $h : \mathbb{R}^m \to U$ be a diffeomorphism transforming the classical standard flat parallelism $\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}\right)$ on $\mathbb{R}^m$ into $A_1|U$. Define a map $f : \mathbb{R}^m \to P$ by $f(x) = A_0(h(x))$. Then $A|U = W^1f(\mathbb{R}^m \times \{e\})$ follows directly from the proof of Proposition 2. The converse assertion can be proved by the same argument. □

Proposition 2 and 3 imply the following result, which compares our approach with the research by Epstein and M. de León [3, 4].

COROLLARY 1. A generalized parallelism is locally flat, if and only if $\Gamma_0 = \Gamma_2$ and Γ_1 is torsion-free.

6. Semiprolongable generalized G-structures

In [5] we have established that the most interesting generalized G-structures are the semiprolongable ones. Consider the jet projection $\pi_0^1 : W^1P \to P$. For every H-structure $Q \subset W^1P$, $H_0 := \pi_0^1(H)$ is a subgroup in G and $Q_0 := \pi_0^1(Q)$ is a reduction of P to H_0. By Sec. 2, the injection $i : Q \to P$ induces an injection $W^1i : W^1Q_0 \to W^1P$. The following concept was introduced in [5], where the reader can find a justification of the terminology.

DEFINITION 7. *An H-structure $Q \subset W^1P$ is called semiprolongable, if $Q \subset W^1Q_0$.*

If we apply this concept to a generalized parallelism on P, we obtain, as a direct consequence of Proposition 2, the following assertion.

COROLLARY 2. A generalized parallelism $A : M \to W^1P$ is semiprolongable, if and only if A is decomposable.

7. The structure function

From the viewpoint of the theory of generalized G-structures, it is interesting that the local flatness of generalized parallelisms can be characterized in terms of the structure function, analogously to the case of a classical parallelism.

On the linear frame bundle P^1M, we have the canonical form $\psi : TP^1M \to \mathbb{R}^m$, [6, 9]. For a classical G-structure $Q \subset P^1M$, a horizontal tangent space means any m-dimensional linear subspace in TQ which is complementary to the vertical tangent space. The structure function τ of Q is defined by restricting the exterior differential $d\psi$ to the horizontal tangent spaces of Q, [7, 9]. This is a map $\tau : Q \to H^{0,2}(\mathfrak{g})$, where $H^{0,2}(\mathfrak{g})$ denotes the Spencer cohomology class of bidegree $(0, 2)$ of the Lie algebra $\mathfrak{g}$ of G. In the case of a classical parallelism $B : M \to P^1M$, the structure function of $B(M)$ is a map $\tau : B(M) \to \mathbb{R}^m \otimes \Lambda^2\mathbb{R}^{m*}$. By the definition of ψ, the structure function of $B(M)$ coincides with the torsion of the integrable connection Γ determined by B. Hence B is locally flat if and only if its structure function vanishes.

On W^1P, we have a canonical form $\theta : TW^1P \to \mathbb{R}^m \oplus \mathfrak{g}$, [6]. Proposition 2 of [5] reads that an H-structure $Q \subset W^1P$ is semiprolongable if and only if the values of the restriction of θ to TQ lie in $\mathbb{R}^m \oplus \mathfrak{h}_0$, where $\mathfrak{h}_0$ is the Lie algebra of $H_0 = \pi_0^1(H) \subset G$. In particular, for a decomposable generalized parallelism $A : M \to W^1P$ the values of the restriction of θ to $T(A(M))$ lie in $\mathbb{R}^m \oplus \{0\}$. For a semiprolongable H-structure Q, we defined its structure function τ by restricting the exterior differential $d\theta$ to certain distinguished horizontal tangent subspaces of Q, [5]. By [5], τ is a map $\tau : Q \to H^{0,2}(\mathfrak{k})$, where $\mathfrak{k}$ is the Lie algebra of the kernel K of the jet homomorphism $\pi_0^1 : H \to H_0$. In particular, the structure function of a decomposable generalized parallelism has values in $\mathbb{R}^m \otimes \Lambda^2\mathbb{R}^{m*}$.

PROPOSITION 4. A decomposable generalized parallelism $A : M \to W^1P$ is locally flat, if and only if its structure function vanishes.

Proof. The bundle projection $W^1P \to M$ identifies $A(M)$ with M. This identifies $T(A(M))$ with TM. By definition, the restriction of θ to $T(A(M))$ corresponds to the restriction of the canonical form ψ of P^1M to $T(A_1(M))$. Then our assertion follows from the above mentioned classical result. □

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