

Fractional calculus and stable probability distributions

*Dedicated to Prof. Henryk Zorski
on the occasion of his 70-th birthday*

R. GORENFLO (BERLIN) and F. MAINARDI (BOLOGNA)

FRACTIONAL calculus allows one to generalize the linear (one-dimensional) diffusion equation by replacing either the first time-derivative or the second space-derivative by a derivative of a fractional order. The fundamental solutions of these generalized diffusion equations are shown to provide certain probability density functions, in space or time, which are related to the relevant class of stable distributions. For the space fractional diffusion, a random-walk model is also proposed.

1. Introduction

THE PURPOSE of this note is to outline the role of fractional calculus in generating stable probability distributions through generalized diffusion equations of fractional order.

For the standard diffusion equation it is well known that the fundamental solution of the Cauchy problem provides the spatial *probability density function* (*pdf*) for the Gaussian or normal distribution, whose variance is proportional to time. For convenience, let us derive this result, using standard notation and leaving out the regularity requirements. The Cauchy problem for the diffusion equation reads

$$(1.1) \quad \frac{\partial}{\partial t} u(x, t) = D \frac{\partial^2}{\partial x^2} u(x, t), \\ -\infty < x < \infty, \quad t \geq 0 \quad \text{with} \quad u(x, 0) = g(x),$$

and can be easily solved making use of the Fourier transform. Adopting the notation $g(x) \div \hat{g}(\kappa)$ with $\kappa \in \mathbb{R}$ and

$$\hat{g}(\kappa) = \mathcal{F}[g(x)] = \int_{-\infty}^{\infty} e^{+i\kappa x} g(x) dx, \\ g(x) = \mathcal{F}^{-1}[\hat{g}(\kappa)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\kappa x} \hat{g}(\kappa) d\kappa,$$

the transformed solution reads

$$(1.2) \quad \hat{u}(\kappa, t) = \hat{g}(\kappa) e^{-Dt\kappa^2}.$$

Then, introducing

$$(1.3) \quad \mathcal{G}_c^d(x, t) \doteq e^{-Dt\kappa^2},$$

the required solution is provided by inversion in terms of the space convolution

$$(1.4) \quad u(x, t) = \int_{-\infty}^{\infty} \mathcal{G}_c^d(\xi, t) g(x-\xi) d\xi, \quad \mathcal{G}_c^d(x, t) = \frac{1}{2\sqrt{\pi D}} t^{-1/2} e^{-x^2/(4Dt)}.$$

Here $\mathcal{G}_c^d(x, t)$ represents the fundamental solution (or Green function) of the Cauchy problem, since it corresponds to $g(x) = \delta(x)$.

The interpretation of such Green function in probability theory is straightforward since we easily recognize

$$(1.5) \quad \mathcal{G}_c^d(x, t) = p_G(x; \sigma) := \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/(2\sigma^2)}, \quad -\infty < x < \infty, \quad \sigma^2 = 2Dt,$$

where $p_G(x; \sigma)$ denotes the well-known *Gauss* or *normal pdf* whose moment of the second order, the *variance*, is σ^2 . The associated cumulative distribution function (*cdf*) is known to be

$$(1.6) \quad \mathcal{P}_G(x; \sigma) := \int_{-\infty}^x p_G(x'; \sigma) dx' = \frac{1}{2} \left[1 + \operatorname{erf} \left(\frac{x}{\sqrt{2}\sigma} \right) \right] \\ = \frac{1}{2} \left[1 + \operatorname{erf} \left(\frac{x}{2\sqrt{Dt}} \right) \right],$$

where $\operatorname{erf}(z) := (2/\sqrt{\pi}) \int_0^z \exp(-u^2) du$ denotes the error function. Furthermore, the moments of even order of the *Gauss pdf* turn out to be

$$(1.7) \quad \int_{-\infty}^{\infty} x^{2n} p_G(x; \sigma) dx = \frac{(2n)!}{2^n n!} \sigma^{2n} \\ = (2n-1)!! \sigma^{2n} = (2n-1)!! (2Dt)^n, \quad n \in \mathbb{N},$$

where \mathbb{N} denotes the set of the positive integers.

Let us show how the fundamental solution of the *signalling* problem in the semi-infinite line provides a time *pdf* related to the unilateral Lévy distribution, a property not so well-known as that for the Cauchy problem. Under fairly general conditions, the *signalling* problem

$$(1.8) \quad \frac{\partial}{\partial t} u(x, t) = D \frac{\partial^2}{\partial x^2} u(x, t), \quad x \geq 0, \quad t \geq 0 \quad \text{with} \quad u(0, t) = \phi(t),$$

can be easily solved by making use of the Laplace transform. Adopting the notation $\phi(t) \div \tilde{\phi}(s)$ with $s \in \mathbb{C}$ and

$$\tilde{\phi}(s) = \mathcal{L}[\phi(t)] = \int_0^\infty e^{-st} \phi(t) dt, \quad \phi(t) = \mathcal{L}^{-1}[\tilde{\phi}(s)] = \frac{1}{2\pi i} \int_{\text{Br}} e^{st} \tilde{\phi}(s) ds,$$

where Br denotes the Bromwich path, the transformed solution reads

$$(1.9) \quad \tilde{u}(x, s) = \tilde{\phi}(s) e^{-(x/\sqrt{D}) s^{1/2}}.$$

Then introducing

$$(1.10) \quad \mathcal{G}_s^d(x, t) \div e^{-(x/\sqrt{D}) s^{1/2}},$$

the required solution is provided by inversion in terms of the time convolution,

$$(1.11) \quad u(x, t) = \int_0^t \mathcal{G}_s^d(x, \tau) \phi(t-\tau) d\tau, \quad \mathcal{G}_s^d(x, t) = \frac{x}{2\sqrt{\pi D}} t^{-3/2} e^{-x^2/(4Dt)}.$$

Here $\mathcal{G}_s^d(x, t)$ represents the fundamental solution (or Green function) of the signalling problem, since it corresponds to $\phi(t) = \delta(t)$. We note that

$$(1.12) \quad \mathcal{G}_s^d(x, t) = p_L(t; \mu) := \frac{\sqrt{\mu}}{\sqrt{2\pi} t^{3/2}} e^{-\mu/(2t)}, \quad t \geq 0, \quad \mu = \frac{x^2}{2D},$$

where $p_L(t; \mu)$ denotes the unilateral Lévy pdf, with cdf

$$(1.13) \quad \mathcal{P}_L(t; \mu) := \int_0^t p_L(t'; \mu) dt' = \text{erfc} \left(\sqrt{\frac{\mu}{2t}} \right) = \text{erfc} \left(\frac{x}{2\sqrt{Dt}} \right),$$

see e.g. FELLER [1]. The Lévy pdf has all moments of integral order infinite, since it decays at infinity as $t^{-3/2}$. However, we note that the absolute moments of real order δ are finite only if $0 \leq \delta < 1/2$. In particular, for this pdf the mean (expectation) is infinite, but the median is finite. In fact, from $\mathcal{P}_L(t_{\text{med}}; \mu) = 1/2$, it turns out that $t_{\text{med}} \approx 2\mu$, since the complementary error function gets the value 1/2 as its argument is approximately 1/2 (a better estimate of the argument is 1/2.1).

Both the Gauss and Lévy laws belong to the important class of stable probability distributions, which are mainly characterized by an index α ($0 < \alpha \leq 2$), called index of stability or characteristic exponent. In particular, the index of the Gauss law is 2, whereas that of the Lévy law is 1/2. A special case of stable distribution with $\alpha = 1$ is provided by the Cauchy law with pdf $p_C(x; \lambda) = \lambda/[\pi(x^2 + \lambda^2)]$, $\lambda > 0$.

For the theory of *stable distributions* we refer to some classical treatises of probability theory as [1–4], to the monographs by ZOLOTAREV [5], JANICKI and WERON [6], SAMORODNITSKY and TAQQU [7] and to the article by SCHNEIDER [8]. Here we limit ourselves to recall the main properties essential for the present analysis.

All the stable *pdf* are *unimodal* and indeed *bell-shaped*, i.e. their n -th derivative has exactly n zeros, see GAWRONSKI [9]. However, while the Gaussian distribution has a finite variance and is symmetric around its mean, for $0 < \alpha < 2$ all the stable distributions have infinite variance and can have an arbitrary degree of skewness. The skewness parameter is varying around zero (*symmetric pdf*) between two extremal symmetrical values (*extremal pdf*); for $0 < \alpha < 1$ the *extremal pdf* turn out to be *unilateral*, i.e. restricted to a semi-infinite real line. For $0 < \alpha < 2$ all the stable distributions exhibit at least one long tail, which decays with the power $(\alpha + 1)$, so their absolute moments of order δ are finite only if $\delta < \alpha$. Note that for asymmetric distributions, the faster tail may decay exponentially.

For our present purposes it is sufficient to draw the attention to the Fourier transform of the *symmetric* stable distributions acting for $x \in \mathbb{R}$, i.e.

$$(1.14) \quad p_S(x; \alpha, a) \div \widehat{p}(\kappa; \alpha, a) = e^{-a|\kappa|^\alpha}, \quad a > 0, \quad 0 < \alpha \leq 2$$

and to the Laplace transform of the *unilateral* stable distributions acting for $t \in \mathbb{R}^+$, i.e.

$$(1.15) \quad p_U(t; \beta, b) \div \widetilde{p}(s; \beta, b) = e^{-bs^\beta}, \quad b > 0, \quad 0 < \beta < 1.$$

In (1.14)–(1.15) the parameters α, β denote the characteristic exponent of the corresponding stable distribution whereas a, b are scaling factors. The singularity of the Fourier and Laplace transforms in the origin corresponds to the power-type long tails of the distribution.

As a matter of fact we note that for the standard diffusion equation, the Green function for the Cauchy problem yields (1.14) with $\alpha = 2$ and $a = Dt$, whereas the Green function for the *signalling* problem yields (1.15) with $\beta = 1/2$ and $b = x/\sqrt{D}$.

In order to reproduce both the classes of stable distributions (1.14)–(1.15), we need to consider separately the Cauchy problem and the *signalling* problem for two different diffusion equations in which the space or time derivatives of integral order are substituted by special pseudo-differential operators, which are shown to be expressed in terms of suitable “fractional derivatives”. For the Cauchy problem we consider the so-called space fractional diffusion equation

$$(1.16) \quad \frac{\partial u}{\partial t} = D(\alpha) \frac{\partial^\alpha u}{\partial |x|^\alpha}, \quad -\infty < x < \infty, \quad t \geq 0 \quad \text{with} \quad u(x, 0) = \delta(x),$$

where $0 < \alpha \leq 2$ and $D(\alpha)$ is a suitable diffusion coefficient depending on α . Here the pseudo-differential operator of space fractional derivative (symmetric in x) is defined by its Fourier representation,

$$(1.17) \quad \frac{\partial^\alpha}{\partial|x|^\alpha} u(x, t) \doteq -|\kappa|^\alpha \hat{u}(\kappa, t).$$

Correspondingly, for the *signalling* problem we consider the so-called time fractional diffusion equation

$$(1.18) \quad \frac{\partial^{2\beta} u}{\partial t^{2\beta}} = D(\beta) \frac{\partial^2 u}{\partial x^2}, \quad x \geq 0, \quad t \geq 0 \quad \text{with} \quad u(0, t) = \delta(t),$$

where $0 < \beta < 1$ and $D(\beta)$ is a suitable diffusion coefficient depending on β . Here the pseudo-differential operator of time fractional derivative is defined by its Laplace representation,

$$(1.19) \quad \frac{\partial^{2\beta} u}{\partial t^{2\beta}} \doteq \begin{cases} s^{2\beta} \tilde{u}(x, s) - s^{2\beta-1} u(x, 0^+), & \text{if } 0 < \beta \leq 1/2, \\ s^{2\beta} \tilde{u}(x, s) - s^{2\beta-1} u(x, 0^+) - s^{2\beta-2} u_t(x, 0^+), & \text{if } 1/2 < \beta \leq 1. \end{cases}$$

We easily recognize from (1.17) and (1.19) that the pseudo-differential operators reduce to the space and time derivatives of integral order entering the standard diffusion equation when $\alpha = 2$ and $\beta = 1/2$, respectively.

In the following sections we shall discuss the two fractional diffusion equations, providing an interpretation of the pseudo-differential operators (1.17) and (1.19) in the framework of some established theories of the fractional calculus.

We begin in Sec. 2 with the time fractional diffusion equation that in recent years has been extensively treated by MAINARDI [10–14]. Other significant treatments of this equation have been given by a number of authors including WYSS [15], SCHNEIDER and WYSS [16], SCHNEIDER [17], FUJITA [18], KOCHUBEI [19], EL-SAYED [20], and ENGLER [21]. In this case the fractional derivative can be interpreted recurring to the Riemann–Liouville approach to the fractional calculus.

For the space fractional diffusion equation, the literature on extensive and clear treatments appears poor, in that the topic is mostly treated only briefly from the mathematical point of view, as in SESHADRI and WEST [22], TAKAYASU [23], ZASLAVSKY [24], COMPTE [25]. In this case, the fractional derivative can be interpreted as recurring to the Riesz approach to the fractional calculus. In Sec. 3 we present a new and interesting analysis of the space fractional diffusion equation, originally started by GORENFLO and MAINARDI [26], which leads to an interpretation through a random-walk model. This analysis has been inspired by a classical (but almost ignored) contribution by FELLER [27] and by a recent paper by SAICHEV and ZASLAVSKY [28].

2. The time fractional diffusion equation

In the time fractional diffusion equation (1.18), the pseudo-differential operator of fractional derivative (1.19) is acting for $t \in \mathbb{R}^+$ and hence we must consider the Riemann–Liouville fractional calculus, which is suitable for causal functions $f(t)$, vanishing for $t < 0$. For details on this calculus the reader is referred to the standard treatises of fractional calculus, which include OLDHAM and SPANIER [29], SAMKO, KILBAS and MARICHEV [30], MILLER and ROSS [31], and to our recent CISM lecture notes [32–33].

Here, for our purposes, we adopt the following definition for the fractional derivative of order α of a causal function $f(t)$,

$$(2.1) \quad \frac{d^\alpha}{dt^\alpha} f(t) := \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau$$

with $m-1 < \alpha < m$, $m \in \mathbb{N}$,

where Γ is the Gamma function and $f^{(m)}(t)$ denotes the derivative of order m , which is assumed to be Laplace transformable. This definition has been originally introduced by CAPUTO [34, 35] in the late sixties, and extensively applied by CAPUTO & MAINARDI [36] for modelling dissipation effects in *Linear Viscoelasticity*. This derivative, that we refer to as the Caputo derivative, represents a sort of regularization in the time origin for the Riemann–Liouville fractional derivative by incorporating the relevant initial conditions. It has been extensively investigated in [32] in view of its major utility in treating physical and engineering problems with standard initial conditions, but has been ignored in the mathematical treatises [29–31]. The Caputo derivative satisfies the relevant properties: in particular, the derivative of any order of a constant is still zero and the Laplace transform of a derivative of non-integral order still requires the initial data for integral derivatives, according to the rule

$$(2.2) \quad \frac{d^\alpha}{dt^\alpha} f(t) \div s^\alpha \tilde{f}(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+) s^{\alpha-1-k}, \quad m-1 < \alpha \leq m.$$

Since the above property is consistent with (1.19) for $\alpha = 2\beta$, and $m = 1, 2$, the Caputo derivative is just the pseudo-differential operator suitable for the time fractional diffusion equation.

The application of the Laplace transform to the *signalling* problem (1.18) allows us to find the corresponding Green function, that we denote by $\mathcal{G}_s(x, t; \beta)$. In fact, after standard manipulations, we obtain the transformed solution

$$(2.3) \quad \tilde{\mathcal{G}}_s(x, s; \beta) = e^{-(x/\sqrt{D(\beta)})s^\beta}, \quad x \geq 0.$$

Introducing the similarity variable $r = x/(\sqrt{D(\beta)}t^\beta)$ and the auxiliary function

$$(2.4) \quad F(r; \beta) := \frac{1}{2\pi i} \int_{\text{Br}} e^{\sigma - r\sigma^\beta} d\sigma,$$

we find

$$(2.5) \quad \mathcal{G}_s(x, t; \beta) = \frac{1}{t} F(r; \beta) \sim c(x) t^{-(1+\beta)}, \quad t \rightarrow \infty,$$

where $c(x)$ is a certain positive function. The definition of $F(r; \beta)$ can be analytically continued from $r > 0$ to any $z \in \mathbb{C}$, by deforming the Bromwich path into the Hankel path. The auxiliary function $F(z)$ turns out as an entire function of order $1/(1-\beta)$, which can be identified with a special function, known as Wright function [37].

In conclusion, the Green function for the *signalling* problem of the time fractional diffusion equation turns out to be a *unilateral* stable distribution in time, with characteristic exponent β and scale factor $b = x/\sqrt{D(\beta)}$, that is expressed in terms a Wright function in the similarity variable. For more details we refer the reader to [10–14], where also the Cauchy problem has been treated.

It is noteworthy to recall here that for the Cauchy problem, the corresponding Green function, obtained by the techniques of the Laplace or Fourier transform, turns out to be a *symmetric pdf* in space, provided by

$$(2.6) \quad \mathcal{G}_c(|x|, t; \beta) = \frac{1}{2\beta|x|} F(|r|; \beta) \sim a(t) |x|^{(\beta-1/2)/(1-\beta)} e^{-b(t)|x|^{1/(1-\beta)}},$$

as $|x| \rightarrow \infty$, where $a(t)$, $b(t)$ are positive functions. Therefore the *pdf* exhibits two branches, for $x > 0$ and $x < 0$, obtained one from the other by reflection. The exponential decay of such *pdf* allows the existence of all the moments; we obtain

$$(2.7) \quad \int_{-\infty}^{\infty} x^{2n} \mathcal{G}_c(x, t; \beta) dx = \frac{\Gamma(2n+1)}{\Gamma(2\beta n+1)} [D(\beta) t^{2\beta}]^n, \quad n = 0, 1, 2, \dots$$

We recognize that the variance is now proportional to $D t^{2\beta}$, which implies a phenomenon of *slow diffusion* (or *sub-diffusion*) if $0 < \beta < 1/2$, *fast diffusion* (or *hyper-diffusion*) if $1/2 < \beta < 1$, and, of course, *normal diffusion* if $\beta = 1/2$. Furthermore, we recognize that for $1/2 < \beta < 1$, any branch of the *pdf* is proportional to the exponential branch of an *extremal* stable distribution of index $1/\beta$.

3. The space fractional diffusion equation

In the space-fractional diffusion equation (1.16) the pseudo-differential operator of fractional derivative (1.16) is acting for $x \in \mathbb{R}$ in a symmetric way. Here we must consider the fractional calculus in the framework of Riesz potentials and define properly the fractional derivative (1.17) as inverse of the Riesz fractional integral. We recall that for $0 < \alpha \leq 2$, with $\alpha \neq 1$ and for a sufficiently well-behaved function $\phi(x)$, $x \in \mathbb{R}$, the Riesz integral operator or Riesz potential I^α and its image in the Fourier domain read

$$(3.1) \quad I^\alpha \phi(x) := \frac{1}{2 \Gamma(\alpha) \cos(\pi\alpha/2)} \int_{-\infty}^{\infty} |x - \xi|^{\alpha-1} \phi(\xi) d\xi \div \frac{\hat{\phi}(\kappa)}{|\kappa|^\alpha}.$$

On its turn the Riesz potential can be written in terms of two Weyl integrals I_\pm^α according to

$$(3.2) \quad I^\alpha \phi(x) = \frac{1}{2 \cos(\pi\alpha/2)} [I_+^\alpha \phi(x) + I_-^\alpha \phi(x)],$$

where

$$(3.3) \quad \begin{aligned} I_+^\alpha \phi(x) &:= \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x - \xi)^{\alpha-1} \phi(\xi) d\xi, \\ I_-^\alpha \phi(x) &:= \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} (\xi - x)^{\alpha-1} \phi(\xi) d\xi. \end{aligned}$$

Then, at least in a formal way, the fractional derivative (1.17) turns out to be

$$(3.4) \quad \frac{d^\alpha}{|dx|^\alpha} \phi(x) := -I^{-\alpha} \phi(x) = -\frac{1}{2 \cos(\pi\alpha/2)} [I_+^{-\alpha} \phi(x) + I_-^{-\alpha} \phi(x)].$$

Notice that (3.4) becomes meaningless if $\alpha = 1$. Here we resist the temptation to dive into the delicate details of the analytical inversion of the Riesz potential but rather refer the interested reader to the specialized treatises by SAMKO, KILBAS and MARICHEV [30] and RUBIN [38], and to the paper by SAICHEV and ZASLAVSKI [28].

Here, for $0 < \alpha \leq 2$, $\alpha \neq 1$, we propose a numerical approach for such inversion, approximating the evolution of the solution $u(x, t)$ of (1.16), interpreted as a probability density, by a (symmetric) *random walk* model, discrete in space and time. We shall see how things become highly transparent, in that we properly generalize the classical random-walk argument of the common diffusion equation to our spatial fractional equation (1.16). So doing we are in a position to

provide in the future a numerical simulation of the related (symmetric) stable distributions in a way analogous to the standard one for the Gaussian law.

The *essential idea* is to approximate the inverse operators $I_{\pm}^{-\alpha}$ by the Grünwald–Letnikov scheme, on which the reader can be informed in the treatises on fractional calculus [29–31] and in [33]. If h denotes a “small” positive step-length, these approximate operators read

$$(3.5) \quad {}_h I_{\pm}^{-\alpha} \phi(x) := \frac{1}{h^{\alpha}} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} \phi(x \mp kh).$$

Assume, for simplicity, $D(\alpha) = 1$, and introduce grid points $x_j = jh$ with $h > 0$, $j \in \mathbf{Z}$, and time instances $t_n = n\tau$ with $\tau > 0$, $n \in \mathbb{N}_0$. Let there be given the probabilities $p_{j,k} \geq 0$ of jumping from point x_j at instant t_n to point x_k at instant t_{n+1} , and define the probabilities $y_j(t_n)$ of the walker being at point x_j at instant t_n . Then, by

$$(3.6) \quad y_k(t_{n+1}) = \sum_{j=-\infty}^{\infty} p_{j,k} y_j(t_n), \quad p_{j,k} = p_{k,j}, \quad \sum_{k=-\infty}^{\infty} p_{j,k} = \sum_{j=-\infty}^{\infty} p_{j,k} = 1,$$

a symmetric random walk (more precisely a symmetric random jump) model is described. With the approximation $y_j(t_n) \approx \int_{(x_j-h/2)}^{(x_j+h/2)} u(x, t_n) dx \approx h u(x_j, t_n)$, and introducing the “scaling parameter” $\mu = \tau/[h^{\alpha} 2 |\cos(\alpha\pi/2)|]$, we have solved

$$(3.7) \quad \frac{y_j(t_{n+1}) - y_j(t_n)}{\tau} = - {}_h I^{-\alpha} y_j(t_n),$$

for $y_j(t_{n+1})$. So we have proved to have a consistent (for $h \rightarrow 0$) symmetric random walk approximation to (1.16) by taking

i) for $0 < \alpha < 1$,

$$(3.8) \quad \begin{aligned} {}_h I^{-\alpha} y_j(t_n) &= \mu \frac{h^{\alpha}}{\tau} \left[{}_h I_{+}^{-\alpha} y_j(t_n) + {}_h I^{-\alpha} y_j(t_n) \right], \\ 0 < \mu &\leq 1/2, \\ p_{j,j} &= 1 - 2\mu, \quad p_{j,j \pm k} = \mu \left| \binom{\alpha}{k} \right|, \quad k \geq 1; \end{aligned}$$

ii) for $1 < \alpha \leq 2$,

$$(3.9) \quad \begin{aligned} {}_h I^{-\alpha} y_j(t_n) &= \mu \frac{h^{\alpha}}{\tau} \left[{}_h I_{+}^{-\alpha} y_{j+1}(t_n) + {}_h I_{-}^{-\alpha} y_{j-1}(t_n) \right], \\ 0 < \mu &\leq 1/(2\alpha), \\ p_{j,j} &= 1 - 2\mu\alpha, \quad p_{j,j \pm 1} = \mu \left[1 + \binom{\alpha}{2} \right], \quad p_{j,j \pm k} = \mu \left| \binom{\alpha}{k+1} \right|, \quad k \geq 2. \end{aligned}$$

In the special case $\alpha = 2$ we recover the well-known three-point approximation of the heat equation, because $p_{j,j\pm k} = 0$ for $k \geq 2$. This means that for approximation of common diffusion, only jumps of one step to the right or one to the left or jumps of width zero occur, whereas, for all other values of α , i.e. $0 < \alpha < 2$ with $\alpha \neq 1$, arbitrary large jumps occur with power-like decaying probability, as it turns out from the asymptotic analysis for the probability coefficients. In fact, using the reflection and Stirling formulas for the Gamma functions entering the binomial coefficients, one finds

$$(3.10) \quad p_{j,j+k} \sim \frac{\mu}{\pi} \Gamma(\alpha + 1) |\sin(\pi\alpha)| k^{-(\alpha+1)}, \quad k \rightarrow \infty.$$

This result thus provides the discrete counterpart of the expected asymptotic behavior for the long power-law tails of the (symmetric) stable distributions when $0 < \alpha < 2$.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE,
FREE UNIVERSITY OF BERLIN

Arnimallee 2-6, D-14195 Berlin, Germany
e-mail: gorenflo@math.fu-berlin.de

DEPARTMENT OF PHYSICS,
UNIVERSITY OF BOLOGNA

Via Irnerio 46, I-40126 Bologna, Italy
e-mail: mainardi@bo.infn.it

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