On two reinterpretations of Cosserat continuum: fiber bundle versus the motor calculus

Dedicated to Prof. Henryk Zorski
on the occasion of his 70-th birthday

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IT IS THE PURPOSE of this paper to reinterprete the original Cosserat continuum from the point of view of both the fiber bundles geometry and the non-Abelian motor calculus. The main ideas of Cosserats are best explained in terms of the moving reper which is synonymous with the procedure of gauging in physics as well as with the procedure of constructing a fiber bundle in pure geometry. On the other hand, the classical (linear) von Mises motor calculus is extended to a non-Abelian case. It also appears that this non-Abelian version of the von Mises concept is fully equivalent with the fiber bundle description.

Notations

A connection,
b   tensor of the second fundamental form,
D   differential operator,
F   curvature,
I   disclination current tensor,
J   dislocation current tensor,
M   motor,
R curvature,
S torsion,
u   translation vector,
v   velocity vector,
X   body force,
Y   body couple,

α dislocation density,
Γ connection,
$\boldsymbol{\gamma}$ strain tensor,
γ time connection,
ε antisymmetric tensor,
θ disclination density,
κ bend-twist tensor,
μ couple-stress tensor,
σ stress tensor,
φ rotation vector,

$\mathcal{A}$ one-form of fiber connection,
$\mathcal{D}$ differential operator,
$\mathcal{F}$ two-form of bundle curvature,
$\mathcal{J}$ base,
$\mathcal{M}$ base manifold,
$\mathcal{P}$ bundle,
$\mathcal{R}$ curvature,
$\mathcal{S}$ torsion.

1. Introduction

STARTING from investigations of KONDO [1] and BILBY *et al.* [2] who identified the Cartan torsion tensor with the dislocation density, development of continuum mechanics is closely associated with differential geometrical ideas and methods. The subject has been extensively developed in Refs. [3–8], among others. The tangent bundle and the connection on it are the fundamental notions of the above-mentioned studies.

The next important step was made by the gauge theory of defects (for example, see [9–11] and others). The gauge theory considers not only the tangent bundle but also more general fiber bundles. It has recently been proposed that a fiber bundle might be of considerable interest in continuum mechanics as a basis for a differential geometrical description of the complex interactions between the recoverable and the dissipative processes.

In order to describe forces and couples, translations and rotations, etc. simultaneously, VON MISES [12, 13] developed the motor algebra, an algebra of vector fields in rigid bodies. OSIPOV [14, 15] and SCHAEFER [16, 17] extended the motor algebra to the motor analysis and introduced the differential operators for motor fields. At the same time, it turns out that there is a close correlation between the motor calculus and the fiber bundle with a group $T(3) \rhd SO(3)$ as a fiber [18–21].

In COSSERAT pioneering paper [22], the geometrization and gauging play a leading role. Recalling the Cosserat results in contemporary language, one can say that during their treatment, the properties of the mathematical model are strictly separated. All geometrical properties are carried by the base manifold, while all the physical properties are embedded within the standard fiber – structural group $T(3) \rhd SO(3)$. The consequences of this are valuable indeed. The objects familiar for the Cauchy continuum, such as displacements, measures of deformation and stresses, equations of motion, etc. are now defined within the space of fiber bundle and, what is characteristic for the Cosserat bundle, they are completely separated from the base.

The aim of the present paper is to show the development of ideas and to reinterpret the original Cosserat continuum in terms of both the non-Abelian motor calculus and the fiber bundles geometry.

2. The Abelian motor calculus

For a given point P of a rigid body, infinitesimal translation and infinitesimal rotation are described by the translation vector $\mathbf{u}(P)$ and the rotation vector $\boldsymbol{\varphi}(P)$ forming a motor

(2.1)
$$\begin{pmatrix} \mathbf{u}(P) \\ \boldsymbol{\varphi}(P) \end{pmatrix},$$

i.e. the ordered pair of two vectors which changes according to the rule

(2.2)
$$\begin{pmatrix} \mathbf{u}(Q) \\ \boldsymbol{\varphi}(Q) \end{pmatrix} = \begin{pmatrix} \mathbf{u}(P) + \boldsymbol{\varphi}(P) \times \overrightarrow{QP} \\ \boldsymbol{\varphi}(P) \end{pmatrix}$$

when changing a reduction point. For the following matrix notation it is convenient to use such an order of vectors forming a motor as that shown in Eq. (2.1), in contrast to that used in the literature.

The gradient and the curl of a motor field have the following form [17]:

(2.3)
$$\operatorname{grad} \begin{pmatrix} \mathbf{W} \\ \mathbf{V} \end{pmatrix} = \begin{pmatrix} \operatorname{grad} \mathbf{W} - \mathbf{V} \times \mathbf{1} \\ \operatorname{grad} \mathbf{V} \end{pmatrix},$$

(2.4)
$$\operatorname{rot} \begin{pmatrix} \mathbf{W} \\ \mathbf{V} \end{pmatrix} = \begin{pmatrix} \operatorname{rot} \mathbf{W} - \mathbf{V}_\times^\times \mathbf{1} \\ \operatorname{rot} \mathbf{V} \end{pmatrix}$$

with the well-known relation

(2.5)
$$\operatorname{rot} \operatorname{grad} \begin{pmatrix} \mathbf{W} \\ \mathbf{V} \end{pmatrix} = 0.$$

In the case of a two-dimensional surface Σ embedded in a three-dimensional space, we have [23]

(2.6)
$$\operatorname{grad}_\Sigma \begin{pmatrix} \mathbf{W}_\Sigma \\ \mathbf{V}_\Sigma \end{pmatrix} = \begin{pmatrix} \operatorname{grad}_\Sigma \mathbf{W}_\Sigma - \mathbf{V}_\Sigma \times \mathbf{1}_\Sigma \\ \operatorname{grad}_\Sigma \mathbf{V}_\Sigma \end{pmatrix},$$

(2.7)
$$\operatorname{rot}_\Sigma \begin{pmatrix} \mathbf{W}_\Sigma \\ \mathbf{V}_\Sigma \end{pmatrix} = \begin{pmatrix} \operatorname{rot}_\Sigma \mathbf{W}_\Sigma - \mathbf{V}_{\Sigma \times}^\times \mathbf{1}_\Sigma \\ \operatorname{rot}_\Sigma \mathbf{V}_\Sigma \end{pmatrix},$$

and

(2.8)
$$\operatorname{rot}_\Sigma \operatorname{grad}_\Sigma \begin{pmatrix} \mathbf{W}_\Sigma \\ \mathbf{V}_\Sigma \end{pmatrix} - \boldsymbol{\varepsilon}_\Sigma \cdot \mathbf{b} \cdot \operatorname{grad}_\Sigma \begin{pmatrix} \mathbf{W}_\Sigma \\ \mathbf{V}_\Sigma \end{pmatrix} = 0,$$

where $\mathbf{1}$ is the three-dimensional metric tensor, $\mathbf{1}_\Sigma$ and $\mathbf{b}$ are the tensors of the first and the second fundamental forms of a surface, $\boldsymbol{\varepsilon}_\Sigma$ is the surface alternating tensor.

The motor calculus is very effective in various investigations of mechanics and in the theories of defects in Cosserat continua [24–28]. For example, the equilibrium equations, the compatibility conditions and the stress-strain relations for a three-dimensional Cosserat continuum with dislocations and disclinations, can be written in very convenient concise form:

$$(2.9) \qquad \mathrm{div}\begin{pmatrix} \mu \\ \sigma \end{pmatrix} = -\begin{pmatrix} \mathbf{Y} \\ \mathbf{X} \end{pmatrix},$$

$$(2.10) \qquad \begin{pmatrix} \alpha \\ \vartheta \end{pmatrix} = \mathrm{rot}\begin{pmatrix} \gamma \\ \kappa \end{pmatrix},$$

$$(2.11) \qquad \begin{pmatrix} \mu \\ \sigma \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{D} \\ \mathbf{C} & 0 \end{pmatrix} : \begin{pmatrix} \gamma \\ \kappa \end{pmatrix},$$

$$(2.12) \qquad \frac{d}{dt}\begin{pmatrix} \alpha \\ \vartheta \end{pmatrix} = -\mathrm{rot}\begin{pmatrix} \mathbf{J} \\ \mathbf{I} \end{pmatrix}.$$

The corresponding set of equations for a Cosserat surface reads

$$(2.13) \qquad \mathrm{div}_\Sigma\begin{pmatrix} \mu_\Sigma \\ \sigma_\Sigma \end{pmatrix} = -\begin{pmatrix} \mathbf{Y}_\Sigma \\ \mathbf{X}_\Sigma \end{pmatrix},$$

$$(2.14) \qquad \begin{pmatrix} \alpha_\Sigma \\ \vartheta_\Sigma \end{pmatrix} = \mathrm{rot}_\Sigma\begin{pmatrix} \gamma_\Sigma \\ \kappa_\Sigma \end{pmatrix} - \varepsilon_\Sigma \cdot \mathbf{b} \cdot \begin{pmatrix} \gamma_\Sigma \\ \kappa_\Sigma \end{pmatrix},$$

$$(2.15) \qquad \begin{pmatrix} \mu_\Sigma \\ \sigma_\Sigma \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{D}_\Sigma \\ \mathbf{C}_\Sigma & 0 \end{pmatrix} : \begin{pmatrix} \gamma_\Sigma \\ \kappa_\Sigma \end{pmatrix},$$

$$(2.16) \qquad \frac{d}{dt}\begin{pmatrix} \alpha_\Sigma \\ \vartheta_\Sigma \end{pmatrix} = -\mathrm{rot}_\Sigma\begin{pmatrix} \mathbf{J}_\Sigma \\ \mathbf{I}_\Sigma \end{pmatrix} + \varepsilon_\Sigma \cdot \mathbf{b} \cdot \begin{pmatrix} \mathbf{J}_\Sigma \\ \mathbf{I}_\Sigma \end{pmatrix},$$

where σ and μ are the stress and the couple-stress tensor, γ and κ denote the strain tensor and the bend-twist tensor, α and θ are the densities of dislocations and disclinations, $\mathbf{J}$ and $\mathbf{I}$ are the dislocation and disclination current tensors, $\mathbf{X}$ and $\mathbf{Y}$ denote the body force and the body couple, $\mathbf{C}$ and $\mathbf{D}$ are the material constant tensors (for details see [28]).

3. The non-Abelian motor calculus

Let us consider six-parameter local transformations of E_3 consisting of three translational parameters $u_a(x^i)$, $a = 1, 2, 3$; $i = 1, 2, 3$, and three rotational parameters $\varphi_A(x^i)$, $A = 1, 2, 3$; $i = 1, 2, 3$. As stated above, in the Abelian approach the ordered pair $(u_a(x^i), \varphi_A(x^i))$ is an Abelian motor. To introduce a concept of the non-Abelian motor [18] we consider the local translation Lie group

$T(3)$, the local special orthogonal Lie group $SO(3)$, and the semi-direct product $T(3) \triangleright SO(3)$ [10]. An element $\mathbf{M}$ of this product is represented by a matrix

$$(3.1) \qquad \begin{bmatrix} 1 & \mathbf{u} \\ 0 & \mathbf{R} \end{bmatrix}_{4 \times 4},$$

where $\mathbf{u} \in T(3)$ is the translation and $\mathbf{R} \in SO(3)$ is the rotation:

$$(3.2) \qquad \mathbf{R}^{\mathrm{T}}\mathbf{R} = 1, \qquad \det \mathbf{R} = 1.$$

The Lie algebra $t(3) \triangleright so(3)$ can be described by six matrices

$$(3.3) \qquad \mathbf{T}_a = \begin{bmatrix} 0 & t_a \\ 0 & 0 \end{bmatrix}_{4 \times 4}, \qquad \mathbf{T}_A = \begin{bmatrix} 0 & 0 \\ 0 & r_A \end{bmatrix}_{4 \times 4}, \qquad a = 1,2,3, \quad A = 1,2,3$$

with the matrices

$$(3.4) \qquad t_1 = (1,0,0), \qquad t_2 = (0,1,0), \qquad t_3 = (0,0,1),$$

$$(3.5) \qquad r_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \qquad r_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad r_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

forming the basis of the Lie algebras $t(3)$ and $so(3)$, respectively.

The Lie brackets in the Lie algebra $t(3) \triangleright so(3)$ have the following properties:

$$(3.6) \qquad \begin{aligned} [\mathbf{T}_a, \mathbf{T}_b] &= 0, & [\mathbf{T}_a, \mathbf{T}_A] &= \epsilon_{aAB}\mathbf{T}_B, \\ [\mathbf{T}_A, \mathbf{T}_b] &= \epsilon_{AbC}\mathbf{T}_C, & [\mathbf{T}_A, \mathbf{T}_B] &= \epsilon_{ABC}\mathbf{T}_C \end{aligned}$$

with the structural constants given by the completely antisymmetric tensor ϵ.

In place of the ordered pair (u_a, φ_A) in the Abelian theory, we consider a motor as an element of the non-Abelian group $T(3) \triangleright SO(3)$

$$(3.7) \qquad \mathbf{M}(x^i) = u^a(x^i)\mathbf{T}_a + \exp\left[\varphi^A(x^i)\mathbf{T}_A\right],$$

where the tensor exponential function realizes the full matrix of a finite rotation $\mathbf{R}$.

An initial reper $\overset{\circ}{\mathbf{R}}$ containing the position vector $\overset{\circ}{\mathbf{r}}$ and the triad $\overset{\circ}{\mathbf{E}}_a$, $a = 1,2,3$, is defined as

$$(3.8) \qquad \overset{\circ}{\mathbf{R}} = \left(\overset{\circ}{\mathbf{r}}, \overset{\circ}{\mathbf{E}}_1, \overset{\circ}{\mathbf{E}}_2, \overset{\circ}{\mathbf{E}}_3\right).$$

Under the action of the motor $\mathbf{M}$, the initial reper $\overset{\circ}{\mathbf{R}}$ transforms into the actual reper $\mathbf{R}$

$$(3.9) \qquad \mathbf{R} = (\mathbf{r}, \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$$

according to

(3.10) $$\mathbf{R} = \mathbf{M}\,\overset{\circ}{\mathbf{R}}.$$

Differentiation of $\overset{\circ}{\mathbf{R}}$ with respect to x^i gives

(3.11) $$\partial_i\,\overset{\circ}{\mathbf{R}} = \overset{\circ}{\mathbf{A}}_i \overset{\circ}{\mathbf{R}}$$

with the six values in the matrix algebra.

The covariant derivative with respect to the undeformed base is

(3.12) $$\overset{\circ}{\mathbf{D}}_i = \partial_i - \overset{\circ}{\mathbf{A}}_i.$$

Thus, the gradient of the motor $\mathbf{M}$ with respect to x^i has the form

(3.13) $$\operatorname{Grad}\mathbf{M} \equiv \mathbf{A} = \overset{\circ}{\mathbf{g}}^i\,\mathbf{A}_i = \overset{\circ}{\mathbf{g}}^i\left[(\partial_i\mathbf{M})\mathbf{M}^{-1} + \mathbf{M}\,\overset{\circ}{\mathbf{A}}_i\,\mathbf{M}^{-1} - \overset{\circ}{\mathbf{A}}_i\right].$$

In a Cartesian coordinate system, by linearization of the rotation with $\mathbf{R} = 1 + \boldsymbol{\varphi} \times \mathbf{1}$ we obtain

(3.14) $$\mathbf{D}_i\mathbf{M} = \begin{bmatrix} 0 & \partial_i\mathbf{u} - \boldsymbol{\varphi}\times\mathbf{1} \\ 0 & \partial_i\boldsymbol{\varphi} \end{bmatrix}$$

which corresponds to the definition (2.3).

Let us define $\mathbf{F}_{ij}$ as a result of the following operation:

(3.15) $$\mathbf{F}_{ij} = \mathbf{D}_i\mathbf{D}_j - \mathbf{D}_j\mathbf{D}_i \equiv [\mathbf{D}_i, \mathbf{D}_j]$$

or

(3.16) $$\mathbf{F}_{ij} = \overset{\circ}{\mathbf{D}}_j\,\mathbf{A}_i - \overset{\circ}{\mathbf{D}}_i\,\mathbf{A}_j + [\mathbf{A}_i, \mathbf{A}_j].$$

Then

(3.17) $$\mathbf{B} = \frac{1}{2}\epsilon^{kij}\mathbf{F}_{ij}\,\overset{\circ}{\mathbf{g}}_k \equiv \operatorname{rot}\mathbf{A} + \frac{1}{2}[(\mathbf{A}\times), \mathbf{A}],$$

and we can realize the non-Abelian extension of the rotor operator

(3.18) $$\operatorname{Rot}\mathbf{M} = \operatorname{rot}\mathbf{M} + \frac{1}{2}[(\mathbf{A}\times), \mathbf{M}],$$

and its linearization coincides with (2.4).

4. The Cosserat bundle

Fiber bundles are the very concise tools for description of the Cosserat continuum. First, we shall very briefly reinterprete some results of [8] in terms of a vector fiber bundle with a four-dimensional base and a three-dimensional fiber.

The one-form of fiber connection is written as

$$(4.1) \qquad \mathbf{A} = \boldsymbol{\gamma}\, dt + \boldsymbol{\Gamma}_1\, dx^1 + \boldsymbol{\Gamma}_2\, dx^2 + \boldsymbol{\Gamma}_3\, dx^3.$$

The two-form of bundle curvature

$$(4.2) \qquad F_{\mu\nu p.}{}^{k} = \partial_\mu A^k_{\nu p} - \partial_\nu A^k_{\mu p} + A^k_{\mu q} A^q_{\nu p} - A^k_{\nu q} A^q_{\mu p}$$

can be split into two parts

$$(4.3) \qquad R_{rsp.}{}^{k} = F_{rsp.}{}^{k}$$

and

$$(4.4) \qquad P_{sp.}{}^{k} \equiv F_{0sp.}{}^{k} = \frac{\partial \Gamma^k_{sp}}{\partial t} - \nabla_s \gamma_{p.}{}^{k},$$

where Roman indices are running over 1, 2, 3, Greek indices over 0, 1, 2, 3.

Equation (4.4) leads to the evolution equation for the torsion tensor

$$(4.5) \qquad \frac{\partial S_{sp.}{}^{k}}{\partial t} = P_{[sp].}{}^{k} + \nabla_{[s}\gamma_{p].}{}^{k}.$$

From the Jacobi identity for three derivative operators

$$(4.6) \qquad [\nabla_0, [\nabla_s, \nabla_r]] + [\nabla_s, [\nabla_r, \nabla_0]] + [\nabla_r, [\nabla_0, \nabla_s]] = 0,$$

we immediately obtain the evolution equation for the curvature tensor

$$(4.7) \qquad \frac{\partial R_{ijk.}{}^{m}}{\partial t} = 2(\nabla_{[i}P_{j]k.}{}^{m} + S_{ij.}{}^{p}P_{pk.}{}^{m} + R_{ij[p.}{}^{m}\gamma_{k].}{}^{m}).$$

As usually, the torsion tensor and the curvature tensor are interpreted in terms of dislocation and disclination densities, respectively, while the time connection $\boldsymbol{\gamma}$ and the tensor $\mathbf{P}$ are related with the dislocation current tensor

$$(4.8) \qquad J_{m.}{}^{n} = \nabla_m v^n - \gamma_{m.}{}^{n},$$

where $\mathbf{v}$ is the velocity vector, and the disclination current tensor

$$(4.9) \qquad I_{m.}{}^{n} = -\frac{1}{2}\epsilon^{npk} P_{mpk}.$$

Being interpreted in such a way, Eqs. (4.5) and (4.7) provide nonlinear general-ization of Eqs. (2.12).

Let us next consider a bundle $\mathcal{P}$ with four-dimensional or three-dimensional base $\mathcal{M}$ and the semi-simple group $G = T(3) \rhd SO(3)$ as a fiber. It is a natural choice of the base $\mathcal{J}_a$, $a = 1, 2, ..., 6$ of algebra G to be the fundamental vector fields defining a vertical tangent to $\mathcal{P}$. The base $\mathcal{M}$ is parametrized by the Gaus-sian coordinates θ^α, ($\alpha = 0, 1, 2, 3$ or $\alpha = 0, 1, 2$, where index 0 corresponds to a time t) and its holonomic base vectors ∇_α. A pair $\partial_M = \{\nabla_\alpha, \mathcal{J}_a\}$, $M = \alpha, a$ spans the base of the tangent space to $\mathcal{P}$, the Greek indices α, β indicate pure geometry and the Latin indices a, b indicate pure physics.

The fundamental operation for both the gauge and the fiber construction is the horizontal lift

$$(4.10) \qquad \partial_M \longrightarrow \ell_M \equiv \{\mathcal{D}_\alpha, \mathcal{J}_a\}.$$

This operation is synonymous with the Utiyama compensating fields or the non-inertial frame of reference in Poisson rigid-body mechanics.

The field

$$(4.11) \qquad \mathcal{A}(\theta^\alpha, \theta^a) = \mathcal{A}^a_\alpha \mathcal{J}_a \mathrm{d}\theta^\alpha$$

is the one-form of fiber connection or the compensating gauge potential. Note also that the vector product

$$(4.12) \qquad [\mathcal{D}_\alpha, \mathcal{D}_\beta] = \mathcal{F}^a_{\alpha\beta} \mathcal{J}_a$$

is purely vertical and can be expanded along $\mathcal{J}_a$ with θ^M-dependend coefficients

$$(4.13) \qquad \mathcal{F}^a_{\alpha\beta} = -\nabla_\alpha \mathcal{A}^a_\beta + \nabla_\beta \mathcal{A}^a_\alpha + \mathrm{C}^a_{bc} \mathcal{A}^b_\alpha \mathcal{A}^c_\beta$$

called the two-form of bundle curvature

$$(4.14) \qquad \mathcal{F} = \mathcal{F}^a_{\alpha\beta} \mathcal{J}_a\, \mathrm{d}\theta^\alpha \wedge \mathrm{d}\theta^\beta,$$

or the gauge strength tensor.

Within $\mathcal{P}$ such objects as the metric g_{MN} connection Γ^K_{MN}, torsion $S_{MN}{}^L$, curvature $\mathcal{R}_{MN}{}^L$, etc. can be prescribed in a natural way:

$$(4.15) \qquad \Gamma^K_{MN} = \frac{1}{2} g^{KL} (\partial_M g_{NL} + \partial_N g_{ML} - \partial_L g_{MN} + C_{LMN} + C_{LNM} - C_{MNL}),$$

$$(4.16) \qquad S_{MN}{}^L = \Gamma_{MN}{}^L - \Gamma_{MN}{}^L - C_{MN}{}^L,$$

$$(4.17) \qquad \mathcal{R}_{MNK}{}^L = \ell_M \Gamma_{NK}{}^L - \ell_N \Gamma_{MK}{}^L - \Gamma_{MK}{}^P \Gamma_{NP}{}^L + \Gamma_{NK}{}^P \Gamma_{MP}{}^L - C_{MN}{}^P S_{PK}{}^L,$$

where $\overset{\circ}{g}_{\alpha\beta} = \nabla_\alpha \cdot \nabla_\beta$ and $\overset{\circ}{g}_{ab} = C^c_{ad} C^d_{cb}$ are the metric of a base and the Killing – Cartan metric on a fiber, respectively; $[\ell_M, \ell_N] = C_{MN}{}^L \ell_L$.

The one-form of fiber connection $\mathcal{A} = \mathcal{A}_\alpha^a \mathcal{J}_a d\theta^\alpha$ can be interpreted as the measures of deformation of the three-dimensional Cosserat continuum or two-dimensional Cosserat surface, while the two-form of bundle curvature $\mathcal{F} = \mathcal{F}_{\alpha\beta}^a \mathcal{J}_a d\theta^\alpha \wedge d\theta^\beta$ can be regarded as the compatibility of these measures (see also [19]).

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