Arch. Mech., 50, 2, pp. 321–340, Warszawa 1998
FIFTY YEARS OF THE ARCHIVES OF MECHANICS

Effective properties of physically nonlinear piezoelectric composites

J.J. TELEGA, A. GAŁKA and B. GAMBIN (WARSZAWA)

FOR PIEZOELECTRIC composites with periodic microstructure and subject to stronger electric fields, an effective model has been prepared. To this end the Γ-convergence theory has been applied. Detailed convergence proof has been given. Specific cases of the internal energy have been suggested. Comments on homogenization in the case of periodically nonuniform microstructure have also been provided.

1. Introduction

IN THE LAST DECADE various approaches were proposed to finding the effective properties of piezoelectric composites, cf. [1–7] and the references cited therein. In 1991 the first author published a paper [1] where he performed non-uniform homogenization of linear piezoelectric composites by using the Γ-convergence method. However, that paper did not contain the proof of convergence. The aim of the present contribution is to perform *nonlinear* homogenization of piezoelectric composite with periodic or non-uniformly periodic microstructure. As it was argued by TIERSTEN [8], for stronger electric fields one has to take into account higher order terms in the electric field $\mathbf{E}$. The form of the electric enthalpy $H(\mathbf{e}, \mathbf{E})$ proposed by this author, being of the third order in $\mathbf{E}$, cannot be *concave* in $\mathbf{E}$, where $\mathbf{e}$ denotes the strain tensor. It should be remembered that $H(\mathbf{e}, \mathbf{E})$ is here understood as a partial *concave* conjugate of the internal energy function $U(\mathbf{e}, \mathbf{D})$ with respect to $\mathbf{D}$, $\mathbf{D}$ being the electric displacement vector. In Sec. 2 we shall briefly discuss a plausible form of the internal energy which accounts for stronger electric fields.

The plan of the paper is as follows. Fundamental relations and nonlinear piezoelectric composites with εY-periodic microstructure are introduced in Sec. 2. In Sec. 3 we recall the basic notions of the Γ-convergence theory, which will next be of primary importance in Sec. 4. The heart of the paper constitutes Sec. 4, where we give the proof of the Γ-convergence of sequence of functionals J_ε defined by (4.2), to the limit functional J_h. Comments on non-uniform homogenization are provided in Sec. 5. The summation convention applies to repeated indices.

2. Basic relations

Let $V \subset \mathbb{R}^3$ be a bounded, sufficiently regular domain such that its closure $\overline{V}$ stands for a considered piezoelectric composite in its natural state. By $\gamma = \partial V$ we denote the boundary of V. If $\mathbf{u} = (u_i)$ is a displacement field, then $e_{ij}(\mathbf{u}) = u_{(i,j)}$ is the strain tensor; $i, j = 1, 2, 3$. By $\mathbf{D} = (D_i)$, $\mathbf{E} = (E_i)$ and $\boldsymbol{\sigma} = (\sigma_{ij})$ we denote the electric displacement vector, the electric field and the stress tensor, respectively. As usual, we set $E_i(\varphi) = -\varphi_{,i}$, where φ is the electric potential [9]. Let $\varepsilon > 0$ be a small parameter and $\varepsilon = l/L$. Here l, L are typical length scales associated with microinhomogeneities and the region V, respectively. The internal energy is $U = U(y, \mathbf{e}, \mathbf{D})$, $y \in Y$. Here Y is a so-called basic cell, cf. [10, 11, 12]. We set

$$U_\varepsilon(x, \mathbf{e}, \mathbf{D}) = U\left(\frac{x}{\varepsilon}, \mathbf{e}, \mathbf{D}\right), \tag{2.1}$$

where $x \in V$, $\mathbf{e} \in \mathbb{E}^3_s$ and $\mathbf{D} \in \mathbb{R}^3$; $\mathbb{E}^3_s$ stands for the space of symmetric 3×3 matrices. Consequently, the piezoelectric material occupying V exhibits the εY-periodic microstructure. We observe that the case of quadratic internal energy has been studied in [1]. In the general case the constitutive equations are given by

$$\boldsymbol{\sigma} = \frac{\partial U}{\partial \mathbf{e}}, \qquad \mathbf{E} = \frac{\partial U}{\partial \mathbf{D}}. \tag{2.2}$$

We make the following assumption:

(A) The function $U : (y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho}) \in \mathbb{R}^3 \times \mathbb{E}^3_s \times \mathbb{R}^3 \to U(y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho}) \in \mathbb{R}$ is measurable and Y-periodic in y, convex in $(\mathbf{e}, \mathbf{D})$ and such that

$$\exists\, c_1 \geq c_0 > 0, \quad c_0(|\boldsymbol{\varepsilon}|^p + |\boldsymbol{\varrho}|^q) \leq U(y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho}) \leq c_1(|\boldsymbol{\varepsilon}|^p + |\boldsymbol{\varrho}|^q), \tag{2.3}$$

for each $(y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho})$. Here $p > 1$ and $q > 1$. As usual, we set $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, $\dfrac{1}{q} + \dfrac{1}{q'} = 1$. The assumption (A) will significantly be weakened in the case of non-uniformly microperiodic composites, cf. Sec. 5 of our paper.

REMARK 1. As a particular case of the internal energy one can consider the following one:

$$U\left(\frac{x}{\varepsilon}, \mathbf{e}, \mathbf{D}\right) = U_1\left(\frac{x}{\varepsilon}, \mathbf{e}, \mathbf{D}\right) + U_2\left(\frac{x}{\varepsilon}, \mathbf{e}, \mathbf{D}\right), \tag{2.4}$$

where $U_1\left(\dfrac{x}{\varepsilon}, \mathbf{e}, \mathbf{D}\right)$ is a positive definite quadratic form in $\mathbf{e}$ and $\mathbf{D}$, typical for linear piezocomposites, cf. [1–3]. On the other hand, the function $U_2\left(\dfrac{x}{\varepsilon}, \mathbf{e}, \mathbf{D}\right)$ collects non-quadratic, higher order terms. Obviously, the function $U_2\left(\dfrac{x}{\varepsilon}, \mathbf{e}, \mathbf{D}\right)$ has still to be convex in $\mathbf{e}$ and $\mathbf{D}$.

A simple example is provided by

$$U_2(y, \mathbf{e}, \mathbf{D}) = \widetilde{U}_2(y, \mathbf{D}) = \frac{1}{4} b_{ijkl}(y) D_i D_j D_k D_l, \qquad y = \frac{x}{\varepsilon}, \tag{2.5}$$

where $b_{ijkl} \in L^\infty(Y)$ is a completely symmetric tensor [13]. Further restrictions on the material functions are imposed by the requirement of $\widetilde{U}_2(y, \mathbf{D})$ being convex in $\mathbf{D}$. The present contribution is confined to small deformations and the internal energy $U(y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho})$ is convex in $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varrho}$. Finite deformations are properly described by a nonconvex internal energy. Nonconvex homogenization is out of scope of the present contribution.

Let us pass to the formulation of the minimum principle. The following boundary conditions are assumed:

$$\mathbf{u} = 0 \quad \text{on } \gamma_0, \qquad \sigma_{ij}n_j = \Sigma_i \quad \text{on } \gamma_1, \tag{2.6}$$

$$\varphi = \varphi_0 \quad \text{on } \gamma_2, \qquad D_i n_i = 0 \quad \text{on } \gamma_3, \tag{2.7}$$

where Σ_i are the surface tractions, $\gamma = \overline{\gamma}_0 \cup \overline{\gamma}_1$, $\gamma_0 \cap \gamma_1 = \emptyset$; $\gamma = \overline{\gamma}_2 \cup \overline{\gamma}_3$, $\gamma_2 \cap \gamma_3 = \emptyset$, and $\mathbf{n} = (n_i)$ is the outward unit normal vector to γ; obviously $\emptyset$ denotes the empty set.

For fixed $\varepsilon > 0$ we set

$$F_\varepsilon(\mathbf{u}, \mathbf{D}) = \int_V U_\varepsilon(x, \mathbf{e}(\mathbf{u}), \mathbf{D})\, dx - L(\mathbf{u}, \mathbf{D}), \tag{2.8}$$

where

$$L(\mathbf{u}, \mathbf{D}) = \int_V b_i u_i\, dx + \int_{\gamma_1} \Sigma_i u_i d\gamma - \int_{\gamma_2} \varphi_0 D_i n_i d\gamma, \tag{2.9}$$

and

$$\mathbf{u} \in \mathbf{W}(V, \gamma_0) = \left\{ \mathbf{v} = (v_i) \mid v_i \in W^{1,p}(V), \ \ \mathbf{v} = \mathbf{0} \ \text{ on } \gamma_0 \right\},$$

$$\mathbf{D} \in \mathbf{W}(\mathrm{div}; V, \gamma_3) = \{ \mathbf{D} = (D_i) \mid D_i \in L^q(V), \ \mathrm{div}\, \mathbf{D} \in L^q(V), \ \mathbf{D}\cdot\mathbf{n} = 0 \text{ on } \gamma_3 \}.$$

For more details on the spaces just introduced the reader is refered to [14, 15].

The equlibrum problem of the physically nonlinear piezocomposites with the ε-Y periodic microstructure means evaluating

$$F_\varepsilon(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon) = \inf \{ F_\varepsilon(\mathbf{u}, \mathbf{D}) \mid \mathbf{u} \in \mathbf{W}(V, \gamma_0), \ \mathbf{D} \in \mathbf{W}(\mathrm{div}; V, \gamma_3) \}. \tag{$\mathcal{P}_\varepsilon$}$$

The assumption (A) implies the existence of unique $(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon) \in \mathbf{W}(V, \gamma_0) \times \mathbf{W}(\mathrm{div}; V, \gamma_3)$ solving the problem $(\mathcal{P}_\varepsilon)$.

3. Γ-convergence

A detailed presentation of the theory of Γ-convergence is provided by ATTOUCH [10] and DAL MASO [11]. ATTOUCH [10] prefers to use the notion of epi-convergence, which in fact is a special case of Γ-convergence. In our specific case these notions coincide.

DEFINITION 1. *Let (X,τ) be a metrisable topological space, and let $\{G_\varepsilon\}_{\varepsilon>0}$ be a sequence of functionals from X into $\overline{\mathbb{R}}$ – the extended reals.*

a. *The $\Gamma(\tau)$-limit inferior, denoted also by G_i, is the functional on X defined by*

$$G_i(u) = \Gamma(\tau)\text{-}\liminf_{\varepsilon\to 0} G_\varepsilon(u) = \min_{\{u_\varepsilon \xrightarrow{\tau} u\}} \liminf_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon).$$

b. *The $\Gamma(\tau)$-limit superior, denoted also by G_s, is the functional on X defined by*

$$G_s(u) = \Gamma(\tau)\text{-}\limsup_{\varepsilon\to 0} G_\varepsilon(u) = \min_{\{u_\varepsilon \xrightarrow{\tau} u\}} \limsup_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon).$$

c. *The sequence $\{G_\varepsilon\}_{\varepsilon>0}$ is said to be $\Gamma(\tau)$-convergent if $G_i = G_s$; we then write*

$$G = \Gamma(\tau)\text{-}\lim_{\varepsilon\to 0} G_\varepsilon.$$

PROPERTIES. Let $G_\varepsilon : (X,\tau) \to \overline{\mathbb{R}}$ be a sequence of $\Gamma(\tau)$-convergent functionals and let $G = \Gamma(\tau)\text{-}\lim_{\varepsilon\to 0} G_\varepsilon$. Then the following properties hold:

(i) The functionals G_i and G_s are τ-lower semicontinuous (τ-l.s.c.).

(ii) If the functionals G_ε are convex, then $G_s = \Gamma(\tau)\text{-}\limsup_{\varepsilon\to 0} G_\varepsilon$ is also a convex functional. Hence the $\Gamma(\tau)$-limit $G = \Gamma(\tau)\text{-}\lim_{\varepsilon\to 0} G_\varepsilon$ is a τ-closed (τ-l.s.c) convex functional.

(iii) If $\Phi : X \to \mathbb{R}$ is a τ-continuous functional, called a perturbation functional, then

$$\Gamma(\tau)\text{-}\lim_{\varepsilon\to 0}(G_\varepsilon + \Phi) = \Gamma(\tau)\text{-}\lim_{\varepsilon\to 0} G_\varepsilon + \Phi = G + \Phi;$$

(iv)

$$G(u) = \Gamma(\tau)\text{-}\lim_{\varepsilon\to 0} G_\varepsilon(u) \Leftrightarrow \begin{cases} \forall\ \{u_\varepsilon \xrightarrow{\tau} u\},\ G(u) \le \liminf_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon),\ u \in X, \\ \forall\ u \in X \quad \exists\ u_\varepsilon \xrightarrow{\tau} u, \text{ such that} \\ G(u) \ge \limsup_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon). \end{cases}$$

Further characterization is given by

THEOREM 1. *Let $G = \Gamma(\tau)\text{-}\lim_{\varepsilon\to 0} G_\varepsilon$, and suppose that there exists a τ-relatively compact subset $X_0 \subset X$ such that $\inf_{X_0} G_\varepsilon = \inf_X G_\varepsilon$ ($\forall \varepsilon > 0$). Then $\inf_X G = \lim_{\varepsilon\to 0}(\inf_X G_\varepsilon)$. Moreover, if $\{u_\varepsilon\}_{\varepsilon>0}$ is such that $G_\varepsilon(u_\varepsilon) - \inf_X G_\varepsilon \underset{\varepsilon\to 0}{\to} 0$, then every τ-cluster point of the sequence $\{u_\varepsilon : \varepsilon \to 0\}$ minimizes G on X.*

REMARK 2. From a practical point of view the following sufficient condition of existence of compact set X_0 is very useful. If $(X, \|\,.\,\|)$ is a Banach space with

τ-relatively compact balls, then a sufficient condition of existence of compact set X_0 is that the sequence $\{G_\varepsilon\}_{\varepsilon>0}$ satisfies the condition of equi-coercivity

$$\lim_{\varepsilon}\sup G_\varepsilon(u_\varepsilon) < +\infty \Longrightarrow \lim_{\varepsilon}\sup \|u_\varepsilon\| < +\infty. \tag{3.1}$$

4. Γ-convergence of the sequence of functionals $\{F_\varepsilon\}_{\varepsilon>0}$

We proceed to find the limit functional

$$\Gamma\left[(s - L^p(V)^3) \times (w - L^q(V)^3)\right] - \lim_{\varepsilon\to 0} F_\varepsilon = F_h\,, \tag{4.1}$$

where $L^p(V)^3 = [L^p(V)]^3$ and $s - L^p(V)^3(w - L^q(V)^3)$ stands for the strong topology of $L^p(V)^3$ (the weak topology of $L^q(V)^3$). The loading functional L may be assumed to be continuous in the topology $\tau = (s - L^p(V)^3) \times (w - L^q(V)^3)$. To this end it is sufficient to assume that $\mathbf{b} \in L^{p'}(V)^3$, $\mathbf{\Sigma} \in L^{p'}(\gamma_1)^3$ and $\varphi_0 \in W^{1-\frac{1}{q'},q'}(\gamma_2)$. As a particular case, one can impose φ_0 being continuous on γ_2 and vanishing on $\partial\gamma_2$. According to the property (iii), the functional L plays the role of a perturbation functional. Consequently it suffices to study the $\Gamma(\tau)$-limit of the following sequence of functionals $\{J_\varepsilon\}_{\varepsilon>0}$ given by

$$J_\varepsilon(\mathbf{u},\mathbf{D}) = \int_V U_\varepsilon(x, \mathbf{e}(\mathbf{u}), \mathbf{D})dx. \tag{4.2}$$

The main result of this paper is formulated as:

THEOREM 2. *Let the assumption (A) be satisfied. The sequence of functionals $\{J_\varepsilon\}_{\varepsilon>0}$ is $\Gamma(\tau)$-convergent to the functional*

$$J_h(\mathbf{u},\mathbf{D}) = \int_V U_h(\mathbf{e}(\mathbf{u}), \mathbf{D})dx, \tag{4.3}$$

where $\mathbf{u} \in W^{1,p}(V)^3$, $\mathbf{D} \in L^q(V)^3$ *and*

$$U_h(\boldsymbol{\varepsilon},\boldsymbol{\varrho}) = \inf\left\{ \frac{1}{|Y|}\int_Y U(y, \mathbf{e}^y(\mathbf{v}(y)) + \boldsymbol{\varepsilon}, \mathbf{d}(y) + \boldsymbol{\varrho})dy \;\Big|\; \mathbf{v} \in W^{1,p}_{\text{per}}(Y)^3,\ \mathbf{d} \in \Delta_{\text{per}}(Y)\right\}. \tag{4.4}$$

Here $\boldsymbol{\varepsilon} \in \mathbb{E}^3_s$, $\boldsymbol{\varrho} \in \mathbb{R}^3$, $e^y_{ij}(\mathbf{v}) = \dfrac{1}{2}\left(\dfrac{\partial v_i}{\partial y_j} + \dfrac{\partial v_j}{\partial y_i}\right)$ *and*

$$W^{1,p}_{\text{per}}(Y)^3 = \{\mathbf{v} \in W^{1,p}(Y)^3 | \mathbf{v} \text{ is } Y\text{−periodic}\}, \tag{4.5}$$

$$\Delta_{\mathrm{per}}(Y) = \left\{ \mathbf{d} \in L^q(Y)^3 \mid \operatorname{div}_y \mathbf{d} = 0 \text{ in } Y,\ \langle \mathbf{d} \rangle = 0,\ \mathbf{d} \text{ is anti-periodic} \right\}, \tag{4.6}$$

$$\langle \mathbf{d} \rangle = \frac{1}{|Y|} \int_Y \mathbf{d}(y)\,dy. \qquad \square$$

REMARK 3. A function $\mathbf{v} \in W^{1,p}_{\mathrm{per}}(Y)^3$ is Y-periodic if the traces of $\mathbf{v}$ on the opposite faces of Y are equal. It means that on these faces, the values of v_i are equal almost everywhere, at least. Similarly, if $\mathbf{d} \in \Delta_{\mathrm{per}}(Y)$, then the traces $\mathbf{d}\cdot\mathbf{N}$ are opposite on the opposite faces of Y. Here $\mathbf{N}$ stands for the outward unit normal vector to ∂Y.

PROPERTIES OF U_h

(i) The function U_h is convex.

Proof. This evident property follows immediately from the convexity of the function $U(y,\cdot,\cdot)$ and the linearity of the operator $\mathbf{e}^y(\cdot)$, cf. [16].

(ii) $\exists\, c_1 \geq c_0' > 0$ such that

$$c_0'(|\boldsymbol{\varepsilon}|^p + |\boldsymbol{\varrho}|^q) \leq U_h(\boldsymbol{\varepsilon}, \boldsymbol{\varrho}) \leq c_1(|\boldsymbol{\varepsilon}|^p + |\boldsymbol{\varrho}|^q),$$

for each $\boldsymbol{\varepsilon} \in \mathbb{E}^3_s$, $\boldsymbol{\varrho} \in \mathbb{R}^3$. The constant c_1 is the same as in (2.3).

Proof. Indeed, from (2.3) and (4.4) we obtain

$$U_h(\boldsymbol{\varepsilon}, \boldsymbol{\varrho}) \leq \langle U(y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho}\rangle \leq c_1(|\boldsymbol{\varepsilon}|^p + |\boldsymbol{\varrho}|^q).$$

Similarly, let $(\widetilde{\mathbf{v}}, \widetilde{\mathbf{d}}) \in W^{1,p}_{\mathrm{per}}(Y)^3 \times \Delta_{\mathrm{per}}(Y)$ be a minimizer of the minimization problem occurring on the r.h.s. of (4.4). This minimizer is unique, provided that $\langle \widetilde{\mathbf{v}} \rangle = 0$. Taking into account (2.3) we have

$$\begin{aligned} U_h(\boldsymbol{\varepsilon}, \boldsymbol{\varrho}) &= \langle U(y, \mathbf{e}^y(\widetilde{\mathbf{v}}) + \boldsymbol{\varepsilon}, \widetilde{\mathbf{d}}(y) + \boldsymbol{\varrho}\rangle \\ &\geq \langle |\mathbf{e}^y(\widetilde{\mathbf{v}}) + \boldsymbol{\varepsilon}|^p + |\widetilde{\mathbf{d}}(y) + \boldsymbol{\varrho}|^q \rangle \geq c_0'(|\boldsymbol{\varepsilon}|^p + |\boldsymbol{\varrho}|^q). \end{aligned} \tag{4.7}$$

Indeed, from (2.3) we conclude

$$c_2(|\boldsymbol{\varepsilon}^*|^{p'} + |\boldsymbol{\varrho}^*|^{q'}) \leq U^*(y, \boldsymbol{\varepsilon}^*, \boldsymbol{\varrho}^*) \leq c_3(|\boldsymbol{\varepsilon}^*|^{p'} + |\boldsymbol{\varrho}^*|^{q'}), \tag{4.8}$$

where $c_3 \geq c_2 > 0$ are constants and $\boldsymbol{\varepsilon}^* \in \mathbb{E}^3_s$, $\boldsymbol{\varrho}^* \in \mathbb{R}^3$. Here U^* denotes Fenchel's conjugate of U.

We recall that if $f \leq g$ then $g^* \leq f^*$ [16, 17]. Since $U^*(y, \boldsymbol{\varepsilon}^*, \boldsymbol{\varrho}^*) \geq 0$ and $U^*(y, \mathbf{0}, \mathbf{0}) = 0$, we may apply Remark 4.3, Chap. I, of EKELAND and TEMAM [17] and (4.8) immediately follows.

By using the formula for the dual effective potential (4.9) below we conclude

$$U_h^*(\boldsymbol{\varepsilon}^*, \boldsymbol{\varrho}^*) \leq \langle U^*(y, \boldsymbol{\varepsilon}^*, \boldsymbol{\varrho}^*)\rangle \leq c_3(|\boldsymbol{\varepsilon}^*|^{p'} + |\boldsymbol{\varrho}^*|^{q'}).$$

Hence

$$U_h(\varepsilon,\varrho) \geq c_0'(|\varepsilon|^p + |\varrho|^q).$$

REMARK 4. For $p = q = 2$ another proof of (4.7) is more straightforward. In this case we have

$$\begin{aligned} U_h(\varepsilon,\varrho) &\geq c_0\langle|\mathbf{e}^y(\widetilde{\mathbf{v}}) + \varepsilon|^2 + |\widetilde{\mathbf{d}}(y) + \varrho|^2\rangle \\ &= \frac{c_0}{|Y|}\int\limits_Y \Big[|\mathbf{e}^y(\widetilde{\mathbf{v}}(y))|^2 + 2\varepsilon : \mathbf{e}^y(\widetilde{\mathbf{v}}) + |\varepsilon|^2 + |\widetilde{\mathbf{d}}(y)|^2 + 2\widetilde{\mathbf{d}}(y)\cdot\varrho + |\varrho|^2\Big]\,dy \\ &\geq \frac{c_0}{|Y|}\int\limits_Y (|\varepsilon|^2 + |\varrho|^2)dy = c_0'(|\varepsilon|^2 + |\varrho|^2), \end{aligned}$$

since $\langle\widetilde{\mathbf{d}}(y)\rangle = 0$ and $\int\limits_Y \mathbf{e}^y(\widetilde{\mathbf{v}}(y))dy = 0$; here $c_0' = c_0/|Y|$. □

Prior to passing to the proof of Theorem 2, we shall formulate two lemmas.

LEMMA 1. The dual macroscopic potential U_h^* is given by

$$U_h^*(\varepsilon^*,\varrho^*) = \inf\Big\{\langle U^*(y, \mathbf{t}(y) + \varepsilon^*,\ \mathbf{E}^y(\xi) + \varrho^*)\rangle \ \Big| \ \mathbf{t} \in \mathcal{S}_{\rm per}(Y),\ \ \xi \in W^{1,q'}_{\rm per}(Y)\Big\} \tag{4.9}$$

where U^* is the Fenchel coniugate of $U(y,\cdot,\cdot)$, $\varepsilon^* \in \mathbb{E}^3_s$, $\varrho^* \in \mathbb{R}^3$ and

$$\mathcal{S}_{\rm per}(Y) = \Big\{\mathbf{t} \in L^{p'}(Y,\mathbb{E}^3_s)|\ \ {\rm div}_y\mathbf{t} = 0 \ \text{ in } Y,\ \ \langle\mathbf{t}\rangle = 0,\ \ \mathbf{t}\cdot\mathbf{N} \text{ is antiperiodic}\Big\}. \tag{4.10}$$

P r o o f. We have

$$\begin{aligned} U_h^*(\varepsilon^*,\varrho^*) &= \sup\Big\{\varepsilon^* : \varepsilon + \varrho^*\cdot\varrho - U_h(\varepsilon,\varrho) \ | \ \varepsilon \in \mathbb{E}^3_s,\ \ \varrho \in \mathbb{R}^3\Big\} \\ &= \sup_{\varepsilon\in\mathbb{E}^3_s,\ \varrho\in\mathbb{R}^3}\Bigg\{\frac{1}{|Y|}\int\limits_Y (\varepsilon^* : \varepsilon + \varrho^*\cdot\varrho)\,dy \\ &\qquad - \inf_{\mathbf{v}\in W^{1,p}(Y)^3,\,\mathbf{d}\in\Delta_{\rm per}(Y)}\int\limits_Y U(y, \mathbf{e}^y(\mathbf{v}(y)) + \varepsilon,\ \mathbf{d}(y) + \varrho)\,dy\Bigg\} \\ &= \sup\frac{1}{|Y|}\Bigg\{\int\limits_Y \Big[(\varepsilon^* : (\mathbf{e}^y(\mathbf{v}(y)) + \varepsilon) + \varrho^*\cdot(\mathbf{d}(y) + \varrho) \\ &\qquad - U(y, \mathbf{e}^y(\mathbf{v}(y)) + \varepsilon,\ \mathbf{d}(y) + \varrho)\Big]\,dy | \ \mathbf{v} \in W^{1,p}(Y)^3, \\ &\qquad \mathbf{d} \in \Delta_{\rm per}(Y),\ \ \varepsilon \in \mathbb{E}^3_s,\ \ \varrho \in \mathbb{R}^3\Bigg\}, \end{aligned} \tag{4.11}$$

since $\int_Y \boldsymbol{\varepsilon}^* : \mathbf{e}^y(\mathbf{v})dy = 0$ and $\langle \mathbf{d}(y)\rangle = 0$. The last relation is written as follows

$$U_h^*(\boldsymbol{\varepsilon}^*, \boldsymbol{\varrho}^*) = \frac{1}{|Y|}\,(j + I_{\mathcal{H}})^*\,(\boldsymbol{\varepsilon}^*, \boldsymbol{\varrho}^*), \tag{4.12}$$

where

$$j(\mathbf{t}, \boldsymbol{\gamma}) = \int_Y U(y, \mathbf{t}(y), \boldsymbol{\gamma}(y))\,dy, \tag{4.13}$$

$$\mathcal{H} = \left(\mathbf{e}^y(W^{1,p}_{\text{per}}(Y)^3) \oplus \mathbb{E}^3_s\right) \times \left(\Delta_{\text{per}}(Y) \oplus \mathbb{R}^3\right). \tag{4.14}$$

Identifying $\boldsymbol{\varepsilon}^*$ with a constant element of $L^{p'}(Y, \mathbb{E}^3_s)$ and $\boldsymbol{\varrho}^*$ with a constant element of $L^{q'}(Y)^3$ we have, cf. [18]

$$(j + I_{\mathcal{H}})^* = (j^* \Box I_{\mathcal{H}^\perp}), \tag{4.15}$$

where $\Box$ denotes the inf-convolution and

$$j^*(\mathbf{t}^*, \boldsymbol{\gamma}^*) = \int_Y U^*(y, \mathbf{t}^*(y), \boldsymbol{\gamma}^*(y))dy,$$

$$\mathcal{H}^\perp = \left[\left(\mathbf{e}^y(W^{1,p}_{\text{per}}(Y)^3)\right)^\perp \cap (\mathbb{E}^3_s)^\perp\right] \times \left[(\Delta_{\text{per}}(Y))^\perp \cap (\mathbb{R}^3)^\perp\right]. \tag{4.16}$$

We find

$$(\mathbb{E}^3_s)^\perp = \left\{\boldsymbol{\tau} \in L^{p'}(Y, \mathbb{E}^3_s) \mid \langle \boldsymbol{\tau}\rangle = 0\right\},$$

$$\left(\mathbf{e}(W^{1,p}_{\text{per}}(Y)^3)\right)^\perp = \left\{\boldsymbol{\tau} \in L^{p'}(Y, \mathbb{E}^3_s) \mid \int_Y \boldsymbol{\tau}(y) : \mathbf{e}^y(\mathbf{v})dy = 0 \ \ \forall \mathbf{v} \in W^{1,p}_{\text{per}}(Y)^3\right\}$$
$$= \Big\{\boldsymbol{\tau} \in L^{p'}(Y, \mathbb{E}^3_s) \ |\text{div}_y \boldsymbol{\tau} \in L^{p'}(Y)^3, \ \text{div}_y \boldsymbol{\tau} = 0 \text{ in } Y;$$
$$\boldsymbol{\tau}\cdot\mathbf{N} \ \text{ is anti-periodic}\Big\},$$

$$(\Delta_{\text{per}}(Y))^\perp = \left\{\boldsymbol{\phi} \in L^{q'}(Y)^3 \mid \int_Y \boldsymbol{\phi}\cdot\mathbf{d}(y)dy = 0 \ \forall \mathbf{d} \in \Delta_{\text{per}}(Y)\right\}$$
$$= \left\{\boldsymbol{\phi} \in L^{q'}(Y)^3 \mid \boldsymbol{\phi} = \mathbf{E}^y(\varphi), \ \varphi \in W^{1,q'}_{\text{per}}(Y)\right\},$$

since

$$\int_Y \mathbf{d}(y)\cdot\, \mathbf{E}^y(\varphi)dy = \int_Y \varphi(\text{div}_y \mathbf{d})dy - \int_{\partial Y} \varphi d_i N_i ds = 0 \ \ \forall \mathbf{d} \in \Delta_{\text{per}}(Y),$$

provided that $\varphi \in W^{1,q'}_{\text{per}}(Y)$. Here $\mathbf{E}^y(\varphi) = -\partial\varphi/\partial y_i$.

Taking into account (4.15) and (4.16) in (4.12) we obtain

$$U_h^*(\boldsymbol{\varepsilon}^*, \boldsymbol{\varrho}^*) = \frac{1}{|Y|}\,(j^* \Box I_{\mathcal{H}^\perp})\,(\boldsymbol{\varepsilon}^*, \boldsymbol{\varrho}^*)$$

$$= \inf \frac{1}{|Y|}\Big\{j^*(\mathbf{t}_1, \boldsymbol{\gamma}_1) + I_{\mathcal{H}^\perp}(\mathbf{t}_2, \boldsymbol{\gamma}_2) \mid \boldsymbol{\varepsilon}^* = \mathbf{t}_1 + \mathbf{t}_2,$$

$$\boldsymbol{\varrho}^* = \boldsymbol{\gamma}_1 + \boldsymbol{\gamma}_2, \ \ \mathbf{t}_\alpha \in \mathcal{S}_{\mathrm{per}}(Y), \ \ \boldsymbol{\gamma} = \mathbf{E}^y(\varphi_\alpha), \ \varphi_\alpha \in W^{1,q'}_{\mathrm{per}}(Y), \ \ \alpha = 1, 2\Big\}$$

$$= \inf \frac{1}{|Y|}\left\{j^*(\boldsymbol{\varepsilon}^* - \mathbf{t}, \ \ \boldsymbol{\varrho}^* - \boldsymbol{\gamma}) \mid \ \mathbf{t} \in \mathcal{S}_{\mathrm{per}}(Y), \ \ \boldsymbol{\gamma} = \mathbf{E}^y(\varphi), \ \varphi \in W^{1,q'}_{\mathrm{per}}(Y)\right\}$$

$$= \inf\left\{\langle U^*(y, \mathbf{t} + \boldsymbol{\varepsilon}^*, \ \mathbf{E}^y(\varphi) + \boldsymbol{\varrho}^*\rangle | \ \ \mathbf{t} \in \mathcal{S}_{\mathrm{per}}(Y), \varphi \in W^{1,q'}_{\mathrm{per}}(Y)\right\},$$

because $\mathcal{S}_{\mathrm{per}}(Y)$ and $W^{1,q'}_{\mathrm{per}}(Y)$ are linear spaces. This establishes the formula (4.9). □

COROLLARY 1. The macroscopic electric enthalpy $H_h(\mathbf{e}, \mathbf{E})$ can be calculated as the partial concave conjugate of U_h, cf. [1]

$$H_h(\mathbf{e}, \mathbf{E}) = \inf\left\{-\mathbf{E}\cdot\mathbf{D} + U_h(\mathbf{e}, \mathbf{E}) \mid \mathbf{D} \in \mathbb{R}^3\right\}. \tag{4.17}$$

Proceeding similarly to the proof of (4.9) we finally obtain

$$H_h(\boldsymbol{\varepsilon}, \boldsymbol{\varrho}^*) = \inf_{\mathbf{v} \in W^{1,p}_{\mathrm{per}}(Y)^3} \ \sup_{\xi \in W^{1,q'}_{\mathrm{per}}(Y)} \ \langle H(y, \mathbf{e}^y(\mathbf{v}) + \boldsymbol{\varepsilon}, \mathbf{E}^y(\xi) + \boldsymbol{\varrho}^*)\rangle, \tag{4.18}$$

where $\boldsymbol{\varepsilon} \in \mathbb{E}^3_s$, $\boldsymbol{\varrho}^* \in \mathbb{R}^3$ and

$$H(y, \mathbf{e}, \mathbf{E}) = \inf\left\{-\mathbf{E}\cdot\mathbf{D} + U(y, \mathbf{e}, \mathbf{E}) \mid \mathbf{D} \in \mathbb{R}^3\right\}, \tag{4.19}$$

is the microscopic electric enthalpy. As we have already mentioned, the homogenization of linear piezocomposites was performed in [1], cf. also [2, 3, 6, 7, 21, 22]. □

LEMMA 2. Let $\mathbf{t} \in \mathcal{S}_{\mathrm{per}}(Y)$, $\xi \in W^{1,q'}_{\mathrm{per}}(Y)$, $\psi \in \mathcal{D}(V)$. Let the bounded sequences $\{\mathbf{u}^\varepsilon\}_{\varepsilon>0} \subset W^{1,p}(V)^3$, $\{\mathbf{D}^\varepsilon\}_{\varepsilon>0} \subset \mathbf{W}(\mathrm{div}, V)$ be such that

$$\mathbf{u}^\varepsilon \to \mathbf{u} \quad \text{strongly in } L^p(V)^3,$$

$$\mathbf{D}^\varepsilon \rightharpoonup \mathbf{D} \quad \text{weakly in } L^q(V)^3,$$

when $\varepsilon \to 0$. Then

$$\lim_{\varepsilon \to 0} \int_V \psi(x) t_{ij}\left(\frac{x}{\varepsilon}\right) e_{ij}(\mathbf{u}^\varepsilon(x))dx = 0, \tag{4.20}$$

$$D_i^\varepsilon(x)E_i\left(\xi\left(\frac{x}{\varepsilon}\right)\right) \rightharpoonup D_i(x)E_i(\langle\xi\rangle) = 0 \quad \text{in } \mathcal{D}'(V) \text{ when } \varepsilon \to 0. \tag{4.21}$$

Here $\mathcal{D}'(V)$ is the space of distributions or the dual of the space $\mathcal{D}(V)$.

P r o o f. To prove (4.20) we set

$$R_\varepsilon = \int_V \psi(x)t_{ij}\left(\frac{x}{\varepsilon}\right)e_{ij}(\mathbf{u}^\varepsilon(x))dx.$$

Using integration by parts we obtain

$$R_\varepsilon = -\int_V \psi_{,j}(x)t_{ij}\left(\frac{x}{\varepsilon}\right)u_i^\varepsilon(x)\,dx - \int_V \psi(x)t_{ij,j}\left(\frac{x}{\varepsilon}\right)u_i^\varepsilon(x)\,dx.$$

After the rescaling $y \to x/\varepsilon$ we have $\dfrac{\partial}{\partial y_i} = \varepsilon\dfrac{\partial}{\partial x_i}$ and consequently, $\operatorname{div}_y \mathbf{t} = 0$ in Y implies $\varepsilon(\operatorname{div}\mathbf{t})(x/\varepsilon) = 0$ or $(\operatorname{div}\mathbf{t})(x/\varepsilon) = 0$ in V. Since $t_{ij,j}(x/\varepsilon) \rightharpoonup \langle t_{ij,j}(y)\rangle = 0$ in $L^{p'}(V)$ weakly as $\varepsilon \to 0$, therefore $\lim\limits_{\varepsilon\to 0} R_\varepsilon = 0$ as claimed. □

To prove (4.21) we shall exploit the following result due to MURAT. [19].

PROPOSITION 1. Let q and q' be such that

$$1 < q,\ q' < \infty, \qquad \frac{1}{q} + \frac{1}{q'} = 1,$$

and let V be a bounded or unbounded domain of $\mathbb{R}^N$. We define the following spaces:

$$\mathbf{W}(\operatorname{div}, V) = \left\{\mathbf{u} \in L^q(V)^N \mid \operatorname{div}\mathbf{u} \in L^q(V)\right\},$$
$$\mathbf{W}(\operatorname{rot}, V) = \left\{\mathbf{v} \in L^{q'}(V)^N \mid \operatorname{rot}\mathbf{v} \in L^{q'}(V, \mathbb{E}^N)\right\},$$

where

$$(\operatorname{rot}\mathbf{v})_{ij} = \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}, \qquad 1 \le i, j \le N,$$

and $\mathbb{E}^N$ is the space of $N \times N$ matrices. The spaces $\mathbf{W}(\operatorname{div}, V)$ and $\mathbf{W}(\operatorname{rot}, V)$ are equipped with the norms:

$$\|\mathbf{u}\|_{\mathbf{W}(\operatorname{div},V)} = \|\mathbf{u}\|_{L^q(V)^N} + \|\operatorname{div}\mathbf{u}\|_{L^q(V)},$$
$$\|\mathbf{v}\|_{\mathbf{W}(\operatorname{rot},V)} = \|\mathbf{v}\|_{L^{q'}(V)^N} + \|\operatorname{rot}\mathbf{v}\|_{L^{q'}(V,\mathbb{E}^N)}.$$

If two sequences $\{\mathbf{u}_n\}_{n\in\mathbb{N}} \subset \mathbf{W}(\operatorname{div}, V)$, $\{\mathbf{v}_n\}_{n\in\mathbb{N}} \subset \mathbf{W}(\operatorname{rot}, V)$ satisfy the conditions

$\mathbf{u}_n$ is bounded in $\mathbf{W}(\operatorname{div}, V)$, $\mathbf{u}_n \rightharpoonup \mathbf{u}$ weakly in $L^q(V)^N$,

$\mathbf{v}_n$ is bounded *in* $\mathbf{W}(\operatorname{rot}, V)$, $\mathbf{v}_n \rightharpoonup \mathbf{u}$ weakly in $L^{q'}(V)^N$, then

$$\mathbf{u}_n \cdot \mathbf{v}_n \rightharpoonup \mathbf{u} \cdot \mathbf{v} \quad \text{in } \mathcal{D}'(V) \text{ when } n \to \infty. \qquad \square$$

Proof of (4.21). We observe that rot $\mathbf{E}^y(\xi) = 0$ and consequently $\mathbf{E}(\xi(\cdot/\varepsilon) \in \mathbf{W}(\mathrm{rot}, V)$. By using Proposition 1 we conclude that

$$D_i^\varepsilon(x) E_i \left(\xi \left(\frac{x}{\varepsilon} \right) \right) \rightharpoonup D_i(x) E_i(\langle \xi \rangle) = 0 \quad \text{in } \mathcal{D}'(V) \text{ when } \varepsilon \to 0,$$

because $\xi(x/\varepsilon) \rightharpoonup \langle \xi(y) \rangle$ weakly in $L^{q'}(V)$, cf. [12, 20]. The proof is complete. $\square$

Now we are in a position to prove Th. 2.

Proof of Th. 2. It falls naturally into two parts.

I. Let us first show that

$$J_i = \Gamma(\tau)\text{-} \liminf_{\varepsilon \to 0} J_\varepsilon \geq J_h.$$

We recall that $\tau = w\text{-}(W^{1,p}(\Omega)^3 \times L^q(\Omega)^3)$. Let $\{\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon\}_{\varepsilon > 0} \subset W^{1,p}(\Omega)^3 \times L^q(\Omega)^3$ be a bounded sequence such that

$$\begin{aligned} \mathbf{u}^\varepsilon &\to \mathbf{u} \quad \text{strongly in } L^p(\Omega)^3, \\ \mathbf{D}^\varepsilon &\rightharpoonup \mathbf{D} \quad \text{weakly in } L^q(\Omega)^3, \end{aligned}$$

when $\varepsilon \to 0$.

We have to show that

$$\begin{aligned} \text{(4.22)} \qquad \liminf_{\varepsilon \to 0} J_\varepsilon(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon) \geq J_h(\mathbf{u}, \mathbf{D}) &= \int_V U_h(\mathbf{e}(\mathbf{u}), \mathbf{D})\, dx \\ &= \sup \left\{ \int_V [\boldsymbol{\sigma} : \mathbf{e}(\mathbf{u}) + \mathbf{D}^* \cdot \mathbf{D} - U_h^*(\boldsymbol{\sigma}, \mathbf{D}^*)]\, dx \;\middle|\; \right. \\ &\qquad \left. \boldsymbol{\sigma} \in L^{p'}(V, \mathbb{E}_s^3), \quad \mathbf{D}^* \in L^{q'}(V)^3 \right\}. \end{aligned}$$

STEP 1. First we take $\boldsymbol{\sigma}$ and $\mathbf{D}^*$ in the following form:

$$\text{(4.23)} \qquad \begin{aligned} \boldsymbol{\sigma}(x) &= \sum_{K \in \mathcal{K}} \chi_{V_K}(x) \boldsymbol{\sigma}^K, \qquad & \boldsymbol{\sigma}^K &\in \mathbb{E}_s^3, \\ \mathbf{D}^*(x) &= \sum_{K \in \mathcal{K}} \chi_{V_K}(x) \mathbf{D}^{*K}, \qquad & \mathbf{D}^{*K} &\in \mathbb{R}^3, \end{aligned}$$

where

$$\chi_{V_K}(x) = \begin{cases} 1 & \text{if } x \in V_K, \\ 0 & \text{if } x \notin V_K. \end{cases}$$

Here $\{V_K\}_{K\in\mathcal{K}}$ is a family of open disjoint sets such that $\overline{V} = \bigcup_{K\in\mathcal{K}} \overline{V}_K$.

For $\delta > 0$ we set

$$V_K^\delta = \{x \in V_K \mid \operatorname{dist}(x, \partial V_K) > \delta\}. \tag{4.24}$$

Let $\psi_K^\delta \in \mathcal{D}(V_K)$ be such that $0 \le \psi_K^\delta \le 1$ and $\psi_K^\delta = 1$ for $x \in V_K^\delta$.

Let $(\mathbf{t}^K, \varphi^K) \in \mathcal{S}_{\rm per}(Y) \times W^{1,q'}_{\rm per}(Y)$, $K \in \mathcal{K}$. By using Lemma 2 and recalling that $U \ge 0$ we obtain

$$\liminf_{\varepsilon\to 0} J_\varepsilon(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon) \ge \liminf_{\varepsilon\to 0}\left\{\int_V U\left(\frac{x}{\varepsilon}, \mathbf{e}(\mathbf{u}^\varepsilon), \mathbf{D}^\varepsilon\right) dx\right.$$

$$\left. - \sum_K \int_{V_K} \psi_K^\delta(x) t_{ij}^K\left(\frac{x}{\varepsilon}\right) e_{ij}(\mathbf{u}^\varepsilon)\,dx - \sum_K \int_{V_K} \psi_K^\delta(x) D_i^\varepsilon(x) E_i\left(\varphi^K\left(\frac{x}{\varepsilon}\right)\right) dx\right\}$$

$$= \sum_K \liminf_{\varepsilon\to 0} \int_{V_K} \psi_K^\delta(x) U_{(\mathbf{t}^K, \mathbf{E}(\varphi^K))}\left(\frac{x}{\varepsilon}, \mathbf{e}(\mathbf{u}^\varepsilon), \mathbf{D}^\varepsilon\right) dx,$$

where

$$U_{(\mathbf{t}^K, \mathbf{E}(\varphi^K))}(y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho}) = U(y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho}) - \mathbf{t}^K : \boldsymbol{\varepsilon} - \mathbf{E}(\varphi^K)\cdot\boldsymbol{\varrho}.$$

Fenchel's inequality applied to $[(\mathbf{e}(\mathbf{u}^\varepsilon), \mathbf{D}^\varepsilon); (\boldsymbol{\sigma}^K, \mathbf{D}^{*K})]$ yields:

$$U^*_{(\mathbf{t}^K, \mathbf{E}^y(\varphi^K))}(y, \boldsymbol{\varepsilon}^*, \boldsymbol{\varrho}^*) \ge \boldsymbol{\sigma}^K : \mathbf{e}(\mathbf{u}^\varepsilon) + \mathbf{D}^{*K}\cdot\mathbf{D} - U_{(\mathbf{t}^K, \mathbf{E}^y(\varphi^K))}(y, \mathbf{e}(\mathbf{u}^\varepsilon), \mathbf{D}^\varepsilon).$$

Hence

$$\liminf_{\varepsilon\to 0} J_\varepsilon(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon)$$
$$\ge \sum_{K\in\mathcal{K}} \liminf_{\varepsilon\to 0} \int_{V_K} \psi_K^\delta\left[\boldsymbol{\sigma}^K : \mathbf{e}(\mathbf{u}^\varepsilon) + \mathbf{D}^{*K}\cdot\mathbf{D}^\varepsilon - U^*_{(\mathbf{t}^K, \mathbf{E}^y(\varphi^K))}(y, \boldsymbol{\sigma}^K, \mathbf{D}^{*K})\right] dx.$$

Standard calculation yields

$$U^*_{(\mathbf{t}^K, \mathbf{E}^y(\varphi^K))}(y, \boldsymbol{\sigma}^K, \mathbf{D}^{*K})$$
$$= \sup\left\{\boldsymbol{\sigma}^K : \boldsymbol{\varepsilon} + \mathbf{D}^{*K}\cdot\boldsymbol{\varrho} - U_{(\mathbf{t}^K, \mathbf{E}^y(\varphi^K))}(y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho}) \mid \boldsymbol{\varepsilon} \in \mathbb{E}^3_s,\ \boldsymbol{\varrho} \in \mathbb{R}^3\right\}$$
$$= \sup\left\{(\boldsymbol{\sigma}^K + \mathbf{t}^K) : \boldsymbol{\varepsilon} + (\mathbf{D}^{*K} + \mathbf{E}^y(\varphi^K))\cdot\boldsymbol{\varrho} - U(y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho}) \mid \boldsymbol{\varepsilon} \in \mathbb{E}^3_s,\ \boldsymbol{\varrho} \in \mathbb{R}^3\right\}$$
$$= U^*(y, \boldsymbol{\sigma}^K + \mathbf{t}^K(y), \mathbf{D}^{*K} + \mathbf{E}^y(\varphi^K)).$$

Thus we have

$$\langle U^*_{(\mathbf{t}^K, \mathbf{E}^y(\varphi^K))}(y, \boldsymbol{\sigma}^K, \mathbf{D}^{*K})\rangle = \langle U^*(y, \boldsymbol{\sigma}^K + \mathbf{t}^K(y), \mathbf{D}^{*K} + \mathbf{E}^y(\varphi^K(y)))\rangle. \tag{4.25}$$

Consequently

$$\liminf_{\varepsilon\to 0} J_\varepsilon(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon) \geq \sum_{K\in\mathcal{K}} \int_{V_K} \psi_K^\delta(x) \cdot \left\{ \boldsymbol{\sigma}^K : \mathbf{e}(\mathbf{u}) + \mathbf{D}^{*K}\cdot\mathbf{D} - \langle U^*[y, \boldsymbol{\sigma}^K + \mathbf{t}^K(y), \mathbf{D}^{*K} + \mathbf{E}^y(\varphi^K(y))]\rangle \right\} dx.$$

Passing to the supremum on the r.h.s. of the last inequality when $(\mathbf{t}^K, \varphi^K)$ runs over $\mathcal{S}_{\text{per}}(Y) \times W^{1,q'}_{\text{per}}(Y)$ we obtain

$$\liminf_{\varepsilon\to 0} J_\varepsilon(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon) \geq \sum_{K\in\mathcal{K}} \int_{V_K} \psi_K^\delta \left[\boldsymbol{\sigma}^K : \mathbf{e}(\mathbf{u}) + \mathbf{D}^{*K}\cdot\mathbf{D} - U_h^*(\boldsymbol{\sigma}^K, \mathbf{D}^{*K}) \right] dx,$$

because $\sup(-\int) = -\inf\int$. Recall that U_h^* is given by (4.9). Since $\psi_K^\delta \geq 0$ and $\boldsymbol{\sigma}(x)$, $\mathbf{D}^*(x)$ are given by (4.23), therefore

$$\liminf_{\varepsilon\to 0} J_\varepsilon(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon) \geq \int_V \sum_{K\in\mathcal{K}} \psi_K^\delta(x) \left[\boldsymbol{\sigma}(x) : \mathbf{e}(\mathbf{u}(x)) + \mathbf{D}^*(x)\cdot\mathbf{D}(x)\right] dx - \int_V \sum_{K\in\mathcal{K}} \psi_K^\delta(x) U_h^*(\boldsymbol{\sigma}(x), \mathbf{D}^*(x))\, dx.$$

The inequality

$$0 \leq \sum_{K\in\mathcal{K}} \psi_K^\delta \leq 1,$$

implies

$$0 \leq \sum_{K\in\mathcal{K}} \psi_K^\delta(x) U_h^*(\boldsymbol{\sigma}(x)), \mathbf{D}^*(x)) \leq U_h^*(\boldsymbol{\sigma}(x), \mathbf{D}^*(x)),$$

because $U_h^* \geq 0$. It follows that

$$\liminf_{\varepsilon\to 0} J_\varepsilon(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon) \geq \int_V \sum_{K\in\mathcal{K}} \psi_K^\delta(x) \left[\boldsymbol{\sigma}(x) : \mathbf{e}(\mathbf{u}(x)) + \mathbf{D}^*(x)\cdot\mathbf{D}(x)\right] dx - \int_V U_h^*(\boldsymbol{\sigma}(x), \mathbf{D}^*(x))\, dx.$$

We pass now to the limit when $\delta \to 0$; $\sum_{K\in\mathcal{K}} \psi_K^\delta(x)$ tends to 1 for a.e. $x \in \Omega$ and consequently we obtain

$$\liminf_{\varepsilon\to 0} J_\varepsilon(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon) \geq \int_V \left[\boldsymbol{\sigma}(x) : \mathbf{e}(\mathbf{u}(x)) + \mathbf{D}^*(x)\cdot\mathbf{D}(x)\right] dx - \int_V U_h^*(\boldsymbol{\sigma}(x), \mathbf{D}^*(x)) dx.$$

STEP 2. For each $\boldsymbol{\sigma} \in L^{p'}(V, \mathbb{E}^3_s)$ and $\mathbf{D}^* \in L^{q'}(V)^3$ there exist sequences $\{\boldsymbol{\sigma}^n\}_{n\in\mathbb{N}} \subset L^{p'}(V, \mathbb{E}^3_s)$ and $\{\mathbf{D}^{*n}\}_{n\in\mathbb{N}} \subset L^{q'}(V)^3$ of simple functions such that

$$\begin{aligned} &\boldsymbol{\sigma}^n \to \boldsymbol{\sigma} \quad \text{in } L^{p'}(V, \mathbb{E}^3_s) \text{ as } n \to \infty, \\ &\mathbf{D}^{*n} \to \mathbf{D}^* \text{ in } L^{q'}(V)^3 \quad \text{as } n \to \infty, \end{aligned}$$

respectively. Here

$$\begin{aligned} \boldsymbol{\sigma}^n(x) &= \sum_{K(n)} \chi^{\delta_n}_{V_{K(n)}}(x)\boldsymbol{\sigma}^{K(n)}, \quad \boldsymbol{\sigma}^{K(n)} \in \mathbb{E}^3_s, \\ \mathbf{D}^{*n}(x) &= \sum_{K(n)} \chi^{\delta_n}_{V_{K(n)}}(x)\mathbf{D}^{*K(n)}, \quad \mathbf{D}^{*K(n)} \in \mathbb{R}^3, \end{aligned}$$

and $\delta_n = 1/n$, $\operatorname{diam} V_{K(n)} \le \delta_n$, $\overline{V} = \bigcup\limits_{K(n)} \overline{V}_{K(n)}$. We conclude by the previous step that

$$\begin{aligned} &\liminf_{\varepsilon\to 0} J_\varepsilon(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon) \\ &\qquad \ge \int_V [\boldsymbol{\sigma}^n(x) : \mathbf{e}(\mathbf{u}(x)) + \mathbf{D}^{*n}(x)\cdot\mathbf{D}(x)]\,dx - \int_V U^*_h(\boldsymbol{\sigma}^n(x), \mathbf{D}^{*n}(x))\,dx. \end{aligned}$$

A passage to the limit on the r.h.s. of the last inequality when $n \to \infty$ finally gives:

$$\liminf_{\varepsilon\to 0} J_\varepsilon(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon) \ge \int_V [\boldsymbol{\sigma}(x) : \mathbf{e}(\mathbf{u}(x)) + \mathbf{D}^*(x)\cdot\mathbf{D}(x) - U^*_h(\boldsymbol{\sigma}(x), \mathbf{D}(x))]\,dx.$$

II. We pass now to demonstrate that for any $(\mathbf{u}, \mathbf{D}) \in W^{1,p}(V)^3 \times L^q(V)^3$ there exists a sequence $\{\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon\}_{\varepsilon>0} \subset W^{1,p}(\Omega)^3 \times L^q(\Omega)^3$ such that

$$\begin{aligned} &\mathbf{u}^\varepsilon \rightharpoonup \mathbf{u} \quad \text{in} \quad W^{1,p}(V)^3 \quad \text{weakly}, \\ &\mathbf{D}^\varepsilon \rightharpoonup \mathbf{D} \quad \text{in} \quad L^q(V)^3 \quad \text{weakly} \end{aligned}$$

when $\varepsilon \to 0$ and

$$J_h(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon) \ge \limsup_{\varepsilon\to 0} J_\varepsilon(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon). \tag{4.26}$$

Obviously, the convergence of $\mathbf{u}^\varepsilon$ to $\mathbf{u}$ is strong in $L^p(V)^3$.

STEP 3. We take

$$u_i(x) = \varepsilon_{ij}x_j + a_i\,, \qquad \boldsymbol{\varepsilon} \in \mathbb{E}^3_s\,, \qquad a_i \in \mathbb{R}, \tag{4.27}$$

whereas $\mathbf{D}$ is an arbitrary element of $\mathbb{R}^3$ treated as a constant function of $L^q(V)^3$.

Next we set

$$
\begin{aligned}
\mathbf{u}^\varepsilon(x) &= \mathbf{u}(x) + \varepsilon\widetilde{\mathbf{v}}\left(\frac{x}{\varepsilon}\right), \\
\mathbf{D}^\varepsilon(x) &= \mathbf{D}(x) + \widetilde{\mathbf{d}}\left(\frac{x}{\varepsilon}\right),
\end{aligned}
\tag{4.28}
$$

where $(\widetilde{\mathbf{v}}, \widetilde{\mathbf{d}})$ solves the local problem. Hence we conclude that

$$
\begin{aligned}
\mathbf{u}^\varepsilon &\to \mathbf{u} \quad \text{in} \quad L^p(V)^3 \quad \text{strongly}, \\
\mathbf{D}^\varepsilon &\rightharpoonup \mathbf{D} \quad \text{in} \quad L^q(V)^3 \quad \text{weakly},
\end{aligned}
$$

as $\varepsilon \to 0$.

Applying Th. 1.5 of DACOROGNA [20, Chap. 2], we obtain

$$
\begin{aligned}
\lim_{\varepsilon\to 0} J_h(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon) &= \lim_{\varepsilon\to 0}\int_V U\left[\frac{x}{\varepsilon}, \boldsymbol{\varepsilon} + \mathbf{e}(\widetilde{\mathbf{v}})\left(\frac{x}{\varepsilon}\right), \mathbf{D} + \widetilde{\mathbf{d}}\left(\frac{x}{\varepsilon}\right)\right] dx \\
&= \int_V \langle U[y, \boldsymbol{\varepsilon} + \mathbf{e}^y(\widetilde{\mathbf{v}}(y)), \mathbf{D} + \widetilde{\mathbf{d}}(y)]\rangle dx \\
&= \int_V U_h(\boldsymbol{\varepsilon}, \mathbf{D})dx = \int_V U_h\left[\mathbf{e}(\mathbf{u}(x)), \mathbf{D}(x)\right] dx,
\end{aligned}
$$

since $\mathbf{u}$ is given by (4.27) and $\mathbf{D} \in L^q(V)^3$ is a constant function.

STEP 4. Let now $\mathbf{u}$ be a continuous affine function as an element of $W^{1,p}(V)^3$ and $\mathbf{D}$ a simple function in the space $L^q(V)^3$:

$$
\mathbf{u}(x) = \boldsymbol{\varepsilon}^K x + \mathbf{a}^K, \qquad x \in V_K, \tag{4.29}
$$

$$
\mathbf{D}(x) = \sum_K \chi_{V_K}(x)\mathbf{D}^K, \qquad \mathbf{D}^K \in \mathbb{R}^3, \tag{4.30}
$$

where $\boldsymbol{\varepsilon}^K \in \mathbb{E}^3_s$, $\mathbf{a}^K \in \mathbb{R}^3$ and $\{V_K\}_{K\in\mathcal{K}}$ is a finite partition of V formed by polyhedral sets.

We set

$$
V_K^\delta = \{x \in V_K \mid \operatorname{dist}(x, \partial V_K) > \delta\}, \qquad \delta > 0.
$$

Let $\psi_K^\delta \in \mathcal{D}(V_K)$ be such that $0 \le \psi_K^\delta \le 1$ and $\psi^\delta_{K|V_K^\delta} = 1$. With every family of functions $(\mathbf{v}^K, \mathbf{d}^K) \in W^{1,p}_{\text{per}}(Y)^3 \times \Delta_{\text{per}}(Y)$ we link the following sequences:

$$
\mathbf{u}^{\varepsilon,\delta}(x) = \mathbf{u}(x) + \varepsilon\sum_{K\in\mathcal{K}}\psi_K^\delta(x)\mathbf{v}^K\left(\frac{x}{\varepsilon}\right), \tag{4.31}
$$

$$
\mathbf{D}^{\varepsilon,\delta}(x) = \mathbf{D}(x) + \sum_{K\in\mathcal{K}}\psi_K^\delta(x)\mathbf{d}^K\left(\frac{x}{\varepsilon}\right). \tag{4.32}
$$

It is evident that

$$\mathbf{u}^{\varepsilon,\delta} \to \mathbf{u} \quad \text{in} \quad L^p(V)^3 \quad \text{strongly,}$$
$$\mathbf{D}^{\varepsilon,\delta} \rightharpoonup \mathbf{D} \quad \text{in} \quad L^q(V)^3 \quad \text{weakly,}$$

as $\varepsilon \to 0$.

We recall that for $N_0 \in \mathbb{N}$ sufficiently large one has:

$$\int\limits_{V_K} \left| \mathbf{v}^K \left(\frac{x}{\varepsilon} \right) \right|^p dx \le N_0 \int\limits_Y |\mathbf{v}^K(y)|^p \, dy.$$

Take $0 < t < 1$. Since $t\psi_K^\delta + t(1-\psi_K^\delta) + (1-t) = 1$, therefore by convexity of $U((x/\varepsilon), \cdot, \cdot)$ we obtain

$$(4.33) \qquad J_\varepsilon(t\,\mathbf{u}^{\varepsilon,\delta}, t\,\mathbf{D}^{\varepsilon,\delta}) = \sum_K \int\limits_{V_K} U\Big[\frac{x}{\varepsilon}, t\,\psi_K^\delta \left(\boldsymbol{\varepsilon}^K + \mathbf{e}(\mathbf{v}^K)\left(\frac{x}{\varepsilon}\right)\right) + t\,(1-\psi_K^\delta)\boldsymbol{\varepsilon}^K$$
$$+ (1-t)\frac{\varepsilon\, t}{1-t}\left(\psi^\delta_{K,(i}(x) v^K_{j)}\left(\frac{x}{\varepsilon}\right)\right), t\,\psi_K^\delta\left(\mathbf{D}^K + \mathbf{d}^K\left(\frac{x}{\varepsilon}\right)\right)$$
$$+ t(1-\psi_K^\delta)\mathbf{D}^K + (1-t)\mathbf{0}\Big] dx$$
$$\le \sum_K \Bigg\{ \int\limits_{V_K} U\left[\frac{x}{\varepsilon}, \boldsymbol{\varepsilon}^K + \mathbf{e}(\mathbf{v}^K)\left(\frac{x}{\varepsilon}\right), \mathbf{D}^K + \mathbf{d}^K\left(\frac{x}{\varepsilon}\right)\right] dx$$
$$+ c_1\left(|\boldsymbol{\varepsilon}^K|^p + |\mathbf{D}^K|^q\right)\int\limits_{V_K}(1-\psi_K^\delta)dx + c_1(1-t)\int\limits_{V_K}\frac{\varepsilon\, t}{1-t}\left|\left(\psi^\delta_{K,(i}(x) v^K_{j)}\left(\frac{x}{\varepsilon}\right)\right)\right|^p dx\Bigg\}.$$

We recall that $U \ge 0$.

Let now ε tend to zero. Then we get

$$\limsup_{\varepsilon\to 0} J_\varepsilon(t\,\mathbf{u}^{\varepsilon,\delta}, t\,\mathbf{D}^{\varepsilon,\delta}) \le \sum_K \Big\{ |V_K| \langle U\left(y, \boldsymbol{\varepsilon}^K + \mathbf{e}^y(\mathbf{v}^K(y)), \mathbf{D}^K + \mathbf{d}^K(y)\right)\rangle$$
$$+ c_1\left(|\boldsymbol{\varepsilon}^K|^p + |\mathbf{D}^K|^q\right)\int\limits_{V_K}(1-\psi_K^\delta)dx\Big\}.$$

Next, let $t \to 1^-$ and $\delta \to 0$. The sequence $\sum\limits_K \psi_K^\delta(x)$ converges to 1 almost everywhere when $\delta \to 0$. We conclude that

$$(4.34) \qquad \limsup_{\substack{\delta\to 0\\ t\to 1^-}} \; \limsup_{\varepsilon\to 0} J_\varepsilon(t\,\mathbf{u}^{\varepsilon,\delta}, t\,\mathbf{D}^{\varepsilon,\delta})$$
$$\le \sum_K |V_K| \langle U\left(y, \boldsymbol{\varepsilon}^K + \mathbf{e}^y(\mathbf{v}^K(y)), \mathbf{D}^K + \mathbf{d}^K(y)\right)\rangle.$$

To proceed further we shall exploit the following lemma due to ATTOUCH [10].

LEMMA 3. Let $\{a_{A,B} \mid A \in \mathbb{N}, B \in \mathbb{N}\}$ be a doubly indexed family in $\overline{\mathbb{R}}$-the extended reals. Then there exists a mapping $A \to B(A)$, increasing to $+\infty$, such that

$$\limsup_{A\to\infty} a_{A,B(A)} \leq \limsup_{B\to\infty} \limsup_{A\to\infty} a_{A,B}. \qquad \square$$

Applying this lemma we construct a mapping $\varepsilon \to (t(\varepsilon), \delta(\varepsilon))$ with $(t(\varepsilon), \delta(\varepsilon)) \to (1^-, 0)$ such that by setting

$$\mathbf{u}^\varepsilon = t(\varepsilon)\, \mathbf{u}^{\varepsilon,\delta}, \qquad \mathbf{D}^\varepsilon = t(\varepsilon)\, \mathbf{D}^{\varepsilon,\delta},$$

we obtain from (4.34)

$$\text{(4.35)} \qquad \limsup_{\varepsilon\to 0} J_\varepsilon(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon) \leq \sum_K |V_K| \left\langle U\left(y, \boldsymbol{\varepsilon}^K + \mathbf{e}^y(\mathbf{v}^K(y)), \mathbf{D}^K + \mathbf{d}^K(y)\right)\right\rangle.$$

Taking the infimum on the right-hand side of the last inequality when $(\mathbf{v}^K, \mathbf{d}^K)$ run over $W^{1,p}_{\mathrm{per}}(Y)^3 \times \Delta_{\mathrm{per}}(Y)$, we get

$$\text{(4.36)} \qquad J_s(\mathbf{u}, \mathbf{D}) \leq \limsup_{\varepsilon\to 0} J_\varepsilon(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon) \leq \sum_K |V_K| U_h(\boldsymbol{\varepsilon}^K, \mathbf{D}^K)$$
$$= \int_\Omega U_h(\mathbf{e}(\mathbf{u}), \mathbf{D})dx = J_h(\mathbf{u}, \mathbf{D}),$$

where J_s stands for the $\Gamma\left[(\mathrm{w}-W^{1,p}(V)^3) \times (\mathrm{w}-L^q(V)^3)\right] - \limsup_{\varepsilon\to 0} J_\varepsilon$. In (4.36) we have used the fact that

$$\begin{array}{lll} \mathbf{u}^\varepsilon \to \mathbf{u} & \text{strongly} & \text{in} \quad L^p(V)^3, \\ \mathbf{D}^{\varepsilon,\delta} \rightharpoonup \mathbf{D} & \text{weakly} & \text{in} \quad L^q(V)^3, \end{array}$$

when $\varepsilon \to 0$.

STEP 5. From Sec. 3 we know that the convexity of J_ε is preserved by the Γ-limit superior. By virtue of the property (ii) of U_h we write

$$J_s(\mathbf{u}, \mathbf{D}) \leq c_1 \int_V \left(|\mathbf{e}(\mathbf{u}(x))|^p + |\mathbf{D}(x)|^q\right)\, dx,$$

where $\mathbf{u} \in W^{1,p}(V)^3$, $\mathbf{D} \in L^q(V)^3$. Being convex and finite, the functional J_s is continuous on the space $W^{1,p}(V)^3 \times L^q(V)^3$. Exploiting the properties of the homogenized potential U_h we readily conclude that J_h is also a convex and continuous functional on this space. By density of piecewise affine continuous functions in $W^{1,p}(V)$ and simple functions in $L^q(V)$, cf. [17], the inequality $J_s(\mathbf{u}, \mathbf{D}) \leq J_h(\mathbf{u}, \mathbf{D})$ is readily extended to $W^{1,p}(V)^3 \times L^q(V)^3$, see [23, 24]. This completes the proof. $\square$

REMARK 5. The proof of Th. 2 remains valid for $\mathbf{u} \in W^{1,p}(V)^3$ with $\mathbf{u} = \mathbf{0}$ on γ_0 and $\mathbf{D} \in \mathbf{W}(\mathrm{div}, V)$ satisfying $\mathrm{div}\, \mathbf{D} = 0$ in V. For the the sake of sinplicity, let us assume that $\gamma_3 = \emptyset$, hence $\gamma_2 = \partial V$. Then we take $\mathbf{u}$ as in (4.29), with $\mathbf{u} = \mathbf{0}$ on γ_0 and

$$\mathbf{D}^{\varepsilon,\delta}(x) = \mathbf{D}^{\varepsilon}(x) = \mathbf{D}(x) + \mathbf{d}^K\left(\frac{x}{\varepsilon}\right), \qquad \mathbf{d}^K = \mathbf{d}.$$

Here $\mathbf{d}$ is an element in $\Delta_{\mathrm{per}}(Y)$. Since $\mathrm{div}_y \mathbf{d}(y) = 0$ in Y therefore $(\mathrm{div}\,\mathbf{d})(x/\varepsilon)$ vanishes in V. Instead of $J_\varepsilon(t\,\mathbf{u}^{\varepsilon,\delta}, t\,\mathbf{D}^{\varepsilon,\delta})$ we now consider $J_\varepsilon(t\,\mathbf{u}^{\varepsilon,\delta}, \mathbf{D}^{\varepsilon})$

REMARK 6. By exploiting the assumption (A) it is not difficult to slow that Th. 1 applies and consequently

$$\inf\{J_h(\mathbf{u},\mathbf{D}) - L(\mathbf{u},\mathbf{D}) \mid (\mathbf{u},\mathbf{D}) \in X\} = \lim_{\varepsilon\to 0}(\inf\{J_\varepsilon(\mathbf{u},\mathbf{D}) - L(\mathbf{u},\mathbf{D}) \mid (\mathbf{u},\mathbf{D}) \in X\})$$

where

$$X = \Big\{(\mathbf{u},\mathbf{D}) \in W^{1,p}(V)^3 \times \mathbf{W}(\mathrm{div}, V) \mid \mathbf{u} = 0 \ \text{ on } \gamma_0, \quad D_{i,i} = 0 \ \text{ in } V, \quad D_i n_i = 0 \ \text{ on } \gamma_3\Big\}. \tag{4.37}$$

5. Comments on non-uniform homogenization

From the point of view of homogenization, the coercivity condition appearing in (2.3) can be significantly weakened. In fact, let now

$$U = U(x, y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho}) : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{E}^3_s \times \mathbb{R}^3 \to [0, +\infty),$$

be a measurable function, Y-periodic in y, continuous in x and convex in $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varrho}$. This function is assumed to satisfy the following conditions, cf. [25–27]:

(i) $|U(x, y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho}) - U(x', y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho})| \le \omega(|x - x'|)\,(a(y) + U(x, y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho}))$,

(ii) $0 \le U(x, y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho}) \le b(x)\,(a(y) + |\boldsymbol{\varepsilon}|^p + |\boldsymbol{\varrho}|^q)$,

for each $x \in V$, $y \in Y$, $\boldsymbol{\varrho} \in \mathbb{R}^3$ and each $\boldsymbol{\varepsilon} \in \mathbb{E}^3_s$. Here $a \in L^1_{loc}(\mathbb{R}^3)$ is a Y-periodic function, $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function, continuous at zero and such that $\omega(0) = 0$ and $b : \mathbb{R}^3 \to \mathbb{R}$ is a continuous and nonnegative function. The assumption (ii) admits internal energies, which are not strictly convex. Thus it may then happen that $U(x, y, \mathbf{e}(\mathbf{u}), \mathbf{D}) = 0$ though either $\mathbf{e}(\mathbf{u})$ or/and $\mathbf{D}$ do not disappear. The dependence of U on the macroscopic variable x means that after homogenization, the effective potential U_h still depends on x (nonuniform homogenization). Indeed, the latter potential takes the form:

$$U_h(x, \boldsymbol{\varepsilon}, \boldsymbol{\varrho}) = \inf\left\{\frac{1}{|Y|}\int_Y U\left(x, y, \mathbf{e}^y(\mathbf{v}) + \boldsymbol{\varepsilon}, \mathbf{d}(y) + \mathbf{D}\right) dy \mid \mathbf{v} \in W^{1,p}_{\mathrm{per}}(Y)^3,\ \mathbf{d} \in \Delta_{\mathrm{per}}(Y)\right\}. \tag{5.1}$$

The macroscopic variable x changes slowly while the local variable y characterizes fast local changes. More general case of the stored energy function was considered in [26, 27]. The last papers were inspired by application to geometrically nonlinear structures such like plates and shells, yet the general nonuniform homogenization procedure can be adapted to our case of nonlinear piezoelectric composites. We observe that for microperiodic composites, U in (5.1) does not depend on the macroscopic variable $x \in V$. The assumption (i) is then trivially satisfied. We conclude from (ii) that though then the problem becomes noncoercive, yet the Γ-convergence yields the same results as for the coercive problem studied in Sec. 4.

6. Final remarks

To model the behaviour of piezoelectric composites with periodic structure and subjected to stronger electric fields, nonlinear homogenization has been used. Our considerations are confined to small deformations. Such a case has its practical value, cf. [8]. Theorem 2 justifies also the homogenization results obtained by the first author in [1] for linear piezocomposites as well as the homogenization formulae used by BISEGNA and LUCIANO [3 – 5], cf. also [2]. The primal and dual effective potential cannot be explicitly found, except in particular cases. For instance, such is the case of layered composites. Hence the need follows for bounding the effective potential U_h from below and from above. To this end, the nonlinear bounding techniques developed by TALBOT and WILLIS [28] can be applied. We observe that bounding techniques for linear piezoelectric composites have been used in [3, 21].

Acknowledgment

The authors were supported by the State Committee for Scientific Research (Poland) through the grant No 7 T07 A016 12.

References

1. J.J. TELEGA, *Piezoelectricity and homogenization. Application to biomechanics*, [in:] Continuum Models and Discrete Systems, vol 2, G.A. MAUGIN [Ed.], pp. 220–229, Longman, Scientific and Technical, Harlow, Essex 1991.
2. A. GAŁKA, J.J. TELEGA and R. WOJNAR, *Some computational aspects of homogenization of thermopiezoelectric composites*, Comp. Assisted Mech. and Eng. Sci., **3**, 133–154, 1996.
3. P. BISEGNA and R. LUCIANO, *Variational bounds for the overall properties of piezoelectric composites*, J. Mech. Phys. Solids, **44**, 583–602, 1996.
4. P. BISEGNA and R. LUCIANO, *Bounds on the off-diagonal coefficients of the constitutive tensor of a composite material*, Mech. Res. Comm., **23**, pp. 239–245, 1996.
5. P. BISEGNA and R. LUCIANO, *On methods for bounding the overall properties of periodic piezoelectric fibrous composites*, J. Mech. Phys. Solids, **45**, 8, 1329–1356, 1997.
6. O. SIGMUND, S. TORQUATO and I.A. AKSAY, *On the design of 1-3 piezocomposites using topology optimization*, DCAM, Report No 542, April 1997.

7. L.V. Gibiansky and S. Torquato, *On the use of homogenization theory to design optimal piezocomposites for hydrophone applications*, J. Mech. Phys. Solids, **45**, 689–708, 1997.
8. H.F. Tiersten, *Equations for the extension and flexure of relatively thin electroelastic plates undergoing large electric fields*, [in:] Mechanics of Electromagnetic Materials and Structures, J.S. Lee, G.A. Maugin, Y. Shindo [Eds.], pp. 21–34, AMD-vol. 161, ASME, New York 1993.
9. G.A. Maugin, *Continuum mechanics of electromagnetic solids*, North-Holland, Amsterdam 1988.
10. H. Attouch, *Variational convergence for functions and operators*, Pitman, Boston 1984.
11. G. Dal Maso, *An introduction to Γ-convergence*, Birkhäuser, Boston 1993.
12. V.V. Jikov, S.M. Kozlov and O.A. Oleinik, *Homogenization of differential operators and integral functionals*, Springer-Verlag, Berlin 1994.
13. S. Jemioło and J.J. Telega, *Representation of tensor functions and applications in continuum mechanics*, IFTR Reports 3/1997.
14. R. Dautray and J.-L. Lions, *Mathematical analysis and numerical methods for science and technology*, vol. 3, Springer-Verlag, Berlin 1990.
15. R. Temam, *Navier–Stokes equations*, North-Holland, Amsterdam 1979.
16. R.T. Rockafellar, *Convex analysis*, Princeton University Press, Princeton 1970.
17. I. Ekeland and R. Temam, *Convex analysis and variational problems*, Amsterdam 1976.
18. P.J. Laurent, *Approximation et optimisation*, Hermann, Paris 1972.
19. F. Murat, *Compacité par compensation*, Annali della Scuola Norm. Sup. di Pisa, Classe di Scienze, Serie IV, **5**, 489–508, 1978.
20. B. Dacorogna, *Direct methods in the calculus of variations*, Springer-Verlag, Berlin 1989.
21. A. Gałka, B. Gambin and J.J. Telega, *Variational bounds on the effective moduli of piezoelectric composites*, Arch. Mech. [in print].
22. K. Kycia, A. Nowicki, O. Gajl and T.D. Hien, *Calculations of effective material tensors and the electromechanical coupling coefficient of a type 1-3 composite transducer*, Arch. Acoustics, **21**, 371–386, 1996.
23. J.J. Telega, *Justification of a refined scaling of stiffnesses of Reissner plates with fine periodic structure*, Math. Models and Math. in Appl. Sci., **2**, 375–406, 1992.
24. T. Lewiński and J.J. Telega, *Plates, laminates and shells: asymptotic analysis and homogenization* [in preparation].
25. A. Braides, *Omogenizzazione di integrali non coercivi*, Ricerche di Matematica, **32**, 347–368.
26. W.R. Bielski and J.J. Telega, *Non-linear moderately thick plates and homogenization*, Bull. Pol. Acad. Sci., Technical Sci., **45**, 197–209, 1997.
27. W.R. Bielski and J.J. Telega, *On Γ-convergence of a class of variational functionals* [submitted].
28. D.R.S. Talbot and J.R. Willis, *Bounds and self-consistent estimates for the overall properties of nonlinear composites*, IMA J. Appl. Math., **39**, 215–240, 1987.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH
e-mail: jtelega@ippt.gov.pl
e-mail: agalka@ippt.gov.pl
e-mail: bgambin@ippt.gov.pl

Received January 9, 1998.