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FIFTY YEARS OF THE ARCHIVES OF MECHANICS

An idea of thin-plate thermal mirror
II. Mirror created by a constant heat flux

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FOLLOWING Part I, an idea of thermal mirrors created on the surfaces of a simply supported thin plane circular plate of an isotropic thermoelastic solid material by a uniform constant heat flux, which is applied to one of the plate surfaces, is presented. Such a thermal mirror is – within the approximations applied for obtaining the solutions of the heat conduction and thermoelasticity equations – an ideal (aberration-free) optical mirror. The optical properties of the thermal mirror and their time evolution are derived in two extreme cases: a) no energy losses through both plate surfaces, and b) no losses through the perturbed surface and the maximum losses through the opposite surface, and discussed in two asymptotical regimes: the short-time and the long-time ones. Theoretical possibilities of application of the thermal mirror to experimental determination of temperature conductivity of a material are discussed.

1. Introduction

IN PART I OF THE PAPER [3], the idea and the theory of thin-plate thermal mirror created on the surfaces of a simply supported thin plane circular plate of an isotropic thermoelastic solid material by a uniform heat pulse applied to one of the plate surfaces was presented. In the present part a similar problem, but with different heat perturbation, is examined, namely: the heat pulse is replaced by a constant heat flux (also uniform across the perturbed surface) applied in the initial moment.

The aim is to calculate the fundamental optical properties of the mirror (i.e. – its aberration characteristic, optical power, and the focal length), and their time evolution. The goal will be achieved in the same way as in Part I, i.e. the temperature field will be found first, next the deformation of the plate surfaces will be determined, and finally the optical properties of the mirror will be calculated and discussed.

The boundary conditions for the temperature field in the plate are assumed in two extremal versions:

a. All the plate surfaces are adiabatically insulated (i.e. all the losses through the plate surfaces are neglected).

b. The perturbed and the side surfaces are adiabatically insulated, and the temperature of the opposite surface is equal to the the temperature of the plate surrounding (i.e. the losses through the perturbed surface are neglected, and the losses through the opposite surface are maximum ones) ([1]), ([2]).

All the remaining general assumptions are the same as in Part I. Also identical are the general formulae determining the displacement field in the plate, those determining the deformation of the plate surfaces, and those determining the general optical properties of the mirror. They will be therefore used here without derivation.

2. The thermal problem

Following the specification of the thermal perturbation, the temperature T (counted from the initial (before perturbation) value) in the material is assumed to be dependent on z and t only: $T = T(z,t)$, where z stands for z-coordinate in the cylindrical coordinate system with the origin located in the center of the plate and with z-axis directed perpendicularly from the plate center toward the disturbed surface, and t stands for time. Therefore, according to the general assumptions adopted, the heat conduction equation (in the dimensionless variables) is

$$\frac{\partial \Theta}{\partial \tau} = \frac{\partial^2 \Theta}{\partial \zeta^2} + \delta\left(\zeta - \frac{1}{2}\right) H(\tau - 0),$$

where

$$\tau = \frac{\kappa}{(2h)^2}\, t, \qquad \zeta = \frac{z}{2h},$$

(2.1)

$$\Theta(\zeta, \tau) = \frac{T[z = z(\zeta), t = t(\tau)]}{w_0}, \qquad w_0 = \frac{(2h)^2}{\kappa \varrho_0 c_p}\, q_0$$

stand, respectively, for: dimensionless time (κ is temperature conductivity of the material (heat conductivity divided by heat capacity per unit volume), $2h$ is the plate thickness); dimensionless z-coordinate as referred to the plate thickness $2h$; and dimensionless temperature (ϱ_0 is density of the material (in unperturbed state), c_p – its specific heat (under constant pressure), $q_0 = \mathrm{const}$ represents the heat source function amplitude); $\delta(x - x_0)$ stands for Dirac delta distribution, and $H(\tau - \tau_0)$ – for unit step function (Heaviside function).

The initial condition is assumed in the form

(2.2) $$\Theta(\tau = 0) = 0.$$

The boundary conditions are assumed in two alternative versions, according to the assumptions adopted in Sec. 1:

(2.3a) $$\frac{\partial \Theta^{(a)}}{\partial \zeta}\left(\zeta = \pm\frac{1}{2}\right) = 0,$$

([1]) The third extreme case: all the surface losses are maximum ones – is not interesting, because in this case the temperature field inside the plate is not perturbed at all.

([2]) The problem of finite surface losses (in the long-time regime) is mentioned in Sec. 7.

(2.3b)
$$\begin{cases} \dfrac{\partial \Theta^{(b)}}{\partial \zeta}\left(\zeta = \dfrac{1}{2}\right) = 0, \\ \Theta^{(b)}\left(\zeta = -\dfrac{1}{2}\right) = 0. \end{cases}$$

The solution of the problem expressed by Eqs. (2.1)–(2.3) is found as follows. The Green function to the (one-dimensional) heat conduction problem in the half-space insulated adiabatically is the doubled Green function of that problem in the whole space, and the latter Green function is known [1]. Applying therefore the method of sources and sinks, one may find the Green function to the thermal problem for the plate:

$$\Theta_{Gr}^{(ab)} = \frac{1}{\sqrt{\pi\tau}} \sum_{m=0}^{\infty} (-1)^{mp} \left\{ \exp\left[-\frac{\left(2m + \frac{1}{2} - \zeta\right)^2}{4\tau} \right] + (-1)^p \exp\left[-\frac{\left(2m + \frac{3}{2} + \zeta\right)^2}{4\tau} \right] \right\},$$

where $p = 0$ in case a and $p = 1$ in case b. Next, convoluting the latter Green function and the heat source function, the solution of the problem considered is found:

(2.4a)
$$\Theta^{(a)} = 2\sqrt{\tau} \sum_{m=0}^{\infty} \left[\operatorname{ierfc} \frac{2m + \frac{1}{2} - \zeta}{2\sqrt{\tau}} + \operatorname{ierfc} \frac{2m + \frac{3}{2} + \zeta}{2\sqrt{\tau}} \right] = \tau + 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{\pi^2 k^2} \left\{ 1 - \exp[-k^2\pi^2\tau] \right\} \cos\left[k\pi \left(\zeta + \frac{1}{2} \right) \right],$$

(2.4b)
$$\Theta^{(b)} = 2\sqrt{\tau} \sum_{m=0}^{\infty} (-1)^m \left[\operatorname{ierfc} \frac{2m + \frac{1}{2} - \zeta}{2\sqrt{\tau}} - \operatorname{ierfc} \frac{2m + \frac{3}{2} + \zeta}{2\sqrt{\tau}} \right] = 8 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\pi^2 (2k-1)^2} \left\{ 1 - \exp\left[-(2k-1)^2 \frac{\pi^2}{4} \tau \right] \right\} \times \sin\left[(2k-1)\frac{\pi}{2} \left(\zeta + \frac{1}{2} \right) \right],$$

where the integral complementary error function is

$$\operatorname{ierfc}(x) = \int_x^\infty \operatorname{erfc}(y)\,dy, \qquad \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp[-y^2]\,dy$$

($\operatorname{erfc}(x)$ stands for the complementary error function), and the first line (in each equation) represents the original solution obtained using the method mentioned [3], and the second one – that solution after expansion into Fourier series [4] (the function $\Theta^{(a)}$ is symmetrical, and the function $\Theta^{(b)}$ is antisymmetrical with respect to $\zeta + (1/2)$).

3. Fundamental characteristics of the thermal mirror

According to the assumptions adopted, the general formulae determining the displacement U of the plate surfaces with respect to their initial (before perturbation) position (see Fig. 1 in Part I) are the same as in the case considered in Part I. Using these formulae and the solutions of the thermal problem, we have (see Part I, Eqs. (4.2)):

$$\begin{aligned} U^u &= \frac{N_T}{E} + U_{\max}\left[1 - \left(\frac{r}{r_0}\right)^2 \frac{1}{(1+\delta^u)^2}\right] \cong \frac{N_T}{E} + U_{\max}\left[1 - \left(\frac{r}{r_0}\right)^2\right], \\ U_l &= -U_{\max}\left[1 - \left(\frac{r}{r_0}\right)^2 \frac{1}{(1-\delta_l)^2}\right] \cong -U_{\max}\left[1 - \left(\frac{r}{r_0}\right)^2\right], \end{aligned} \tag{3.1}$$

where the superscripts u and l refer to the disturbed and the opposite surfaces of the plate (the upper and the lower surfaces in Fig. 1 in Part I), respectively; r stands for the coordinate of a given point in the cylindrical coordinate system mentioned earlier; r_0 is the plate radius;

$$U_{\max} = \frac{3r_0^2}{4h^3 E} M_T, \tag{3.2}$$

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(3) The same results are obtainable by applying the Laplace transformation method to solve the following equivalent problems:

$$\frac{\partial\Theta}{\partial\tau} = \frac{\partial^2\Theta}{\partial\zeta^2}, \qquad \Theta(\tau=0) = 0, \qquad \frac{\partial\Theta}{\partial\zeta}\left(\zeta = \frac{1}{2}\right) = H(\tau - 0),$$

$$\begin{cases} \dfrac{\partial\Theta}{\partial\zeta}\left(\zeta = -\dfrac{1}{2}\right) = 0, & \text{in case a,} \\ \Theta\left(\zeta = -\dfrac{1}{2}\right) = 0, & \text{in case b.} \end{cases}$$

(4) The same results are obtainable by applying the Fourier method of separation of independent variables to solve the equivalent problems mentioned in the previous footnote (and expanding the functions $\zeta + (1/2)$ and $(\zeta + (1/2))^2$ into Fourier series).

$$\delta_l^u = \frac{1}{2hE}\left[\pm N_T + \frac{3}{h}M_T\right]. \tag{3.3}$$

Here the upper and the lower signs refer to the perturbed and the opposite surfaces of the plate (the upper and the lower ones in Fig. 1 in Part I), respectively; E stands for the Young modulus (see Part I, Eqs. (4.3) and (4.40)); and (cf. Part I, Eqs. (4.1))

$$N_T = E\alpha \int_{-h}^{h} T\,dz, \tag{3.4}$$

$$N_T^{(a)} = 2hE\alpha w_0 \tau, \tag{3.4a}$$

$$\begin{aligned} N_T^{(b)} &= 2hE\alpha w_0 \tau \left[1 - 8\sum_{m=0}^{\infty}(-1)^m\, \mathrm{i}^2\mathrm{erfc}\frac{2m+1}{2\sqrt{\tau}}\right] \\ &= hE\alpha w_0 \left\{1 - \frac{32}{\pi^3}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^3}\exp\left[-(2k-1)^2\frac{\pi^2}{4}\tau\right]\right\}; \end{aligned} \tag{3.4b}$$

$$M_T = E\alpha \int_{-h}^{h} zT\,dz, \tag{3.5}$$

$$\begin{aligned} M_T^{(a)} &= 2h^2E\alpha w_0 \tau \left[1 - \frac{8}{3\sqrt{\pi}}\sqrt{\tau} + 32\sqrt{\tau}\sum_{m=1}^{\infty}(-1)^{m+1}\, \mathrm{i}^3\mathrm{erfc}\frac{m}{2\sqrt{\tau}}\right] \\ &= \frac{1}{6}h^2E\alpha w_0 \left\{1 - \frac{96}{\pi^4}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}\exp[-(2k-1)^2\pi^2\tau]\right\}, \end{aligned} \tag{3.5a}$$

$$\begin{aligned} M_T^{(b)} &= 2h^2E\alpha w_0 \tau \left\{1 - \frac{8}{3\sqrt{\pi}}\sqrt{\tau} + 8\sum_{m=0}^{\infty}(-1)^m \left[\mathrm{i}^2\mathrm{erfc}\frac{2m+1}{2\sqrt{\tau}} \right.\right. \\ &\qquad \left.\left. +\; 4\sqrt{\tau}\,\mathrm{i}^3\mathrm{erfc}\frac{m+1}{\sqrt{\tau}}\right]\right\} \\ &= \frac{1}{3}h^2E\alpha w_0 \left\{1 - \frac{96}{\pi^4}\sum_{k=1}^{\infty}\frac{4-(-1)^{k+1}(2k-1)\pi}{(2k-1)^4} \right. \\ &\qquad \left. \times \exp\left[-(2k-1)^2\frac{\pi^2}{4}\tau\right]\right\}, \end{aligned} \tag{3.5b}$$

where, in turn, α stands for (linear) heat expansion coefficient;

$$\mathrm{i}^n\mathrm{erfc}(x) = \int_x^{\infty} \mathrm{i}^{n-1}\mathrm{erfc}(y)\,dy, \qquad n \geq 2,$$

and $\operatorname{ierfc}(x)$ was defined at the end of Sec. 2; and the approximations (with an accuracy to an assumed small number O^*) hold if

$$|\delta_l^u| \leq \frac{1}{2}O^* \tag{3.6}$$

(for detailed argumentation for this criterion see Part I, Appendix).

The general formulae determining the aberration characteristic $\varepsilon = \varepsilon(r)$, the optical power D and the focal length f of the thermal mirror considered are the same as in the case examined in Part I. The deflection angle ε is defined as an angle between incident testing light beam parallel to the symmetry axis and this ray after reflection from the mirror (see Fig. 2 in Part I). The optical power is defined as a reciprocal of the focal length f, and the latter quantity is defined as a distance of the focal point from the mirror along the mirror symmetry axis (see Fig. 2 in Part I). The deflection angle, the optical power and the focal length are understood to be negative in the case of defocusing mirror (the perturbed, or the upper surface in our case), and positive in the case of focusing mirror (the opposite, or the lower surface in our case).

Using the general formulae mentioned above we have (in both cases a and b):

$$\varepsilon_l^u = \mp 2\arctan\left[\frac{2U_{\max}}{r_0}\frac{r}{r_0}\frac{1}{(1\pm\delta_l^u)^2}\right] \cong \mp 2\arctan\left[\frac{2U_{\max}}{r_0}\frac{r}{r_0}\right] \cong \mp\frac{4U_{\max}}{r_0}\frac{r}{r_0}, \tag{3.7}$$

$$D_l^u = \frac{1}{f_l^u} = \mp\frac{4}{r_0^2}U_{\max}\frac{1}{(1\pm\delta_l^u)^2} \cong \mp\frac{4}{r_0^2}U_{\max}, \tag{3.8}$$

where the upper and the lower signs refer to the disturbed (upper) and the opposite (lower) surfaces, respectively; $U_{\max}$ and δ_l^u are given by Eqs. (3.2) and (3.3), respectively, with Eqs. (3.4) and (3.5); the first approximation in Eqs. (3.7) and (3.8) holds, if Ineq. (3.6) is satisfied; the second approximation in Eq. (3.7) (the so-called paraxial optics approximation) is valid if (in addition)

$$\left(\frac{2U_{\max}}{r_0}\right)^2\frac{r^2}{r_0^2} \leq \frac{3O^*}{1+O^*} \cong 3O^*. \tag{3.9}$$

The results expressed by Eqs. (3.7) and (3.8) denote, that the mirrors under considerations (both the upper and the lower one, in both the cases a and b) are – within the approximations applied – the ideal (parabolic) ones, i.e. they are free of optical aberrations (their optical power and focal length are independent of r). In principle, no paraxial optics approximation is therefore needed to idealize them.

As it is seen from the formulae given above, the time evolution of the displacement function U and the optical properties of the thermal mirror is governed by the dependence of the functions: N_T (Eqs. (3.4)), $U_{\max}$ (Eqs. (3.2) and (3.5)), and δ (Eqs. (3.3) with (3.4) and (3.5)) – on time. This dependence is complicated and difficult for a simple interpretation. Significant simplification can be obtained for sufficiently short or long time and under the additional condition that the term δ can be neglected in comparison with unity in the suitable formulae.

4. The short-time regime

For sufficiently short time ($\tau \le \tau_{\text{short}}$) and for not very strong heat perturbation ($w_0 \le w_{0,\text{short}}$), the characteristics of the thermal mirrors (Eqs. (3.2), (3.1), (3.7) and (3.8)) can be approximated (with an accuracy to $(1+O^*)^2 - 1 \cong 2O^*$ in case a, and $(1+O^*)^3 - 1 \cong 3O^*$ in case b, where O^* is an assumed small number) by the following formulae (in both cases a and b; cf. Eqs. (7.5) and (7.6) in Part I):

$$U_{\max} = \dot{U}_{\max}(0)\, t\, \varphi(\tau), \tag{4.1}$$

$$\begin{aligned} U^u &= \dot{U}_{\max}(0)\, t \left\{ \frac{1}{3}\left(\frac{2h}{r_0}\right)^2 + \left[1 - \left(\frac{r}{r_0}\right)^2\right] \varphi(\tau) \right\}, \\ U_l &= -\dot{U}_{\max}(0) \left[1 - \left(\frac{r}{r_0}\right)^2\right] t\, \varphi(\tau), \end{aligned} \tag{4.2}$$

$$\varepsilon_l^u = \mp 2 \arctan\left[\frac{2\dot{U}_{\max}(0)}{r_0} \frac{r}{r_0} t\varphi(\tau)\right] \cong \mp 4 \frac{\dot{U}_{\max}(0)}{r_0} \frac{r}{r_0} t\varphi(\tau), \tag{4.3}$$

$$D_l^u = \frac{1}{f_l^u} = \mp \frac{4}{r_0^2} \dot{U}_{\max}(0) t\varphi(\tau), \tag{4.4}$$

where (see Eqs. $(2.1)_{1,4}$):

$$\dot{U}_{\max}(0) = \frac{3r_0^2}{2h} \alpha w_0 \frac{\kappa}{(2h)^2} = 3\alpha \left(\frac{r_0}{2h}\right)^2 \frac{2hq_0}{\varrho_0 c_p} = 3\alpha \left(\frac{r_0}{2h}\right)^2 \frac{\dot{Q}_{\text{tot}}}{\pi r_0^2 \varrho_0 c_p} \tag{4.5}$$

(here, in turn, $\dot{Q}_{\text{tot}}$ stands for the total power applied to the perturbed surface),

$$\varphi(\tau) = 1 - \frac{8}{3\sqrt{\pi}} \sqrt{\tau}, \tag{4.6}$$

and the approximation in Eq. (4.3) (the paraxial optics approximation) holds, if

$$\frac{6\alpha}{(2h)^2} \frac{2hq_0}{\varrho_0 c_p} r t\varphi(\tau) \le \sqrt{\frac{3O^*}{1+O^*}} \cong \sqrt{3O^*}.$$

Thus, the functions: $U_{\max}$, U_l^u, $\tan(\varepsilon_l^u/2)$, D_l^u and f_l^u divided by t are linear functions of $\sqrt{t}$ (see Eq. $(2.1)_1$) in the short-time regime.

The approximate formulae given above are the same in both cases a and b. The main difference depends on different criteria of applicability of the short-time approximation. These criteria will be therefore deduced and specified separately for each case.

4.1. Criteria of applicability of the short-time approximation – Case a

The analysis will be performed according to the following program:

- first, Ineq. (3.6) is assumed to be satisfied;
- second, a possibility of simplification of the function M_T for simplifying the equations for U_l, ε_l^u, and $D_l^u = 1/f_l^u$ will be analyzed, since – after neglecting the functions δ – these quantities depend on time through the function M_T only (see Eqs. $(3.1)_2$, (3.2), (3.5), (3.7) and (3.8));
- third, a possibility of simplification of equation for the function U^u will be examined;
- fourth, consequence of Ineq. (3.6) will be analyzed, with comments on additional condition(-s), which should be taken into account in connection with this point, and also – in connection with the time approximation discussed.

Thus, at the beginning Ineq. (3.6) is assumed to be satisfied.

Concerning the function M_T let us note that for sufficiently short time, the sum in the brackets in Eq. $(3.5a)_1$ can be truncated after the second term. Because $\mathrm{i}^3\mathrm{erfc}(x)$ is a monotonically decreasing function, therefore for $\tau < \frac{9}{64}\pi \cong 0.44$ this sum represents the Leibniz series ([5]). Then, the sum considered can be approximated by the first two terms only, with an accuracy to O^*, if

$$32\sqrt{\tau}\,\mathrm{i}^3\mathrm{erfc}\frac{1}{2\sqrt{\tau}} \le O^*\left(1 - \frac{8}{3\sqrt{\pi}}\sqrt{\tau}\right). \tag{4.7}$$

([5]) The Leibniz series (LS) is understood to be a converging series of the type

$$LS = \sum_{m=0}^{\infty}(-1)^m a_m, \qquad a_m > a_{m+1} > 0.$$

Such a series can be precisely estimated as follows (Leibniz's theorem):

$$\sum_{m=0}^{2n-1}(-1)^m a_m < LS < \sum_{m=0}^{2n}(-1)^m a_m, \qquad \text{and} \qquad \sum_{m=0}^{2n}(-1)^m a_m > LS > \sum_{m=0}^{2n+1}(-1)^m a_m.$$

In particular case one may obtain:

$$a_0 - a_1 + a_2 > LS > a_0 - a_1,$$

therefore $LS \cong a_0 - a_1$ with an accuracy to O^*, if $a_2 \le O^*(a_0 - a_1)$; also

$$a_0 > LS > a_0 - a_1,$$

therefore $LS \cong a_0$ with an accuracy to O^*, if $a_1 \le \frac{O^*}{1+O^*}a_0$.

This inequality is satisfied, if

(4.8)
$$\tau \leq \tau_{\text{short}}^{(a)} = \frac{1}{4x_a^2},$$

where x_a stands for a solution of the equation $\mathrm{i}^3\mathrm{erfc}\, x = O^* \left(\frac{x}{16} - \frac{1}{12\sqrt{\pi}}\right)$ with respect to x.

Assuming, for example
- $O^* = 0.01$

one may find (6)

(4.9)
$$\tau \leq \tau_{\text{short}}^{(a)} \cong 0.11.$$

Assuming, in addition
- $\kappa \doteq (10^{-7} - 10^{-4})\ \mathrm{m}^2/\mathrm{s}$,

where the sign $\doteq$ reads: "is of the order of", and the first value in the brackets refers to the worst temperature conductors and the second one – to the best ones, one may rewrite the criterion expressed by Ineq. (4.9) in the following dimensional form (using Eq. $(2.1)_1$) (7):

$$t \leq t_{\text{short}}^{(a)} \cong \begin{cases} 1\cdot(1 - 10^{-3})\,\mathrm{s}, & \text{for} \quad 2h = 10^{-3}\,\mathrm{m}, \\ 1\cdot(10^2 - 10^{-1})\,\mathrm{s}, & \text{for} \quad 2h = 10^{-2}\,\mathrm{m}. \end{cases}$$

Concerning the function U^u let us note – after substituting Eqs. (3.4a) and $(3.5\mathrm{a})_1$ into Eq. $(3.1)_1$ and using Eq. (4.5) – that this function is also the Leibniz series (for $\tau \leq \frac{9}{64}\pi$) and it can be written in the form:

$$\frac{U^u}{\dot{U}_{\max}(0)t} = \left[\frac{1}{3}A^2 + \left(1 - \overline{r}^2\right)\right] - \left[\left(1 - \overline{r}^2\right)\frac{8\sqrt{\tau}}{3\sqrt{\pi}}\right] + \left[32\sqrt{\tau}\left(1 - \overline{r}^2\right)\mathrm{i}^3\mathrm{erfc}\,\frac{1}{2\sqrt{\tau}}\right] - \dots,$$

where $A = 2h/r_0$, and $\overline{r} = r/r_0$. The right-hand side of this formula can be truncated after the second term (with an accuracy to O^*), if (cf. Ineq. (4.7))

$$32\sqrt{\tau}\left(1 - \overline{r}^2\right)\mathrm{i}^3\mathrm{erfc}\,\frac{1}{2\sqrt{\tau}} \leq O^*\left\{\left[\frac{1}{3}A^2 + \left(1 - \overline{r}^2\right)\right] - \left[\left(1 - \overline{r}^2\right)\frac{8\sqrt{\tau}}{3\sqrt{\pi}}\right]\right\}.$$

This inequality is always satisfied for $\overline{r} = 1$. For $\overline{r} < 1$ it can be rewritten in the form:

$$32\sqrt{\tau}\,\mathrm{i}^3\mathrm{erfc}\,\frac{1}{2\sqrt{\tau}} \leq O^*\left\{\left[1 + \frac{1}{3}\frac{A^2}{1 - \overline{r}^2}\right] - \frac{8\sqrt{\tau}}{3\sqrt{\pi}}\right\}.$$

(6) For $O^* = 0.001$ (0.0001) one can find: $\tau_{\text{short}}^{(a)} \cong 0.071$ (0.050).

(7) For $O^* = 0.001$ (0.0001) the coefficient 1 before the brackets is replaced by 0.7 (0.5).

By comparing this inequality and Ineq. (4.7) it is seen, that it is weaker than Ineq. (4.7). If therefore the criterion expressed by Ineq. (4.7) is satisfied, then also the function U^u can be approximated in the same way as the functions U_l, ε_l^u, and $D_l^u = 1/f_l^u$, as it is expressed by Eqs. (4.2)–(4.4).

Concerning the functions δ_l^u let us note that, according to Eq. (3.3), the behaviour of these functions in the case examined is determined by the functions $N_T^{(a)}$ (Eq. (3.4a)) and $M_T^{(a)}$ (Eq. (3.5a)). As it is seen from Eqs. (3.4a) and $(3.5\text{a})_2$, both the latter functions are positive and are monotonically increasing from 0 at $\tau = 0$ ([8]).

The function $\delta^{(a)u}(\tau)$ is therefore also increasing from 0 as $\tau = 0$ to $\delta^{(a)u}(\tau_{\text{short}}^{(a)})$ in the short-time regime; for $\tau_{\text{short}}^{(a)} \cong 0.11$ ($O^* = 0.01$ – see Eq. (4.9)) one can obtain: $0 \le \delta^{(a)u}(\tau) \le \delta^{(a)u}(\tau_{\text{short}}^{a}) \cong 0.28\alpha w_0$ ([9]). The function $\delta_l^{(a)}(\tau)$ is also positive-valued in the short-time regime specified above, however it is not monotonic. It starts from 0 as $\tau = 0$, and approaches a maximum value about $0.059\alpha w_0$ at $\tau = \tau_{l,m}^{(a)} \cong 0.090$.

Thus, the criterion expressed by Ineq. (3.6) for neglecting the functions δ in the short-time regime may be written in the form:

$$
\begin{aligned}
\delta^{u(a)} &\le \delta^u(\tau_{\text{short}}^{(a)}) \le 0.5\,O^* ,\\
\delta_l^{(a)} &\le \delta_l(\tau^*) \qquad \le 0.5\,O^* ,
\end{aligned}
\tag{4.10}
$$

where $\tau^* = \tau_{l,m}^{(a)} \cong 0.090$, if $\tau_{\text{short}}^{(a)} \ge \tau_{l,m}^{(a)}$, or $\tau^* = \tau_{\text{short}}^{(a)}$, if $\tau_{\text{short}}^{(a)} \le \tau_{l,m}^{(a)}$. If $O^* = 0.01$ (and therefore $\tau_{\text{short}}^{(a)} \cong 0.11$), then ([10])

$$
\begin{aligned}
\delta^{u(a)} &\le \delta^u(\tau_{\text{short}}^{(a)}) \cong 0.28\alpha w_0 \;\; \le 0.5O^*,\\
\delta_l^{(a)} &\le \delta_l(\tau_{l,m}^{(a)}) \quad \cong 0.059\alpha w_0 \le 0.5O^*.
\end{aligned}
\tag{4.11}
$$

Under the assumption:

- $\alpha \doteq 10^{-5}$ 1/K,

this inequalities read ([11]):

$$
w_0 \le \begin{cases} 18\cdot 10^2\text{K} & \text{for the perturbed (the upper) surface,} \\ 85\cdot 10^2\text{K} & \text{for the opposite (the lower) surface.} \end{cases}
$$

([8]) The series in Eq. $(3.5\text{a})_2$ at $\tau = 0$ is equal to unity (see [2]).

([9]) For $\tau_{\text{short}}^{(a)} = 0.071$ (0.050) ($O^* \cong 0.001$ (0.0001)) the coefficient 0.28 is replaced by 0.20 (0.15).

([10]) If $O^* = 0.001$ (0.0001), then the coefficient 0.28 is replaced by 0.20 (0.15), and the coefficient 0.59 – by 0.057 (0.049).

([11]) For $O^* = 0.001$ (0.0001) the coefficient 18 is replaced by 2.5 (0.33), and the coefficient 85 – by 8.8 (1.0).

Using Eq. $(2.1)_4$ and assuming (in addition to the assumptions adopted above)

• $\varrho_0 c_p \doteq 5\cdot 10^6$ J/(m^{3}K),

one may rewrite the inequality given above as the following criterion for the disturbing heat flux density (equivalent to the disturbing heat source):

$$2hq_0 \le \begin{cases} B(10^6 - 10^9)\,\mathrm{W/m^2}, & \text{for} \quad 2h = 10^{-3}\,\mathrm{m}, \\ B(10^5 - 10^8)\,\mathrm{W/m^2}, & \text{for} \quad 2h = 10^{-2}\,\mathrm{m}, \end{cases}$$

where $B \cong 0.9$ for the perturbed (the upper) surface, and $B \cong 4$ for the opposite (the lower) one ([12]).

Concerning the limitation of the heat perturbation, an additional condition should be taken into account. The heat conduction and the mechanical processes are treated in the linear approximation. It is therefore reasonable to require, that the heat perturbation should not significantly change the properties of the material. Let the maximum allowable temperature be T^*. Because the temperature approaches its highest value at the perturbed surface, therefore this requirement may be written in the form $T^{(a)}(z = h) \le T^*$. According to Eqs. $(2.1)_{2,3}$ and $(2.4\mathrm{a})_1$ we have

$$T^{(a)}(z = h, t) = w_0\, \Theta^{(a)}\left(\zeta = \frac{1}{2}, \tau\right) = 2w_0\sqrt{\tau}\left[\frac{1}{\sqrt{\pi}} + 2\sum_{m=1}^{\infty} \operatorname{ierfc}\frac{m}{\sqrt{\tau}}\right].$$

The function $\operatorname{ierfc}(x)$ is a decreasing function of its argument, therefore the temperature of the perturbed surface increases monotonically with time. Under the assumption

• $T^* = 100$ K,

the criterion analyzed may be therefore written in the following approximate form:

$$w_0 \le \frac{\sqrt{\pi}\,T^*}{2\sqrt{\tau^{(a)}_{\mathrm{short}}}} \cong 2.7\cdot 10^2\,\mathrm{K}, \tag{4.12}$$

where $\tau^{(a)}_{\mathrm{short}}$ was assumed to be equal 0.11 ([13]). Using Eq. $(2.1)_4$ one may rewrite this inequality as the following criterion for the disturbing heat flux density (equivalent to the disturbing heat source) ([14]):

$$2hq_0 \le \begin{cases} 0.1\cdot(10^6 - 10^9)\,\mathrm{W/m^2}, & \text{for} \quad 2h = 10^{-3}\,\mathrm{m}, \\ 0.1\cdot(10^5 - 10^8)\,\mathrm{W/m^2}, & \text{for} \quad 2h = 10^{-2}\,\mathrm{m}. \end{cases}$$

([12]) If $O^* = 0.001$ (0.0001), then $B \cong 0.1$ (0.02) or $B \cong 0.4$ (0.05) for the perturbed (the upper) surface or for the opposite (the lower) one, respectively.

([13]) If $\tau^{(a)}_{\mathrm{short}}$ is put to be equal to 0.071 (0.050), then the coefficient 2.7 is replaced by 3.3 (4.0).

([14]) If $\tau^{(a)}_{\mathrm{short}}$ is put to be equal to 0.071 (0.050), then the coefficient 0.1 before the brackets is replaced by 0.2 (0.2).

Comparing these results with inequalities for w_0 and q_0 obtained earlier as a criterion for neglecting the functions δ in the short-time regime, one may see that the former inequalities are weaker than the latter ones for $O^* = 0.01$; if therefore the temperature criterion of linearization is satisfied, then the functions δ may be neglected in the suitable formulae in the short-time regime. The reverse situation takes place for $O^* = 0.0001$.

At the end of this subsection two additional conditions should be mentioned. The first concerns the assumption that all the mechanical phenomena are treated in the quasi-static approximation, i.e. the observation time τ can not be too short: $\tau \geq \tau_{\min}$. The suitable criterion of this kind was proposed and commented in Part I.

The second remark concerns the initial condition. The form of this condition indicates that the time, during which the heat perturbation is switched on, should be sufficiently short in the time scale applied.

4.2. Criteria of applicability of the short-time approximation – Case b

The analysis will be performed according to the same program as in Case a (mentioned at the beginning of Subsec. 4.1).

Thus, at the beginning Ineq. (3.6) is assumed to be satisfied.

Concerning the function M_T let us note that for sufficiently short time, the sum in the brackets in Eq. $(3.5\mathrm{b})_1$ can be truncated after the second term. Because the functions $\mathrm{i}^2\mathrm{erfc}(x)$ and $\mathrm{i}^3\mathrm{erfc}(x)$ are both monotonically decreasing ones, therefore the sum examined can be treated as a Leibniz series([5]) for $\tau \leq \dfrac{9\pi}{64} \cong 0.44$, and it can be approximated by its two first terms only with an accuracy to O^*, if

$$\mathrm{i}^2\mathrm{erfc}\,\frac{1}{2\sqrt{\tau}} + 4\sqrt{\tau}\,\mathrm{i}^3\mathrm{erfc}\,\frac{1}{\sqrt{\tau}} \leq O^* \left(\frac{1}{8} - \frac{\sqrt{\tau}}{3\sqrt{\pi}}\right). \tag{4.13}$$

This inequality is satisfied if

$$\tau \leq \tau^{(b)}_{\mathrm{short}} = \frac{1}{4x_b^2}, \tag{4.14}$$

where x_b stands for a solution of the equation

$$x\,\mathrm{i}^2\mathrm{erfc}\,x + 2\,\mathrm{i}^3\,\mathrm{erfc}\,2x = \frac{O^*}{8}\left(x - \frac{4}{3\sqrt{\pi}}\right)$$

with respect to x.

Assuming, for example, $O^* = 0.01$ one may find ([15]):

$$\tau \leq \tau^{(b)}_{\mathrm{short}} \cong 0.083 \tag{4.15}$$

([15]) For $O^* = 0.001$ (0.0001) one can find $\tau^{(b)}_{\mathrm{short}} \cong 0.053$ (0.038).

(cf. Ineq. (4.9)). Assuming, in addition $\kappa \doteq (10^{-7} - 10^{-4})\,\mathrm{m^2/s}$ (where the first value in brackets refers to the worst temperature conductors and the second one – to the best ones), one may rewrite the criterion expressed by Ineq. (4.15) in the following dimensional form (using Eq. $(2.1)_1$) ([16]):

$$t \le t_{\mathrm{short}}^{(b)} \cong \begin{cases} 0.8\cdot(1-10^{-3})\,\mathrm{s}, & \text{for} \quad 2h = 10^{-3}\,\mathrm{m}, \\ 0.8\cdot(10^{2}-10^{-1})\,\mathrm{s}, & \text{for} \quad 2h = 10^{-2}\,\mathrm{m}. \end{cases}$$

Concerning the function U^u let us note – after substituting Eqs. $(3.4\mathrm{b})_1$ and $(3.5\mathrm{b})_1$ into Eq. $(3.1)_1$ and using Eq. (4.5) – that this function also represents the Leibniz series ([5]) (for $\tau \le \dfrac{9\pi}{64}$), and it can be written in the form:

$$\frac{U^u}{\dot{U}_{\max}(0)\,t} = \left[\frac{1}{3}A^2 + \left(1-\bar{r}^2\right)\right] - \left[\frac{8}{3}A^2 \mathrm{i}^2\mathrm{erfc}\,\frac{1}{2\sqrt{\tau}} + \left(1-\bar{r}^2\right)\frac{8}{3\sqrt{\pi}}\sqrt{\tau}\right]$$
$$+ \left[\frac{8}{3}A^2 \mathrm{i}^2\mathrm{erfc}\,\frac{3}{2\sqrt{\tau}} + 8\left(1-\bar{r}^2\right)\left(\mathrm{i}^2\mathrm{erfc}\,\frac{1}{2\sqrt{\tau}} + 4\sqrt{\tau}\,\mathrm{i}^3\mathrm{erfc}\,\frac{1}{\sqrt{\tau}}\right)\right] - \ldots\,,$$

where $A = 2h/r_0$, and $\bar{r} = r/r_0$. The right-hand side of this formula can be truncated after the second term (with an accuracy to O^*), if

$$\frac{8}{3}A^2\,\mathrm{i}^2\mathrm{erfc}\,\frac{3}{2\sqrt{\tau}} + 8\left(1-\bar{r}^2\right)\left(\mathrm{i}^2\mathrm{erfc}\,\frac{1}{2\sqrt{\tau}} + 4\sqrt{\tau}\;\mathrm{i}^3\mathrm{erfc}\,\frac{1}{\sqrt{\tau}}\right)$$
$$\le O^*\left\{\left[\frac{1}{3}A^2 + \left(1-\bar{r}^2\right)\right] - \left[\frac{8}{3}A^2\,\mathrm{i}^2\mathrm{erfc}\,\frac{1}{2\sqrt{\tau}} + \left(1-\bar{r}^2\right)\frac{8}{3\sqrt{\pi}}\sqrt{\tau}\right]\right\}.$$

This inequality is satisfied if the following inequalities are fulfilled (sufficient conditions):

$$\mathrm{i}^2\mathrm{erfc}\,\frac{3}{2\sqrt{\tau}} + O^*\,\mathrm{i}^2\mathrm{erfc}\,\frac{1}{2\sqrt{\tau}} \le \frac{1}{8}O^*,$$
$$\mathrm{i}^2\mathrm{erfc}\,\frac{1}{2\sqrt{\tau}} + 4\sqrt{\tau}\,\mathrm{i}^3\mathrm{erfc}\,\frac{1}{2\sqrt{\tau}} \le O^*\left(\frac{1}{8} - \frac{\sqrt{\tau}}{3\sqrt{\pi}}\right).$$

The second inequality is identical with Ineq. (4.13). The first inequality is weaker than the second one.

Thus, if Ineq. (4.14) is satisfied, then the function U^u can be approximated by its first two terms (see formula for this function given above). Further approximation depends on neglecting the function $\mathrm{i}^2\mathrm{erfc}\,\frac{1}{2\sqrt{\tau}}$. The maximum possible error of the latter approximation does not exceed O^*, if

$$\frac{8}{3}A^2\,\mathrm{i}^2\mathrm{erfc}\,\frac{1}{2\sqrt{\tau}} \le \frac{O^*}{1+O^*}\left[\frac{1}{3}A^2 + \left(1-\bar{r}^2\right)\left(1 - \frac{8\sqrt{\tau}}{3\sqrt{\pi}}\right)\right].$$

([16]) If $O^* = 0.001$ (0.0001), then the coefficient 0.8 before the brackets is replaced by 0.5 (0.4).

Because the right-hand side of this inequality is a decreasing function of $\bar{\tau}$ (in the short-time regime), therefore it is fulfilled for each $\bar{\tau}$, if

$$\mathrm{i}^2\mathrm{erfc}\,\frac{1}{2\sqrt{\tau}} \leq \frac{1}{8}\,\frac{O^*}{1+O^*}\,.$$

Because the left-hand side of the latter inequality is an increasing function of τ, therefore it is sufficient to satisfy this inequality for $\tau = \tau^{(b)}_{\text{short}}$. In fact, if Ineq. (4.14) is satisfied, then the inequality considered is satisfied too.

Thus, if the criterion expressed by Ineq. (4.15) is fulfilled, then the function U^u can be approximated in the same way as functions U_l, ε^u_l, and $D^u_l = 1/f^u_l$, as it is expressed by Eqs. (4.2)–(4.4).

Concerning the functions δ^u_l let us note – after substituting Eqs. $(3.4\text{b})_2$ and $(3.5\text{b})_2$ into Eq. (3.3) – that both these functions are positive-valued in the full time interval $(0 \leq \tau \leq \infty)$. The function δ^u is monotonically increasing from 0 (at $\tau = 0$) to αw_0 (at $\tau = \infty$), therefore its maximum value in the short-time regime is equal to $\delta^u(\tau^{(b)}_{\text{short}})$. The function δ_l starts from 0 (at $\tau = 0$), approaches the maximum value about 0.062 at $\tau = \tau^{(b)}_{l,m} \cong 0.115$, and next is monotonically decreasing to 0 (at $\tau = \infty$). The criterion expressed by Ineq. (3.6) can be therefore written in the case under consideration in the same form as in case a (Ineq. (4.10)) with only the subscript (a) replaced by the subscript (b), $\tau^{(a)}_{\text{short}}$ – by $\tau^{(b)}_{\text{short}}$ (see Eqs. (4.14) and (4.15)), and $\tau^{(a)}_{l,m}$ – by $\tau^{(b)}_{l,m} \cong 0.115$. If $O^* = 0.01$ (and therefore $\tau^{(b)}_{\text{short}} \cong 0.083$), then ([17]):

$$\begin{aligned} \delta^{u(b)} &\leq \delta^u(\tau^{(b)}_{\text{short}}) \cong 0.23\alpha w_0 \leq 0.5 O^*\,, \\ \delta^{(b)}_l &\leq \delta_l(\tau^{(b)}_{\text{short}}) \cong 0.060\alpha w_0 \leq 0.5 O^*\,. \end{aligned} \tag{4.16}$$

Under assumption $\alpha \doteq 10^{-5}$ 1/K, this inequalities read ([18])

$$w_0 \leq \begin{cases} 22\cdot 10^2\,\text{K} & \text{for the perturbed (the upper) surface,} \\ 83\cdot 10^2\,\text{K} & \text{for the opposite (the lower) surface.} \end{cases} \tag{4.17}$$

Using Eq. $(2.1)_4$ and assuming (in addition to the assumptions adopted above) $\varrho_0 c_p \doteq 5\cdot 10^6$ J/(m^{3}K), one may rewrite the inequality given above as the following criterion for the disturbing heat flux density (equivalent to the disturbing heat source) ([19]):

$$2hq_0 \leq \begin{cases} C(10^6 - 10^9)\,\text{W/m}^2, & \text{for} \quad 2h = 10^{-3}\,\text{m}, \\ C(10^5 - 10^8)\,\text{W/m}^2, & \text{for} \quad 2h = 10^{-2}\,\text{m}, \end{cases}$$

([17]) If $\tau^{(b)}_{\text{short}} \cong 0.053$ (0.038) ($O^* = 0.001$ (0.0001)), then the coefficient 0.23 is replaced by 0.16 (0.12), and the coefficient 0.060 – by 0.051 (0.043).

([18]) For $O^* = 0.001$ (0.0001) the coefficient 22 is replaced by 3.2 (0.42), and the coefficient 83 – by 9.8 (1.2).

([19]) If $O^* = 0.001$ (0.0001), then $C \cong 0.2$ (0.02) for the perturbed (the upper) surface, and $C \cong 0.5$ (0.06) for the opposite (the lower) one, respectively.

where $C \cong 1$ for the perturbed (the upper) surface, and $C \cong 4$ for the opposite (the lower) one.

Concerning the restriction of the heat perturbation, which follows from the requirement for the temperature value not exceeding an assumed value T^*, the criterion has the same general form as in the previous case: $T^{(b)}(z = h) \leq T^*$. According to Eqs. $(2.1)_{2,3}$ and $(2.4\text{b})_1$, we have

$$T^{(b)}(z = h, t) = w_0\, \Theta^{(b)}\left(\zeta = \frac{1}{2}, \tau\right) = 2w_0\sqrt{\tau}\left[\frac{1}{\sqrt{\pi}} - 2\sum_{m=1}^{\infty}(-1)^{m+1}\,\mathrm{ierfc}\,\frac{m}{\sqrt{\tau}}\right].$$

This is an increasing function of time. Under the assumption $T^* = 100\,\mathrm{K}$, the criterion analyzed may be written in the identical approximate form as in case a (Ineq. (4.12) for w_0 and the next inequality for $2hq_0$), with only $\tau^{(a)}_{\text{short}}$ replaced by $\tau^{(b)}_{\text{short}}$, the coefficient 2.7 – by 3.1 (for $\tau^{(b)}_{\text{short}} = 0.083$) ([20]), and the coefficient 0.1 – by 0.15 (for $\tau^{(b)}_{\text{short}} = 0.083$) ([21]). Conclusions on the role of this criterion as compared to the criteria obtained earlier for neglecting the functions δ are the same as in the previous case.

At the end of this subsection we note, that two remarks made at the end of the previous subsection concern also the case considered here.

4.3. Conclusions

Estimations given in the two previous subsections show that the short-time approximation seems to be realistic (except for very thin plates with the best temperature conductors) and offering simple interpretation of the time evolution of the properties of the mirror considered. Comparison of the two cases considered shows, that for sufficiently short time there are no significant differences between both the cases examined, i.e. – the energy losses through the lower surface can be neglected.

5. The long-time regime

For sufficiently long time ($\tau \geq \tau_{\text{long}}$) and for not very strong heat perturbation ($w_0 \leq w_{0,\text{long}}$), the characteristics of the thermal mirrors (Eqs. (3.1), (3.2), (3.7) and (3.8)) can be approximated (with an accuracy to $(1+O^*)^2 - 1 \cong 2O^*$ in both cases a and b, where O^* is an assumed small number) by the following formulae (cases a and b are distinguished by the superscript $(j) = (a), (b)$, respectively; cf. Eqs. (8.3) and (8.4) in Part I):

$$U^{(j)}_{\max} = U^{(j)}_{\max}(\infty)\,\phi^{(j)}_M(\tau), \tag{5.1}$$

([20]) Or by 3.8 (4.5) (for $\tau^{(b)}_{\text{short}} = 0.053\ (0.038)$).

([21]) Or by 0.2 (0.2) (for $\tau^{(b)}_{\text{short}} = 0.053\ (0.038)$).

$$\text{(5.2)}\qquad \begin{aligned} U^{u(j)} &= U^{(j)}_{\max}(\infty)\left\{\left(\frac{2h}{r_0}\right)^2 \phi_N^{(j)}(\tau) + \left[1-\left(\frac{r}{r_0}\right)^2\right]\phi_M^{(j)}(\tau)\right\},\\ U_l^{(j)} &= -U^{(j)}_{\max}(\infty)\left[1-\left(\frac{r}{r_0}\right)^2\right]\phi_M^{(j)}(\tau), \end{aligned}$$

$$\text{(5.3)}\qquad \varepsilon_l^{u(j)} = \mp 2\arctan\left[\frac{2U^{(j)}_{\max}(\infty)}{r_0}\,\frac{r}{r_0}\phi_M^{(j)}(\tau)\right] \cong \mp 4\frac{U^{(j)}_{\max}(\infty)}{r_0}\,\frac{r}{r_0}\phi_M^{(j)}(\tau),$$

$$\text{(5.4)}\qquad D_l^{u(j)} = \frac{1}{f_l^{u(j)}} = \mp\frac{4}{r_0^2}U^{(j)}_{\max}(\infty)\phi_M^{(j)}(\tau),$$

where (see Eqs. $(2.1)_{1,4}$)

$$\text{(5.5)}\qquad U^{(a)}_{\max}(\infty) = \frac{1}{2}U^{(b)}_{\max}(\infty) = \frac{r_0^2}{8h}\alpha w_0 = \alpha\frac{r_0^2}{4}\,\frac{2hq_0}{\kappa\varrho_0 c_p} = \alpha\frac{\dot{Q}_{\text{tot}}}{4\pi\kappa\varrho_0 c_p},$$

$$\text{(5.6a)}\qquad \phi_N^{(a)} = 4\tau, \qquad\qquad \phi_M^{(a)} = 1-\frac{96}{\pi^4}\exp[-\pi^2\tau],$$

$$\text{(5.6b)}\qquad \phi_N^{(b)} = 1-\frac{32}{\pi^3}\exp[-\frac{1}{4}\pi^2\tau], \qquad \phi_M^{(b)} = 1-\frac{96}{\pi^4}(4-\pi)\exp\left[-\frac{1}{4}\pi^2\tau\right],$$

and the approximation in Eq. (5.3) (the paraxial optics approximation) holds, if

$$\frac{2U^{(j)}_{\max}(\infty)}{r_0}\,\frac{r}{r_0}\phi_M^{(j)}(\tau) \le \sqrt{\frac{3O^*}{1+O^*}} \cong \sqrt{3O^*}\,.$$

The approximate formulae given above are different in each case (a and b). Criteria of applicability of the long-time approximation are also different in each case. These criteria will be therefore deduced and specified separately for each case.

5.1. Criteria of applicability of the long-time approximation – Case a

The analysis will be performed according to the same program as in Case a for the short-time approximation (as it was mentioned at the beginning of Subsec. 4.1).

Thus, at the beginning Ineq. (3.6) is assumed to be satisfied.

Concerning the function M_T let us note that for sufficiently long time, the sum in the brackets in Eq. $(3.5a)_2$ can be truncated after the second term. For this purpose it is sufficient to require:

• the third term of the whole sum to be much smaller than the sum of the first and the second terms in the following sense:

$$\frac{96}{\pi^4}\,\frac{1}{81}\exp[-9\pi^2\tau] \le \frac{0.9O^*}{1+O^*}\left(1-\frac{96}{\pi^4}\exp[-\pi^2\tau]\right),$$

• and the n-th term of the sum ($(k+1)$-th term of the series), $n \geq 3$ ($k \geq 2$), to be not larger than 0.1 of the $(n-1)$-th term of the sum (the k-th term of the series):

$$\exp[-8\pi^2 k\tau] \leq 0.1 \left(\frac{2k+1}{2k-1}\right)^4 .$$

The first inequality is satisfied for

$$\tau \geq \tau_{\text{long}}^{(a)} = -\frac{1}{\pi^2} \ln x_a \, , \tag{5.7}$$

where x_a is the suitable solution of the equation $x^9 + 81\frac{0.9O^*}{1+O^*}x - \frac{81}{96}\pi^4\frac{0.9O^*}{1+O^*} = 0$.

The second inequality is the strongest one for $k = 4$, and then it is much weaker than the former one for $O^* \leq 0.01$.

Thus, assuming (as previously) $O^* = 0.01$ one can obtain ([22]):

$$\tau \geq \tau_{\text{long}}^{(a)} \cong 0.021 \, ; \tag{5.8}$$

assuming also (as previously) $\kappa \doteq (10^{-7}\text{–}10^{-4})\ \text{m}^2/\text{s}$, one may rewrite the criterion expressed by Ineq. (5.8) in the following dimensional form (using Eq. $(2.1)_1$) ([23]):

$$t \geq t_{\text{long}}^{(a)} \cong \begin{cases} 0.2\cdot(1 - 10^{-3})\ \text{s}, & \text{for} \quad 2h = 10^{-3}\ \text{m}, \\ 0.2\cdot(10^2 - 10^{-1})\ \text{s}, & \text{for} \quad 2h = 10^{-2}\ \text{m}. \end{cases}$$

Concerning the function U^u let us note – after substituting Eq. (3.4a) and $(3.5\text{a})_2$ into Eq. $(3.1)_1$ and using Eq. (5.5) – that applying analogous criteria for truncation the function U^u after the third term to the criteria used above for truncation of the function $M_T^{(a)}$, we obtain also two inequalities. The first one is always satisfied for $r = r_0$, and it is weaker for $r < r_0$ than the suitable inequality for truncation the function $M_T^{(a)}$. The second inequality is identical with that in the case of the function $M_T^{(a)}$. Thus, if the criteria for truncating the function $M_T^{(a)}$ after the second term are satisfied, than also the suitable sum in the expression for the function U^u can be truncated after the third term.

Concerning the functions δ_l^u in the case under consideration let us note that – according to Eqs. (3.3), (3.4a) and (3.5a) – the function δ^u monotonically increases with time from 0 at $\tau = 0$ to infinity as $\tau \to \infty$, being asymptotically limited as follows: $\delta^u(\tau) \leq \alpha w_0\,((1/4) + \tau)$ – for arbitrary time. Because of exponential dependence on time, it seems to be reasonable to limit the long-time interval by the values: $\tau_{\text{long}}^{(a)} - 10\tau_{\text{long}}^{(a)}$. Then Ineq. (3.6) can be rewritten in the form $\delta^{u(a)}(\tau) \leq \delta^{u(a)}(10\,\tau_{\text{long}}^{(a)}) \leq \frac{1}{2}\, O^*$.

([22]) For $O^* = 0.001$ (0.0001) $\tau_{\text{long}}^{(a)} \cong 0.041$ (0.064).

([23]) For $O^* = 0.001$ (0.0001) the coefficient 0.2 before the brackets is replaced by 0.4 (0.6).

The function δ_l initially increases from 0 at $\tau = 0$ to the maximum value of about $0.059\alpha w_0$ at $\tau = \tau_{l,m}^{(a)} \cong 0.090$, and next monotonically decreases to minus infinity as $\tau \to \infty$ (asymptotically as $\alpha w_0 ((1/4) - \tau)$). Inequality (3.6) can be therefore rewritten in the long-time interval bounded as above in the form (for both functions $\delta_l^{u(a)}$ and $\delta_l^{(a)}$):

$$
\begin{aligned}
\delta^{u(a)} &\le \delta^{u(a)}(10\tau_{\text{long}}^{(a)}) \le 0.5O^*\,,\\
|\delta_l^{(a)}| &\le |\delta_l(\tau^*)| \qquad\quad \le 0.5O^*\,,
\end{aligned}
\tag{5.9}
$$

where $\tau^* = \tau_{l,m}^{(a)} \cong 0.090$, if $\delta_l(\tau_{l,m}^{(a)}) \cong 0.059\alpha w_0 > |\delta_l^{(a)}(10\tau_{\text{long}}^{(a)})|$, or $\tau^* = 10\tau_{\text{long}}^{(a)}$ in the opposite case. If $O^* = 0.01$ (and therefore $\tau_{\text{long}}^{(a)} \cong 0.021$), then approximately (cf. Ineq. (4.11)) ([24])

$$
\begin{aligned}
\delta^{u(a)} &\le 0.44\alpha w_0 \;\;\le 0.5O^*\,,\\
|\delta_l^{(a)}| &\le 0.059\alpha w_0 \le 0.5O^*\,.
\end{aligned}
\tag{5.10}
$$

Under the assumption $\alpha \doteq 10^{-5}\,1/\text{K}$ this inequality reads ([25])

$$
w_0 \le \begin{cases} 11\cdot 10^2\text{K (for the perturbed (the upper) surface)},\\ 85\cdot 10^2\text{K (for the opposite (the lower) surface)}. \end{cases}
$$

Using Eq. $(2.1)_4$ and assuming: $\varrho_0 c_p \doteq 5\cdot 10^6\ \text{J}/(\text{m}^3\text{K})$, one may rewrite this inequality as the following criterion for the disturbing heat flux density (equivalent to the disturbing heat source):

$$
2hq_0 \le \begin{cases} D(10^6 - 10^9)\,\text{W}/\text{m}^2, & \text{for} \quad 2h = 10^{-3}\,\text{m},\\ D(10^5 - 10^8)\,\text{W}/\text{m}^2, & \text{for} \quad 2h = 10^{-2}\,\text{m}, \end{cases}
$$

where $D \cong 0.6$ for the perturbed (the upper) surface, and $D \cong 4$ for the opposite (the lower) one ([26]).

As far as the limitation of the heat perturbation in the long-time regime is concerned, two additional conditions should be taken into account. The first such condition follows from the requirement, that the temperature cannot exceed a certain value T^*, as it was mentioned in Subsecs. 4.1 and 4.2. The second condition follows from the requirement that τ cannot be too large, thus allowing

([24]) If $O^* = 0.001$ (0.0001) $(\tau_{\text{long}}^{(a)} \cong 0.041$ $(0.064))$, then the coefficient 0.44 is replaced by 0.66 (0.89), and the coefficient 0.059 – by 0.17 (0.39).

([25]) For $O^* = 0.001$ (0.0001) the coefficient 11 is replaced by 0.76 (0.056), and the coefficient 85 – by 3.0 (0.13).

([26]) If $O^* = 0.001$ (0.0001), then $D \cong 0.04$ (0.003) for the perturbed (the upper) surface, and $D \cong 0.15$ (0.006) for the opposite (the lower) one, respectively.

to neglect thermal losses through the surfaces (first of all – the radiation losses), as it was mentioned in Part I.

The discussion of the first condition, performed in an analogous way as it was done in Subsecs. 4.1 and 4.2 taking into account the previously adopted limits of the long-time regime interval, leads to following approximate criterion:

$$w_0 \leq \frac{T^*}{\frac{1}{4} + 10\tau_{\text{long}}^{(a)} - \frac{2}{\pi^2}\exp[-\pi^2 10\tau_{\text{long}}^{(a)}]} \cong \frac{T^*}{\frac{1}{4} + 10\tau_{\text{long}}^{(a)}} \cong 2.2\cdot 10^2\,\text{K}, \tag{5.11}$$

where T^* was assumed to be equal to 100 K, and $\tau_{\text{long}}^{(a)}$ – to be equal to 0.021 ([27]).

Using Eq. $(2.1)_4$ one may rewrite this inequality as the following criterion for the disturbing heat flux density (equivalent to the disturbing heat source) ([28]):

$$2hq_0 \leq \begin{cases} 0.1\cdot(10^6 - 10^9)\,\text{W/m}^2, & \text{for} \quad 2h = 10^{-3}\,\text{m}, \\ 0.1\cdot(10^5 - 10^8)\,\text{W/m}^2, & \text{for} \quad 2h = 10^{-2}\,\text{m}. \end{cases}$$

Comparing these results with inequalities for w_0 and q_0 obtained earlier as a criterion for neglecting the functions δ in the long-time regime, one may see that the former inequalities are weaker than the latter ones for $O^* = 0.01$; if therefore the temperature criterion of linearization is satisfied, then the functions δ may be neglected in the suitable formulae in the long-time regime. The reverse situation takes place for $O^* = 0.0001$.

The second condition mentioned above concerns the assumption concerning the adiabatic insulation of the plate. In fact, the plate loses its energy at least by thermal radiation through the perturbed and the opposite surfaces. These radiation losses can be neglected, if τ is not too large. The discussion of this problem, performed in an analogous way as it was done in Part I (Sec. 8) ([29]),

([27]) If $\tau_{\text{long}}^{(a)}$ is put to be equal to 0.041 (0.064), then the coefficient 2.2 is replaced by 1.5 (1.1).

([28]) If $\tau_{\text{long}}^{(a)}$ is put to be equal to 0.041 (0.064), then the coefficient 0.1 before the brackets is replaced by 0.07 (0.05).

([29]) For the problem:

$$\frac{\partial\Theta}{\partial\tau} = \frac{\partial^2\Theta}{\partial\zeta^2}, \qquad \Theta(\tau=0)=0, \qquad \frac{\partial\Theta}{\partial\zeta}\left(\zeta=\frac{1}{2}\right) = H(\tau-0) - \beta\Theta\left(\zeta=\frac{1}{2}\right),$$

$$\frac{\partial\Theta}{\partial\zeta}\left(\zeta=-\frac{1}{2}\right) = \beta\Theta\left(\zeta=-\frac{1}{2}\right),$$

where

$$\beta = \frac{2h}{\kappa}\,\frac{4b\sigma_{SB}T_0^3}{\varrho_0 c_p}$$

stands for dimensionless coefficient of radiation losses as obtained from the linearized Stefan–Boltzmann law (b stands here for a correction factor for a real body as compared with the perfectly black one, σ_{SB} – for the Stefan–Boltzmann constant, and T_0 – for the absolute initial temperature (before the perturbation)).

leads to the following criterion of neglecting the radiation losses through the surfaces [30]:

$$\tau \leq \tau_{\max} := O^* \tau_{\mathrm{rad}} := \frac{O^*}{4\beta} = O^* \frac{\kappa}{2h} \frac{\varrho_0 c_p}{16 b \sigma_{SB} T_0^3},$$

where β and the remaining symbols used here are defined in footnote 29. Assuming $O^* = 0.01$, $\kappa \doteq (10^{-7} - 10^{-4})\,\mathrm{m/s^2}$, $\varrho_0 c_p \doteq 5\cdot 10^6\,\mathrm{J/(m^3K)}$, and

- $b = 0.1$,
- $\sigma_{SB} \cong 5.67\cdot 10^{-8}\,\mathrm{J/(m^2sK^4)}$,
- $T_0 = 300\,\mathrm{K}$,

we have (in dimensionless and in dimensional forms) [31]:

$$\tau \leq \tau_{\max} \doteq \begin{cases} 2\cdot(1 - 10^3), & \text{for} \quad 2h = 10^{-3}\,\mathrm{m}, \\ 2\cdot(10^{-1} - 10^2), & \text{for} \quad 2h = 10^{-2}\,\mathrm{m}, \end{cases}$$
$$t \leq t_{\max} \doteq \begin{cases} 2\cdot 10\,\mathrm{s}, & \text{for} \quad 2h = 10^{-3}\,\mathrm{m}, \\ 2\cdot 10^2\,\mathrm{s}, & \text{for} \quad 2h = 10^{-2}\,\mathrm{m}. \end{cases} \tag{5.12}$$

Comparing this result and Ineq. (5.8) one can see, that the criterion expressed by Ineq. (5.12) can significantly restrict the applicability of the long-time approximation in the case under consideration, especially if high accuracy of the approximation is required for thin plates and bad temperature conductors.

At the end of this subsection we note, that two remarks done at the end of Subsec. 4.1 concern the case considered here too, although in the long-time regime they are not so important as previously.

5.2. Criteria of applicability of the long-time approximation – Case b

The analysis will be performed according to the same programe as in Case a for the short-time approximation (as it was mentioned at the beginning of Subsec. 4.1).

Thus, at the beginning Ineq. (3.6) is assumed to be satisfied.

In connection with the function M_T let us note that, for sufficiently long time, the sum in the brackets in Eq. $(3.5\mathrm{b})_2$ can be truncated after the second term. In fact, after merging the unity and the first term of the series one obtains a new series, which can be treated as the Leibniz series. Then, the sum under consideration can be truncated after the second term with an error not exceeding O^*, if [5]

$$\tau \geq \tau_{\mathrm{long}}^{(b)} = -\frac{4}{\pi^2} \ln x_b, \tag{5.13}$$

[30] Note that in Part I there is a numerical mistake: number 8 in the formulae for τ_{rad}, $\tau_{\max}$ and $t_{\max}$ should be replaced by the number 16, and number 4 in Eqs. (8.5) – by the number 2.

[31] For $O^* = 0.001$ (0.0001) the coefficient 2 is replaced by 0.2 (0.02).

where x_b is the suitable solution of the equation

$$\frac{1+O^*}{O^*}x^9 + 81\frac{4-\pi}{3\pi+4}x - \frac{81}{3\pi+4}\frac{\pi^4}{96} = 0.$$

Assuming (as previously) $O^* = 0.01$ one can obtain (32)

$$\tau \geq \tau_{\text{long}}^{(b)} \cong 0.16\,; \tag{5.14}$$

assuming also (as previously) $\kappa \doteq (10^{-7} - 10^{-4})\,\text{m}^2/\text{s}$, one may rewrite the criterion exepressed by Ineq. (5.14) in the following dimensional form (using Eq. (2.1)) (33):

$$t \geq t_{\text{long}}^{(b)} \cong \begin{cases} 2\cdot(1-10^{-3})\,\text{s}, & \text{for} \quad 2h = 10^{-3}\,\text{m}, \\ 2\cdot(10^2 - 10^{-1})\,\text{s}, & \text{for} \quad 2h = 10^{-2}\,\text{m}. \end{cases}$$

Concerning the function U^u let us note, that the sum in the brackets in Eq. $(3.4\text{b})_2$ can be written in the form:

$$S_N = 1 - a_1 + a_2 - \dots\,,$$

where a_k stands for the absolute value of the k-th term of the series in these brackets, and the sum in Eq. $(3.5\text{b})_2$ – in the form:

$$S_M = [1 - b_1] - b_2 + \dots\,,$$

where b_k stands for the absolute value of the k-th term of the series in these brackets, and the term $[1 - b_1]$ is treated as the first term of the series S_M. The function $U^{u(b)}$ can therefore be written in the form (after neglecting the function δ):

$$U^{u(b)} = U_{\max}^{(b)}(\infty)S, \qquad S = A^2 S_N + (1 - \overline{r}^2)S_M\,,$$

where $A = 2h/r_0$, and $\overline{r} = r/r_0$.

Both series S_N and S_M represent the Leibniz series (5) in the long-time regime specified above. They are limited as follows:

$$1 - a_1 \leq S_N \leq 1 - a_1 + a_2\,,$$
$$[1 - b_1] - b_2 \leq S_M \leq [1 - b_1],$$

and therefore the series S may be estimated as follows:

$$A^2(1-a_1) + (1-\overline{r}^2)([1-b_1] - b_2) \leq S \leq A^2(1 - a_1 + a_2) + (1-\overline{r}^2)[1-b_1].$$

(32) If $O^* = 0.001$ (0.0001), then $\tau_{\text{long}}^{(b)} \cong 0.26$ (0.35).

(33) If $O^* = 0.001$ (0.0001), then the coefficient 2 before the brackets is replaced by 3 (4).

The series S_M is approximated (with an accuracy to O^*) by its first term $S_M \cong [1-b_1]$, which specifies the long-time regime and requires b_2 to satisfy the condition:

$$b_2 \leq \frac{O^*}{1+O^*}[1-b_1].$$

The series S_N is approximated (with an accuracy to O_1^*) by its first two terms $S_N \cong 1-a_1$, which requires a_2 to satisfy the condition $a_2 \leq O_1^*(1-a_1)$; one may verify that $O_1^* < O^*$ in the long-time regime specified above, therefore the condition for a_2 can be written in the form:

$$a_2 < O^*(1-a_1).$$

The series S can be approximated as follows:

$$S \cong A^2(1-a_1) + (1-\overline{r}^2)(1-b_1)$$

with the maximum possible (relative) error not exceeding

$$O^*_{\rm tot} = \frac{1}{S}[A^2 a_2 + (1-\overline{r}^2)\, b_2];$$

one can prove, using inequalities for S, a_2, b_2 given above, that

$$O^*_{\rm tot} < O^*$$

in the long-time regime.

Thus, if the functions: $U_l^{(b)}$, $\varepsilon_l^{u(b)}$, $D_l^{u(b)}$, $f_l^{u(b)}$ can be approximated (with an accuracy to O^*) as it is expressed by Eqs. (5.1), $(5.2)_2$, (5.3), (5.4), then also the function $U^{u(b)}$ can be approximated as it is expressed by Eq. $(5.2)_1$ (with an accuracy not worse than O^*).

In turn, because of the time evolution of the functions δ, as it was mentioned in Subsec. 4.2, Ineq. (3.6) in the case examined in the long-time regime specified above may be written in the form analogous to Ineq. (4.10), with the superscript (a) replaced by (b), $\tau_{\rm short}^{(a)}$ – by $10\,\tau_{\rm long}^{(b)}$, and τ^* – by $\tau_{\rm long}^{(b)}$. If $O^* = 0.01$ (and therefore $\tau_{\rm long}^{(b)} \cong 0.16$), then [34]

$$(5.15) \qquad \begin{aligned} \delta^{u(b)} &\leq \delta^u(10\tau_{\rm long}^{(b)}) \cong 0.98\alpha w_0 \ \ \leq 0.5 O^*\,, \\ \delta_l^{(b)} &\leq \delta_l(\tau_{\rm long}^{(b)}) \quad\ \cong 0.059\alpha w_0 \leq 0.5 O^*\,. \end{aligned}$$

Under the assumptions $O^* = 0.01$, and $\alpha \doteq 10^{-5}\,1/{\rm K}$, these inequalities read [35]:

$$w_0 \leq \begin{cases} 5.0\cdot 10^2\,{\rm K}, & \text{for the perturbed (the upper) surface,} \\ 84\cdot 10^2\,{\rm K}, & \text{for the opposite (the lower) surface.} \end{cases}$$

---

(34) For $O^* = 0.001$ (0.0001) ($\tau_{\rm long}^{(b)} \cong 0.26$ (0.35)) the coefficient 0.98 is replaced by 1.0 (1.0), and the coefficient 0.059 – by 0.049 (0.039).

(35) For $O^* = 0.001$ (0.0001) the coefficient 5.0 is replaced by 0.50 (0.050), and the coefficient 84 – by 10 (1.3).

Using Eq. $(2.1)_4$ and assuming $\varrho_0 c_p \doteq 5\cdot 10^6\,\mathrm{J/(m^3K)}$, one may rewrite the inequality given above as the following criterion for the disturbing heat flux density (equivalent to the disturbing heat source):

$$2hq_0 \leq \begin{cases} E(10^6 - 10^9)\,\mathrm{W/m^2}, & \text{for} \quad 2h = 10^{-3}\,\mathrm{m}, \\ E(10^5 - 10^8)\,\mathrm{W/m^2}, & \text{for} \quad 2h = 10^{-2}\,\mathrm{m}, \end{cases}$$

where $E \cong 0.25$ for the perturbed (the upper) surface, and $E \cong 4$ for the opposite (the lower) one ([36]).

As far as the limitation of the heat perturbation is concerned in the long-time regime, two additional conditions mentioned in Subsec. 5.1 should be taken into account in the case under consideration. The first such condition (following from the requirement that the temperature can not exceed a certain value T^*) examined in an analogous way as it was done in Subsecs. 4.1, 4.2 and 5.1 along with the fact that the temperature of the perturbed (the upper) surface is a limited function of time (as it is seen in Eq. $(2.4\mathrm{b})_2$ for $\zeta = 1/2$ ([37])), leads to the following approximate criterion:

$$w_0 \leq \frac{T^*}{1 - \dfrac{8}{\pi^2}\exp\left[-\dfrac{\pi^2}{4}10\tau^{(b)}_{\mathrm{long}}\right]} \cong 1.0\cdot 10^2\,\mathrm{K}, \tag{5.16}$$

where T^* was assumed to be equal to 100 K, and $\tau^{(b)}_{\mathrm{long}}$ to be equal to 0.16 (or more). Using Eq. $(2.1)_4$ one may rewrite this inequality as the criterion for the disturbing heat flux density (equivalent to the disturbing heat source), obtaining the same result as expressed by the inequality following Ineq. (5.11) with the coefficient 0.1 replaced by 0.05. Comparison of these results with inequalities for w_0 and q_0 obtained earlier as the criterion for neglecting the functions δ in the long-time regime, leads to the same conclusions as those in the previous case (see Subsec. 5.1).

The second condition concerning the limitation of the heat perturbation in the long-time regime represents the criterion for neglecting the radiation losses in the case considered. The discussion of this problem, performed in an analogous way as it was done in Part I (Sec. 8) ([38]), leads to the criterion which is weaker

([36]) If $O^* = 0.001$ (0.0001), then $E \cong 0.025$ (0.0025) or $E \cong 0.5$ (0.06) for the perturbed (the upper) surface or for the opposite (the lower) one, respectively.

([37]) For the value of the number series occurring here – see [2].

([38]) For the problem:

$$\frac{\partial\Theta}{\partial\tau} = \frac{\partial^2\Theta}{\partial\zeta^2}, \qquad \Theta(\tau = 0) = 0,$$

$$\frac{\partial\Theta}{\partial\zeta}\left(\zeta = \frac{1}{2}\right) = H(\tau - 0) - \beta\,\Theta\left(\zeta = \frac{1}{2}\right), \qquad \Theta\left(\zeta = -\frac{1}{2}\right) = 0,$$

where β is explained in footnote 29.

(by the factor of 2) than the analogous criterion in case a (i.e. the numbers 4 and 16 in the denominators in the inequality precedeing Ineqs. (5.12) are replaced by the numbers 2 and 8, respectively, and the coefficient 2 in Ineqs. (5.12) – by 4). The conclusion concerning the role of this criterion in limiting the applicablity of the long-time approximation is also the same as that applied to the previous case.

At the end of this subsection we note, that two remarks made at the end of Subsec. 4.1 concern the case considered here too, although in the long-time regime they are not so important as in the case of the short-time regime.

5.3. Conclusions

Estimates given in the two previous subsections show that the long-time regime seems to be a realistic and useful (except the cases of thin plates with bad temperature conductors, when higher accuracy is required, especially in Case b – see and compare estimates of $\tau_{\max}$ in Cases a and b). It starts relatively quickly (especially in Case a). The function $U_{\max}$ increases significantly with time (at least twice when τ increases from τ_{long} to $10\,\tau_{\text{long}}$ – in both the cases). By comparing Ineqs. (4.9) and (5.8) one may see that in Case a for $O^* = 0.01$ and 0.001, both regimes – the short- and the long-time ones – cover the full time range from $\tau_{\min}$ to $\tau_{\max}$; for $O^* = 0.0001$ the gap between the two regimes is relatively small. Comparison of Ineqs. (4.15) and (5.14) shows that in Case b situation is not so comfortable – the gap between the two regimes is quite large (the smaller O^*, the larger the gap).

6. Estimates for possible experiments

The thermal mirror considered may be experimentally studied by investigating at least one of the funtions $F = \{U, \varepsilon, D = 1/f\}$. Each of these functions can be experimentally investigated and interpreted using the theoretical scheme presented, if some conditions are fulfilled. Almost all such general conditions were mentioned in the previous sections. Here the last such condition will be noticed and shortly discussed. This condition is rather obvious: each of the functions F can be observable in a given time τ and at least on an assumed level $F^* = \{U^*, \epsilon^*, D^* = 1/f^*\}$, i.e. $|F| \geq F^*$, if the heat perturbation is sufficiently strong.

The observability conditions for the functions F will be examined in terms of the quantities (see Eqs. (3.1), (3.7), (3.8), (2.1) and (3.2) with Eqs. (3.5)): $U_{\max}$ (the functions U), $4U_{\max}/r_0$ (the functions ε), and $4U_{\max}/r_0^2$ (the functions $D = 1/f$). The heat perturbation level will be characterized alternatively by one of the quantities $G^F = \{(w_0)^F, (2hq_0)^F, (\dot{Q}_{\text{tot}})^F, (Q_{\text{tot}})^F)\}$ for each of the functions F, where: $2hq_0 = \kappa\varrho_0 c_p/(2h)$ (see Eq. $(2.1)_4$) stands for the perturbing heat flux density (equivalent to the perturbing heat source) applied to the perturbed surface, $\dot{Q}_{\text{tot}} = \pi r_0^2 2hq_0$ (see Eq. (4.5)) – for the total perturbing power applied to the perturbed surface, and $Q_{\text{tot}} = \dot{Q}_{\text{tot}}\, t$ – for the total perturbing energy applied to the perturbed surface during the time $t = \tau\,(2h)^2/\kappa$.

The observability conditions considered will be formulated separately in the short- and long-time regimes, and – also separately – in Cases a and b. Thus, using Eqs. (4.1) with Eq. (4.6) and Eqs. (5.1) with Eqs. $(5.6)_2$, one can write down the inequalities: $|F| \geq F^*$ in the form of 12 inequalities (3 functions F in 2 time regimes in two cases) in 4 alternative versions (for 4 quantities G). All these 48 inequalities can be written in the following form:

$$
G^F = \begin{cases} \left.\begin{cases} (w_0)^F \\ (2hq_0)^F \\ (\dot{Q}_{\text{tot}})^F \end{cases}\right\} \geq F^*\psi(\tau) S^F_{G'}\,, \\ (Q_{\text{tot}})^F \qquad \geq F^*\psi(\tau) S^F_{Q_{\text{tot}}}\tau, \end{cases} \tag{6.1}
$$

where

$$
\psi(\tau) = \begin{cases} [\tau\varphi(\tau)]^{-1} \quad \text{(in Cases a and b)} \quad \text{– in the short-time regime,} \\ \left.\begin{cases} 12\left[\phi_M^{(a)}(\tau)\right]^{-1} & \text{(in Case a)} \\ 6\left[\phi_M^{(b)}(\tau)\right]^{-1} & \text{(in Case b)} \end{cases}\right\} \text{– in the long-time regime} \end{cases} \tag{6.2}
$$

(the functions: $\varphi(\tau)$ and $\phi_M^{(j)}(\tau)$ are defined by Eqs. (4.6) and $(5.6)_2$, respectively), and

$$
\left\{ \begin{aligned} S^U_{w_0} &= \frac{2h}{3\alpha r_0^2}\,, \\ S^U_{2hq_0} &= \frac{\kappa\varrho_0 c_p}{3\alpha r_0^2}\,, \\ S^U_{\dot{Q}_{\text{tot}}} &= \frac{\pi\kappa\varrho_0 c_p}{3\alpha}\,, \\ S^U_{Q_{\text{tot}}} &= \frac{(2h)^2\pi\varrho_0 c_p}{3\alpha}\,, \end{aligned} \qquad S^\varepsilon_G = \frac{r_0}{4} S^U_G\,, \qquad S^{1/f}_G = \frac{r_0^2}{4} S^U_G \right\}. \tag{6.3}
$$

For the numerical exemplification of the criteria expressed by Ineqs. (6.1), it is assumed that

$$
\tau = \tau_{\text{short}}\,.
$$

This assumption gives sufficient conditions for $|F| \geq F^*$ at the end of the short-time regime and at least in a part of the long-time regime in Case a or in the whole this regime in Case b [39]. Now, assuming (as previously): $\alpha \doteq 10^{-5}$ 1/K; $\kappa \doteq (10^{-7} - 10^{-4})\,\mathrm{m}^2/\mathrm{s}$, where the first value in the brackets concerns the worst and the second one – the best temperature conductors; $\varrho_0 c_p \doteq 5\cdot 10^6\,\mathrm{J/(m^3K)}$; $r_0 \doteq 10\cdot 2h$; and

- $U^* \doteq 10^{-6}$ m;
- $\varepsilon^* \doteq 10^{-4}$ rad;
- $f^* \doteq 40$ m;
- $\tau^{(a)}_{\mathrm{short}} = 0.11$, $\tau^{(b)}_{\mathrm{short}} = 0.083$ (which corresponds to $O^* \cong 0.01$ [39]);

one may obtain the minimum perturbation power (or energy) as it is given in the table in Case a (the minimum values of w_0, $2hq_0$ and $\dot{Q}_{\mathrm{tot}}$ are by a dozen percent larger, and that value of Q_{tot} is by a dozen percent smaller in Case b) [40].

	for $U_{\max} \geq U^*$	for $\dfrac{4U_{\max}}{r_0} \geq \epsilon^*$	for $\dfrac{4U^{\max}}{r_0^2} \geq \dfrac{1}{f^*}$	for $2h \doteq$
$\dfrac{w_0}{\mathrm{K}} \geq \dfrac{w_{0,\min}}{\mathrm{K}} \doteq$	$\begin{cases} 6 \\ 0.6 \end{cases}$	$\begin{cases} 1.5 \\ 1.5 \end{cases}$	$\begin{cases} 4 \\ 40 \end{cases}$	10^{-3} m 10^{-2} m
$\dfrac{2hqo}{\mathrm{W/m^2}} \geq \dfrac{2hq_{0,\min}}{\mathrm{W/m^2}} \doteq$	$\begin{cases} 3\cdot(10^3 - 10^6) \\ 3\cdot(10 - 10^4) \end{cases}$	$\begin{cases} 0.8\cdot(10^3 - 10^6) \\ 8\cdot(10 - 10^4) \end{cases}$	$\begin{cases} 2\cdot(10^3 - 10^6) \\ 200\cdot(10 - 10^4) \end{cases}$	10^{-3} m 10^{-2} m
$\dfrac{\dot{Q}_{\mathrm{tot}}}{\mathrm{W}} \geq \dfrac{\dot{Q}_{\mathrm{tot,min}}}{\mathrm{W}} \doteq$	$\begin{cases} 1\cdot(1 - 10^3) \\ 1\cdot(1 - 10^3) \end{cases}$	$\begin{cases} 0.25\cdot(1 - 10^3) \\ 2.5\cdot(1 - 10^3) \end{cases}$	$\begin{cases} 0.6\cdot(1 - 10^3) \\ 60\cdot(1 - 10^3) \end{cases}$	10^{-3} m 10^{-2} m
$\dfrac{Q_{\mathrm{tot}}}{\mathrm{J}} \geq \dfrac{Q_{\mathrm{tot,min}}}{\mathrm{J}} \doteq$	$\begin{cases} 1 \\ 10^2 \end{cases}$	$\begin{cases} 0.25 \\ 2.5\cdot 10^2 \end{cases}$	$\begin{cases} 0.6 \\ 60\cdot 10^2 \end{cases}$	10^{-3} m 10^{-2} m

Comparison of the values of $Q_{\mathrm{tot,min}}$ given above with analogous values given in Part I (Eqs. $(9.3)_1$, $(9.4)_1$, $(9.5)_1$) shows, that the observability conditions (in terms of the total perturbation energy) in the cases considered here are nearly the same as those in the case considered in Part I. However, the continuous heat flux perturbation may be more convenient for experimental organization than the pulsed one.

[39] Note that: $U_{\max}$ is a monotonically increasing function (and $\psi(\tau)$ is a decreasing function) of time; and τ_{short} decreases and τ_{long} inreases as the quantity O^* decreases (see also the remark on the relation between both the regimes given in Subsec. 5.3).

[40] Let us note that $w_{0,\min}$, $2hq_{0,\min}$ and $\dot{Q}_{\mathrm{tot,min}}$ decrease with τ, increase as O^* decreases in the short-time regime and decrease as O^* decreases in the long-time regime, and $Q_{\mathrm{tot,min}}$ – inversely. The observability conditions in terms of $w_{0,\min}$, $2hq_{0,\min}$ and $\dot{Q}_{\mathrm{tot,min}}$ become therefore stronger (weaker), and those in terms of $Q_{\mathrm{tot,min}}$ become weaker (stronger) as τ decreases (increases, resp.).

7. Possible applications to determination of the temperature conductivity (and the surface losses coefficients)

As it is seen from the suitable formulae given above (after returning back to dimensional time $t = \tau\,(2h)^2/\kappa$), the time evolution of the thermal mirror depends, among others, on the temperature conductivity κ of the material. Measuring suitable properties of the mirror it is therefore possible to determine κ. However, as it is seen from the formulae mentioned, such a procedure performed in an arbitrary conditions may require some additional information (which should be known or measured), and may prove to be complicated for interpretation.

The problem simplifies in the short-time and the long-time regimes. In fact, as it follows from Eqs. (4.2), (4.3), (4.4) and (4.6), in the short-time regime the quantities $[U^u(r=0) - U^u(r)]/t$, U_l/t, $[tan(\varepsilon_l^u/2)]/t$ and $D_l^u/t = 1/(f_l^u\,t)$, as referred to their values at $t = 0$ (which may be determined by an extrapolation of the suitable experimental data to $t = 0$) are linear functions of $\sqrt{t}$ with the coefficient (at $\sqrt{t}$) equal to $4\sqrt{\kappa}/(2h\sqrt{\pi})$. Measuring the evolution of these quantities one may therefore determine this coefficient and, knowing it and the plate thickness $2h$ of the plate – find the values of κ of a given material.

Analogously, as it follows from Eqs. (5.2), (5.3), (5.4) and (5.6), logarithms of the absolute values of the time derivatives of the quantities $U^u(r=0) - U^u(r)$, U_l, $tan(\varepsilon_l^u/2)$ and $D_l^u = 1/f_l^u$ in the long-time regime are linear functions of t with the coefficient (at t) equal to $\pi^2\kappa/(2h)^2$ (in Case a) or $\pi^2\kappa/[4(2h)^2]$ (in Case b). Measuring the evolution of these quantities one may therefore determine this coefficient and, knowing it and the plate thickness $2h$ – determine κ of a given material.

Additionally let us briefly note that one may think also on applying the thermal mirror considered to experimental determination of the surface losses, if the temperature conductivity of a given material is known. In such a case the thermal problem is formulated as follows:

$$\frac{\partial\Theta}{\partial\tau} = \frac{\partial^2\Theta}{\partial\zeta^2}, \qquad \Theta(\tau=0), \qquad \frac{\partial\Theta}{\partial\zeta}\left(\zeta=\frac{1}{2}\right) = H(\tau-0) - \beta_1\,\Theta\left(\zeta=\frac{1}{2}\right),$$

$$\begin{cases} \dfrac{\partial\Theta}{\partial\zeta}\left(\zeta=-\dfrac{1}{2}\right) = \beta_2\,\Theta\left(\zeta=-\frac{1}{2}\right) & \text{in Case a,} \\ \Theta\left(\zeta=-\frac{1}{2}\right) = 0 & \text{in Case b.} \end{cases}$$

Here $H(\tau=0)$ stands for the Heaviside step function, and β – for the dimensionless surface losses coefficients (in particular case, when the plate loses its energy through its main surfaces by heat radiation only, $\beta_1 = \beta_2 = \beta$, where, in turn, β is defined in Footnote 29). Solving this problem (by applying the Fourier method of separating the independent variables) and calculating the optical characteristics by means of the general formulae given earlier, one may conclude that for a sufficiently long time, logarithms of the absolute values of the time derivatives of the quantities $U^u(r = 0) - U^u(r)$, U_l, $\tan(\varepsilon_l^u/2)$ and $D_l^u = 1/f_l^u$ are linear functions of t with the coefficient (at t) equal to $\mu_1^2\,\kappa/(2h)^2$, where μ_1 is the first positive solution of the following characteristic equation: $\tan\mu = \mu\,(\beta_1 + \beta_2)/(\mu^2 - \beta_1\,\beta_2)$ (in case a) or: $\tan\mu = -\mu/\beta_1$ (in case b). From measurements of the time evolution of the quantities mentioned one may therefore determine the quantity μ_1. Then from the characteristic equation for μ one may determine $\beta_2 = \mu_1\,\tan\mu_1$, if $\beta_1 = 0$ (an ideal thermal insulation on the perturbed surface), or $\beta_1 = \mu_1\,\tan\mu_1$, if $\beta_2 = 0$ (an ideal thermal insulation on the opposite surface), or $\beta_1 = -\mu_1/\tan\mu_1$, if $\beta_2 = \infty$ (ideal losses on the opposite surface, realized for instance by a thermostate, as it was assumed in case b).

8. Final remarks

Remark on the criteria for neglecting the distortion of properties of optical mirrors due to absorption of light, as well as general conclusions are analogous to those presented in Part I.

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