Arch. Mech., 50, 1, pp. 91–112, Warszawa 1998
FIFTY YEARS OF THE ARCHIVES OF MECHANICS

Plastic propagation of band-waves in solids

K.C. VALANIS (VANCOUVER)

HEREWITH WE PRESENT a thermodynamic theory of "plastic propagation" of strain bands. The theory results in a *Wave equation* where the time of propagation is *not* the Newtonian time t but *the intrinsic time* z (in the sense of Valanis), the increment of which is proportional to the norm of the increment of the plastic strain tensor. To our knowledge the concept of plastic propagation is new in theoretical mechanics. The equation is solved exactly for the case of a flat semi-infinite strip under axial tension. The solution depicts the propagation of bands, upon plastic extension of the strip, in the axial direction, very much in accordance with observation.

1. Introduction

NUMEROUS INVESTIGATORS, see for instance LUBAHN and FELGAR [1], have reported the formation of "adiabatic" bands, commonly near the loaded boundary of an axial specimen in tension, and their propagation in the interior of the material domain. The bands become prominent, in the sense of being observable with the naked eye, when the stress reaches the vicinity of yield, and propagate in the interior as plastic deformation increases. The propagation is not time-dependent, in that it stops when the plastic deformation ceases. Therefore what we are witnessing is the propagation of plastic waves whose position in space depends parametrically on some measure of plastic deformation.

We infer on the basis of this observation, that the propagation of the bands obeys a wave equation with a "plastic inertia" term which is proportional to the second derivative of the wave function with respect to an *intrinsic* time z, *and not the clock time* t, where z is a non-decreasing measure of plastic deformation in the sense of VALANIS [2]. This inference has led to the present paper whose object is to arrive at the defining equation that describes the phenomenon of "plastic" propagation of adiabatic bands.

2. Thermodynamics

The thermodynamic theory of formation of bands – single bands or deformation patterns – in the presence of *uniform surface tractions*, was given previously in the context of a Helmholtz as well as a Gibbs formulation of *gradient thermodynamics of internal variables*, under conditions of uniform temperature and infinitesimal deformation, VALANIS [3, 4]. A cogent and detailed derivation of the equations, pertinent to the gradient theory, is also given in the Appendix.

The thermodynamic treatment in Ref. [4] addressed the general case where the Gibbs free energy density ϕ, in a material domain D, is a function of stress and three different classes of internal variables, $\mathbf{p}$, $\mathbf{q}$ and $\boldsymbol{\xi}$, each with a different physical behavior and thus, a fundamentally different form of evolution equation. The pertaining form of ϕ is given in Eq. (2.1):

$$\phi = \phi(\sigma_{ij};\ p_{ij};\ q_{i,j};\ q_i;\ \xi_{i,j};\ \xi_i), \tag{2.1}$$

where $\boldsymbol{\sigma}$ is the stress tensor, $\mathbf{p}$ stands for the class $\mathbf{p}^r$ $(r = 1, 2, ..., n_p)$, $\mathbf{q}$ for the class $\mathbf{q}^r$ $(r = 1, 2, ..., n_q)$ and $\boldsymbol{\xi}$ for the class $\mathbf{p}$ $(r = 1, 2, ..., n_\xi)$. Note that $\mathbf{p}$ are local while $\mathbf{q}$ and $\boldsymbol{\xi}$ are not-local in that their gradients are also constitutive variables of ϕ.

Following VALANIS [4], the strain in D is given by Eq. (2.2):

$$\varepsilon_{ij} = -\partial\phi/\partial\sigma_{ij}\,. \tag{2.2}$$

The stress tensor is symmetric and obeys Newton's law of motion, i.e., Eq. (2.2′) just as in the case of local theories:

$$\sigma_{ij,j} + f_i = \partial^2 u_i/\partial t^2, \tag{2.2′}$$

where f_i and u_i stand for the body force and displacement field, respectively. However, the evolution or field equation that pertains to a specific internal variable depends on whether it is local or non-local. Of the three classes, the first, p_{ij}, the local class, are tensors of the second order and obey a local evolution equation, i.e.,

$$\partial\phi/\partial p_{ij} + b_{ijkl}\,\partial p_{kl}/\partial z = 0. \tag{2.3}$$

This equation is predicated on the position that the internal force $P_{ij}(= -\partial\phi/\partial p_{ij})$ is linearly related to $\partial p_{ij}/\partial z$ through the positive definite resistance tensor b_{ijkl} which is symmetric in the indices i and j, the indices k and l and the pairs of indices (i, j) and (k, l). These variables are dissipative since the inner product $\mathbf{P}\cdot\partial\mathbf{q}/dz$ is not zero, and in fact positive – unless $\|\partial\mathbf{q}/dz\| = 0$ and/or $\|\mathbf{Q}\| = 0$, double bars denoting the Euclidean norm.

We remark that z is the intrinsic time scale in endochronic plasticity, proposed previously by VALANIS [2]. Specifically for rate-indifferent materials,

$$dz = d\zeta/f, \tag{2.3′}$$

where f is the isotropic hardening function and $d\zeta$ is the length of the increment of the plastic strain tensor, with respect to a material metric tensor $\mathbf{P}$ such that

$$d\zeta^2 = P_{ijkl}\,d\varepsilon^p_{ij}\,d\varepsilon^p_{kl}\,. \tag{2.3″}$$

In the event that the material is plastically incompressible, then:

$$d\zeta = k\|de^{p}_{ij}\|, \tag{2.3'''}$$

where e^{p}_{ij} is the strain deviator and k is an arbitrary constant, unity if convenient.

The second class q_i are *non-local* (vectorial) internal variables in that ϕ depends on the variables q_i as well as their gradients $q_{i,j}$. These variables obey the field equation (2.4):

$$(\partial\phi/\partial q_{i,j})_{,j} - \partial\phi/\partial q_i = Q_i\,, \tag{2.4}$$

where Q_i is the internal resistance force. The derivation of this equation as well as the complete set of equations pertaining to the gradient theory of internal variables, is given in the Appendix.

In this paper as well as in our previous work, we stipulate that Q_i is linearly related to $\partial q_i/\partial z$ through the resistance tensor b_{ij} symmetric in i and j and such that $\|\mathbf{b}\| \neq 0$. Hence the variables q_i obey the partial differential equation (2.5):

$$(\partial\phi/\partial q_{i,j})_{,j} - \partial\phi/\partial q_i = b_{ij}\,\partial q_j/\partial z. \tag{2.5}$$

We note that q_i are also dissipative, since again $Q_i\partial q_i/\partial z \neq 0$. Of fundamental importance is the stipulation that the evolution of the q-class of internal variables is expressed with respect to the intrinsic time z, i.e., the length of the plastic strain path, generated strictly only by virtue of the change in the local internal variables p_{ij}.

The third class ξ_i is again vectorial and non-local but inviscid, in that the internal resistance force $\Xi = 0$. These variables are either associated with dislocation glides and other changes in material conformations that take place with negligible (ideally zero) internal resistance, or are descriptors of terminal equilibrium states. Their spatial variation in the material domain D is given by Eq. (2.6):

$$(\partial\phi/\partial\xi_{i,j})_{,j} - \partial\phi/\partial\xi_i = 0. \tag{2.6}$$

REMARK. Of note is the fact that all generic internal variables of the gradient type, say ζ_i, obey one and the same fundamental equation, i.e., Eq. (2.7), where Z_i is the internal resistive force (see VALANIS [3, 4] for details):

$$(\partial\phi/\partial\zeta_{i,j})_{,j} - \partial\phi/\partial\zeta_i = Z_i\,. \tag{2.7}$$

The constitutive equation pertaining to the different sub-types of such variables depends on the nature of the resistive force Z_i. In the case of the q-class, $Z_i = Q_i = b_{ij}\partial q_j/\partial z$, while in the case of the ξ-class, $Z_i = \Xi_i = 0$.

2.1. Boundary conditions on ζ_i

Previously, VALANIS [5, 6], we demonstrated that the physical mechanism of particle migration and hence the non-affine deformation of the associated material sub-domain, is a cause for dissipative behavior. On the other hand it was also demonstrated, VALANIS [3], that internal variables ζ_i also arise as a result of heterogeneity of the internal material structure. While the field equations that describe the spatial distribution of ζ_i as well as their temporal variation are the same irrespective of their physical origin, the boundary conditions on the material surface S depend on the underlying physics. We, therefore, distinguish two different cases: (i) structural and (ii) migratory internal variables.

2.2. Structural internal variables

The surface is composed of two parts: S_T on which tractions are prescribed, and its complement S_u where displacements are prescribed. In this case the boundary conditions are as follows (VALANIS [3]):

$$\text{on } S_T: \qquad \partial\phi/\partial\zeta_{i,j}n_j = 0; \tag{2.8}$$

$$\text{on } S_u: \qquad \zeta_i = 0, \tag{2.9}$$

provided that the tractions and displacements on the boundary are "locally homogeneous" in the sense of VALANIS [3].

2.3. Migratory internal variables

Here the surface S is again composed of two parts: the permeable surface S_p on which the migratory displacements are unknown, and impermeable surface S_i on which the migratory displacements are zero. The boundary conditions are therefore as follows:

$$\text{on } S_p: \qquad \partial\phi/\partial\zeta_{i,j}n_j = 0; \tag{2.10}$$

$$\text{on } S_i: \qquad \zeta_i = 0. \tag{2.11}$$

The derivation of these equations is given in the Appendix.

REMARK. There may exist physical circumstances, though these are difficult to ascertain experimentally, where a surface is impermeable in the normal but permeable in a tangential direction; in other words, surface diffusion is possible. To deal with this situation, we construct a system of surface coordinates x'_i such that x'_1 is normal to the surface and x'_2 and x'_3 are tangential. We also let ζ'_i be the components of $\boldsymbol{\zeta}$ in the primed system and let S_{ij} be the coordinate transformation such that:

$$\zeta_i = S_{ij}\,\zeta'_j \qquad \zeta'_j = S_{ij}\,\zeta_i\,. \tag{2.12}$$

Following VALANIS [6], the appropriate variational boundary condition is:

$$\int_S \partial\phi/\partial\zeta_{i,j} n_j \delta\zeta_i \, dS = 0 \tag{2.13}$$

for all arbitrary variations $\delta\zeta_i$ such that $Z_i\delta\zeta_i \geq 0$. Or, in view of Eq. (2.12):

$$\int_S \partial\phi/\partial\zeta_{i,j} n_j S_{im} \delta\zeta'_m \, dS = 0. \tag{2.14}$$

Since $\zeta'_1 = 0$, it follows that $\delta\zeta'_1 = 0$ and Eq. (2.14) holds for $m = 1$. However $\delta\zeta'_2$ and $\delta\zeta'_3$ are arbitrary since ζ'_2 and ζ'_3 are not prescribed. Thus, if Eq. (2.14) is to hold for $m = 2$ and $m = 3$, then:

$$\partial\phi/\partial\zeta_{i,j} n_j S_{i2} = 0, \qquad \partial\phi/\partial\zeta_{i,j} n_j S_{i3} = 0. \tag{2.15}$$

It was demonstrated, VALANIS [3, 4], that the band formation is attributable to the presence of a non-local internal field. If such multiple fields are present, the material domain is then an ensemble of sub-domains, the behavior of each of which is governed by its own internal field. The geometric extent of each sub-domain is the same as the entire domain D.

3. Wave variables

In this paper we add a *fourth* class of vectorial variables, which we call *wave variables*. These come in pairs, i.e., for each "primary" variable ω_i with a resistance force Ω_i, there is a "dual" variable ω'_i with associated resistance force Ω'_i. Of importance is the fact that though $\|\Omega_i\|$ and $\|\Omega'_i\|$ are different from zero in the course of plastic deformation ($\|\mathbf{p}\| \neq 0$), the scalar product $\Omega_i\delta\omega_i + \Omega'_i\delta\omega'_i$ is zero, for all arbitrary variations $\delta\omega_i$ and $\delta\omega'_i$, and thus *the dissipation of the dual pair is zero*. However, ω_i is non-local while ω'_i is local. Thus ϕ depends on ω_i as well as on $\omega_{i,j}$ and on ω'_i (and not its gradient).

As we shall demonstrate, the primary variable ω_i has an evolution equation which is a pertial differential equation of the wave (hyperbolic) type – the desired form for the propagation of plastic bands.

The Gibbs free energy density ϕ now has the augmented form:

$$\phi = \phi(\sigma_{ij};\ p_{ij}\ q_{i,j};\ q_i;\ \xi_{i,j}\ \xi_i;\ \omega_{i,j};\ \omega_i;\ \omega'_i) \tag{3.1}$$

with the resulting additional equations (3.2) and (3.3) that govern the wave-variables ω_i and ω'_i:

$$\Omega_i = (\partial\phi/\partial\omega_{i,j})_{,j} - \partial\phi/\partial\omega_i \tag{3.2}$$

$$\Omega'_i = -\partial\phi/\partial\omega'_i\,. \tag{3.3}$$

A coupling between ω_i and ω'_i is now introduced through a relation between Ω_i and Ω'_i on one hand, and $\partial\omega_i/\partial z$ and $\partial\omega'_i/\partial z$ on the other, by means of the resistance matrix $\boldsymbol{\eta}$ in the manner of Eqs. (3.4):

$$(3.4)\qquad \begin{aligned} \Omega_i &= \eta_{11}\partial\omega_i/\partial z + \eta_{12}\partial\omega'_i/\partial z, \\ \Omega'_i &= \eta_{21}\partial\omega_i/\partial z + \eta_{22}\partial\omega'_i/\partial z. \end{aligned}$$

We remark that the Onsager form of η_{rs} is symmetric. Here, however, it is not.

We now recall that the *pair* of variables ω_i and ω'_i is *inviscid* giving rise to zero dissipation. This being the case, the matrix η_{ij} must be such that:

$$(3.5)\qquad \Omega_i\partial\omega_i/\partial z + \Omega'_i\partial\omega'_i/\partial z = 0.$$

This is accomplished by letting $\boldsymbol{\eta}$ be an *antisymmetric* matrix, in which event $\eta_{11} = \eta_{22} = 0$; $\eta_{12} = -\eta_{21} \equiv \beta$. Equations (3.4) now reduce to the form:

$$(3.6)\qquad \Omega_i = \beta\partial\omega'_i/\partial z,$$

$$(3.7)\qquad \Omega'_i = -\beta\partial\omega_i/\partial z.$$

Note that the pair (ω_i, ω'_i) is in fact inviscid since Eqs. (3.6) and (3.7) satisfy the inviscid condition (3.5) identically.

In the light of Eqs. (3.2) and (3.3), the field equations for the wave variables ω_i and ω'_i are now the following:

$$(3.8)\qquad (\partial\phi/\partial\omega_{i,j})_{,j} - \partial\phi/\partial\omega_i = \beta\partial\omega'_i/\partial z,$$

$$(3.9)\qquad \partial\phi/\partial\omega'_i = \beta\partial\omega_i/\partial z.$$

REMARK. We remark that, in irreversible thermodynamics, the resistance matrix, such as η_{rs} for instance, that relates the internal resistance forces to their dual internal velocities, need not be symmetric, provided that the work done by these forces is non-negative to satisfy the constraint of positive rate of dissipation. In our specific case above, where the material is inviscid, their inner product is zero.

It is of interest that the "resistance" matrix is now antisymmetric. The off-diagonal terms play the role of thermal inertia giving rise, as we shall see shortly, to a wave equation, instead of a diffusion equation, which is the case when the resistance matrix is symmetric – and positive definite.

4. The wave equation

To avoid complexities associated with the generality of dependence of ϕ on the variables of the various mechanisms, we make the reasonable stipulation that the energies of the mechanisms are additive, in which event Eq. (3.1) has the partitioned form:

$$\phi = \phi_p + \phi_q + \phi_\xi + \phi_\omega\,, \tag{4.1}$$

where

$$\phi_p = \phi(\boldsymbol{\sigma}, \mathbf{p}), \qquad \phi_q = \phi_q(\boldsymbol{\sigma}, \nabla\mathbf{q}, \mathbf{q}), \qquad \phi_\xi = \phi_\xi(\boldsymbol{\sigma}, \nabla\boldsymbol{\xi}, \boldsymbol{\xi}), \tag{4.2}$$

$$\phi_\omega = \phi_\omega(\boldsymbol{\sigma}, \nabla\boldsymbol{\omega}, \boldsymbol{\omega}, \boldsymbol{\omega}'). \tag{4.3}$$

The strain in the domain D consists of the contributions of the various mechanisms that are active during loading. In fact, and in view of Eqs. (2.2′) and (4.1), the strain $\boldsymbol{\varepsilon}$ is the *sum* of the strains produced by the individual mechanisms. Thus:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_p + \boldsymbol{\varepsilon}_q + \boldsymbol{\varepsilon}_\xi + \boldsymbol{\varepsilon}_\omega\,. \tag{4.4}$$

We remark that the suffix p denotes the strain contribution of the local mechanism and that $\boldsymbol{\varepsilon}_p$ will consist of the elastic strain $\boldsymbol{\varepsilon}^{(e)}$ plus the plastic strain $\boldsymbol{\varepsilon}^{(p)}$.

Equations (3.8) and (3.9) now become:

$$\begin{aligned} (\partial\phi_\omega/\partial\omega_{i,j})_{,j} - \partial\phi_\omega/\partial\omega_i &= \beta\partial\omega_i'/\partial z, \\ \partial\phi_\omega/\partial\omega_i' &= \beta\partial\omega_i'/\partial z. \end{aligned} \tag{4.5}$$

4.1. The wave equation in one dimension

Because, to begin with, we are interested in the essential physics of the problem, we do the analysis in a one-dimensional domain D signified by the variable x, $-\infty < x < \infty$ and in the context of a linear theory and, therefore, a quadratic form of ϕ_ω.

Historically, quadratic forms of the free energy densities, Helmholtz or Gibbs, have been powerful in giving insight to the constitutive behavior of solids. For instance, in local viscoelasticity a quadratic form of the Helmholtz free energy density ψ results in the stress being a linear hereditary functional of the strain, while a quadratic Gibbs free energy density ϕ results in the strain being a linear hereditary functional of the stress. The same situation arises in plasticity except that the functionals are taken with respect to the intrinsic time ζ – in the manner of endochronic plasticity, VALANIS [7].

We begin, therefore, with ψ because under isothermal conditions it has the physical significance of stored energy, with the attending stability condition that it must be a positive function of its arguments. Thus, in one dimension, we let ψ_ω have the form shown in Eq. (4.6):

$$\psi_\omega = (1/2)A_\omega(u_x^\omega)^2 - B_\omega u_x^\omega\omega_x + (1/2)C_\omega\omega_x^2 + (1/2)F_\omega(\omega^2 + \omega'^2), \tag{4.6}$$

where a suffix x denotes differentiation with respect to x, u^ω is the displacement field, u^ω_x is the (uniaxial) strain field ε_ω and A_ω, B_ω and C_ω are material constants. Quite clearly and without loss of generality we may put B_ω equal to unity, since we may define a new variable $\omega^* = B_\omega \omega$, substitute in Eq. (4.6) and drop asterisks. Thus:

$$\psi_\omega = (1/2)A_\omega (u^\omega_x)^2 - u^\omega_x \omega_x + (1/2) C_\omega \omega_x^2 + (1/2) F_\omega (\omega^2 + \omega'^2). \tag{4.7}$$

Note that ω' is of the *local* type so that its gradient does not appear in the expression for ψ. Also because ψ is a positive function,

$$A_\omega > 0, \qquad C_\omega > 0, \qquad F_\omega > 0 \qquad \text{and} \qquad A_\omega C_\omega > 1. \tag{4.8}$$

Our purpose is to use Eq. (4.7) to obtain the ancillary form of ϕ_ω by means of the Legendre transformation:

$$\phi_\omega = \psi - \sigma_\omega \varepsilon_\omega \tag{4.9}$$

in the light of the fact that

$$\sigma_\omega = \partial \psi_\omega / \partial \varepsilon_\omega \, . \tag{4.10}$$

Following a simple computation and since $\sigma_\omega = \sigma$, common to all mechanisms in the light of Eq. (4.2), we find that:

$$\phi_\omega = -\sigma^2/2A_\omega - (\sigma/A_\omega)\omega_x + (C^*_\omega/2)\omega_x^2 + (F_\omega/2)(\omega^2 + \omega'^2), \tag{4.11}$$

where we have set $C^*_\omega = C_\omega - A_\omega^{-1}$. In view of the inequality $(4.8)_4$: $C^*_\omega > 0$.

Equations (4.4) and (4.5) in one dimension become:

$$(\partial \phi_\omega / \partial u^x_\omega)_x - \partial \phi_\omega / \partial \omega = \beta \partial \omega' / \partial z, \tag{4.12}$$

$$\partial \phi_\omega / \partial \omega' = \beta \partial \omega / \partial z \tag{4.13}$$

while, in view of Eq. (2.2') and for quasi-static processes without a body force field:

$$\sigma_x = 0. \tag{4.14}$$

At this juncture and as a result of Eqs. (4.10), (4.11), (4.12) and (4.13), the following equations for ω and ω' are obtained:

$$C^*_\omega \omega_{xx} - F\omega = \beta \partial \omega' / \partial z, \tag{4.15}$$

$$F\omega' = \beta \partial \omega / \partial z. \tag{4.16}$$

Elimination of ω' from Eqs. (4.15) and (4.16) gives the following *wave equation* for ω:

$$C^*_\omega \omega_{xx} - F\omega = \varrho_\omega \partial^2 \omega / \partial z^2, \tag{4.17}$$

where $\varrho_\omega = \beta^2/F$. Thus ϱ_ω plays the role of the plastic inertia which is not related to the mass density but to the off-diagonal coefficient of the resistance matrix that relates the internal forces to the rates of the dual variables ω and ω'.

The strain contribution ε_ω of the internal field ω to the total strain in D, in the light of Eqs. (2.2) and (3.1) is given by Eq. (4.18):

$$u_x^\omega = -\partial\phi/\partial\sigma. \tag{4.18}$$

In view of Eq. (4.11):

$$u_x^\omega = \sigma/A_\omega + \omega_x/A_\omega\,. \tag{4.19}$$

Note that u_x^ω consists of the elastic "local" contribution σ/A_ω and the "gradient" contribution ω_x/A_ω. Also note that the dual variable ω' does not contribute to the strain in D.

It is also pointed out that ϕ_ω in Eq. (4.11) consists of the "local term" $-\sigma^2/2A_w$ and the non-local remainder. Furthermore and in view of Eq. (4.1), the local terms in that equation are additive to one grand total local term, which in conjunction with ϕ_p constitutes the local part of ϕ, which as it happens, is of the same form as ϕ_p. Thus in conclusion we may say that ϕ may be partitioned in the form of Eq. (4.1), but with the proviso that the terms ϕ_q, ϕ_ξ and ϕ_ω contain $\boldsymbol{\sigma}$ as a variable, but only in algebraic terms that are coupled with the gradients of the internal field variables.

4.2. The wave equation in three dimensions

With the above discussion we have laid the foundation for the three-dimensional generalization of the expression for ϕ_ω as it appears in Eq. (4.11). We thus set:

$$\phi_w = -A_w^{-1}\sigma_{ij}\omega_{i,j} + (C_\omega/2)\omega_{i,j}\omega_{i,j} + (F_\omega/2)(\omega_i\omega_i + \omega_i'\omega_i'). \tag{4.20}$$

This is not the most general form of ϕ_ω but it will serve us well in the present study of the physics of propagation of band-waves in solids.

At this point in the light of Eq. (2.2′) and in the absence of body and acceleration forces, Eqs. (4.5) become:

$$C_\omega\omega_{i,jj} - F_\omega\omega_i = \beta\partial\omega_i'/\partial z, \tag{4.21}$$

$$F_\omega\omega_i' = \beta\partial\omega_i/\partial z. \tag{4.22}$$

The variable ω_i' is now eliminated from Eqs. (4.21) and (4.2) and the following wave equation for ω_i is thereby obtained:

$$C_\omega\omega_{i,kk} - F_\omega\omega_i = \varrho_\omega\partial^2\omega_i/\partial z^2, \tag{4.23}$$

where as before $\varrho_\omega = \beta^2/F$.

4.3. Boundary conditions

The boundary conditions depend on whether ω_i is a structural or a migratory variable. The following cases arise.

(i) Structural variables. We recall Eqs. (2.8) and (2.9) according to which and in conjunction with Eq. (4.11):

$$\text{(4.24)} \qquad \text{on } S_T: \qquad \omega_{i,j}n_j = (1/A_\omega C_\omega)T_i\,,$$

$$\text{(4.25)} \qquad \text{on } S_u: \qquad \omega_i = 0.$$

(ii) Migratory variables. Substitute p for subscript T in Eq. (4.24) and i for u in Eq. (4.25). For more complex conditions follow the discussion subsequent to Eq. (2.14).

REMARK. The propagation of the "plastic" wave generated by the wave Eq. (4.23) is in terms of the intrinsic time z – where z is the length of the plastic strain path, within a proportionality constant – and not in terms of the clock time t. When the deformation is brought about by application of tractions (or tractions and displacements) on the surface of the material domain D the plastic wave will propagate from the surface toward the interior of the domain and will stop either when the surface tractions (or displacements) cease to increase, or when these change in a manner that constitutes "elastic unloading". We illustrate the above plenomenon by solving the problem of plastic strain bands propagating in a longitudinal direction in a flat strip under axial loading.

5. Band propagation in a flat thin strip under uniform axial traction

We consider the domain D to be a semi-infinitive flat strip, $|x_1| \leq a$, $0 \leq x_2 < \infty$, $|x_3| \leq b$. At $x_2 = 0$ a uniform tensile traction $T(z)$ – independent of x_1 and x_3 – is applied while the edges $|x_1| = a$ and $|x_3| = b$ are stress-free. The initial conditions in D are such that ω_i and $\partial\omega_i/\partial z$ are equal zero at $z = 0$. One can verify that $\omega_1 \equiv 0$ and $\omega_3 \equiv 0$ on the basis of the fact that Eq. (4.23), the initial conditions and the boundary condition (4.24) are satisfied identically. Furthermore, the boundary condition at $x_2 = 0$, i.e.,

$$\text{(5.1)} \qquad A_\omega C_\omega \omega_{2,2} = T(z)$$

and the boundary condition at $|x_1| = a$ and $|x_3| = b$, i.e.,

$$\text{(5.2)} \qquad \omega_{2,1} = 0,$$

$$\text{(5.3)} \qquad \omega_{2,3} = 0,$$

require that ω_2 be a function of x_2 and z only.

Briefly, Eq. (5.1) demands that ω_2 be of the form:

$$\omega_2 = X_1(x_1, x_3, z) + X_2(x_2, z), \tag{5.4}$$

where X_1 and X_2 are arbitrary functions, while Eq. (5.2) and (5.3) demand that X_1 be, at most, a constant field. Thus:

$$\omega_2 = \omega_2(x_2, z). \tag{5.4'}$$

At this point we change the notation to simplify the presentation of the analysis and set: $\omega_2 \equiv \omega$ and $x_2 \equiv y$, $A_\omega \equiv A$, $C_\omega \equiv C$ and $F_\omega \equiv F$. Equations (4.23) and (4.24) then reduce to the following equations. In D:

$$C\partial^2\omega/\partial y^2 - F\omega = \varrho_\omega \partial^2\omega/\partial z^2. \tag{5.5}$$

On S at $y = 0$:

$$AC\, \partial\omega/\partial y = T(z) \tag{5.6}$$

while at $y = \infty$, ω is bounded.

5.1. Solution by Laplace transform

Let $\overline{\omega}(y,p)$ be the Laplace transform of $\omega(y,z)$. Then in view of the initial conditions, Eq. (5.5) becomes:

$$C\, \partial\overline{\omega}^2/\partial y^2 - F\overline{\omega} = \varrho_\omega p^2 \overline{\omega}. \tag{5.7}$$

Let

$$c^2 = C/\varrho_\omega\,, \qquad a^2 = F/\varrho_\omega\,, \qquad x = y/c, \tag{5.8}$$

where c is the wave velocity. Then Eq. (5.5) becomes:

$$\partial\overline{\omega}^2/\partial x^2 - (a^2 + p^2)\overline{\omega} = 0 \tag{5.9}$$

while Eq. (5.6) leads to the boundary conditions:

$$\Gamma \partial\overline{\omega}/\partial x = \overline{T}(p), \tag{5.9'}$$

where $\Gamma = AC/c$. In view of the required boundedness of the solution at $y = \infty$, it follows from Eq. (5.9) that

$$\overline{\omega} = K \exp\left\{-(a^2 + p^2)^{1/2} x\right\}, \tag{5.10}$$

where K is to be determined from the boundary condition at $y = 0$.

It thus follows from Eqs. (5.9′) and (5.10) that:

$$\overline{\omega} = -\frac{\overline{T}(p)}{\Gamma\sqrt{(a^2 + p^2)}} e^{-x\sqrt{(a^2+p^2)}}, \tag{5.11}$$

where $\overline{T}(p)$ is th Laplace transform of $T(z)$.

Discussion. Before we consider, in its generality, the inverse transform of $\overline{\omega}$, we discuss some special cases of interest. Specifically we pose the case where the local constitutive behavior of the material is that of an elastic perfectly plastic solid, in which event the function $T(z)$ is a Heaviside step function, i.e.,

$$T(z) = T^0 H(z), \tag{5.12}$$

where T^0 is the value of yield stress. In this event $\overline{T} = T^0/p$. Noting that the Laplace transform of $\partial\omega/\partial z$ is $p\overline{\omega}$ $(\omega(0) = 0)$, then:

$$p\overline{\omega} = -\frac{T^0}{\Gamma\sqrt{(a^2+p^2)}} e^{-x\sqrt{(a^2+p^2)}}. \tag{5.13}$$

The inverse transform of Eq. (5.13) is now standard and in fact,

$$\partial\omega/\partial z = (T^0/\Gamma) J_0 \left\{a(z^2 - x^2)^{1/2}\right\} \qquad (z \geq x), \tag{5.14}$$

where J_0 is the Bessel function of order zero. One must note, however, that because the intrinsic time z is the length of the plastic strain path, $\partial\omega/\partial z$ is not a "velocity" in the normal sense, but the rate of change of the internal displacement ω with respect to the "magnitude" of the plastic *local* strain. Specifically at $x = 0$:

$$\partial\omega/\partial z = (T^0/\Gamma) J_0(az). \tag{5.14'}$$

Equation (5.14) is that of a wave propagating into the interior of the strip at a speed c, since $x = y/c$.

In reference to Eq. (5.14) and because of the properties of J_0, we observe that at fixed z, $\partial\omega/\partial z$ is an alternating function of x with decreasing amplitude, i.e., its absolute value is largest at the wavefront but diminishes as x approaches the boundary $x = 0$. A plot of $\partial\omega/\partial z$ vs. z, for $a = 10$, is given in Fig. 1.

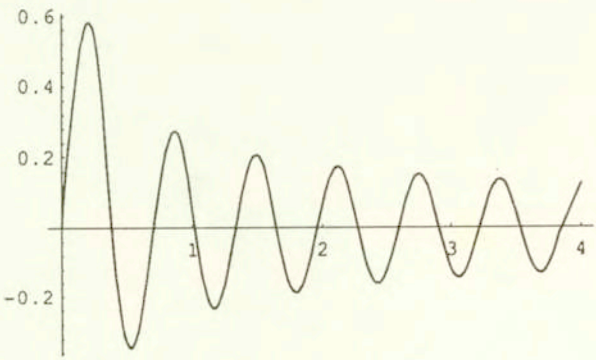

FIG. 1. Displacement (arbitrary units) vs. z. Displacement history at $x = 0$ $(a = 10)$.

The displacement field is found by integrating the right-hand side of Eq. (5.14) with respect to z. Thus:

$$\omega(x,z) = (T_0/\Gamma)\int_x^z \left\{a\sqrt{(z_1^2 - x^2)}\right\} dz_1 . \tag{5.15}$$

The lower limit of integration reflects the nature of the solution whereby $\partial\omega/\partial z$ is null for $z_1 \le x$. Of particular interest is the displacement history at $y = 0$, i.e. at $x = 0$. Then by simple integration:

$$\omega(x,z) = (T^0/a\Gamma)J_1(az), \tag{5.15'}$$

where J_1 is the Bessel function of order one.

The remarkable feature of the solution is that the displacement is highly oscillatory – particularly for large values of a – even though the traction at $x = 0$ is a constant of z. Figure 2 shows a plot of ω vs. z for $a = 10$.

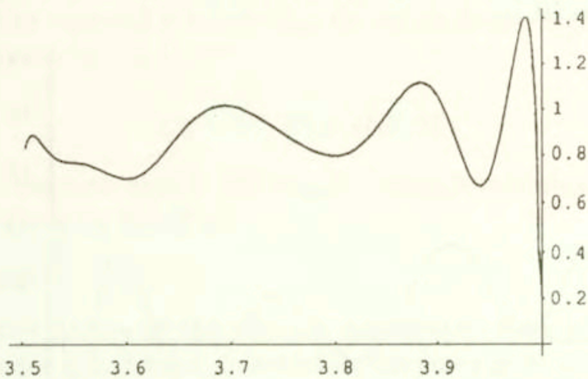

FIG. 2. Displacement vs. x. Displacement distribution close to the propagating wave-front.

5.2. The strain field

The strain is found upon using Eqs. (2.2) and (4.20). Thus:

$$\varepsilon_{ij}^{\omega} = -\partial\phi/\partial\sigma_{ij} = (1/A_\omega \omega_{(i,j)} , \tag{5.16}$$

where $\omega_{(i,j)}$ is the symmetric part of $\omega_{i,j}$. In the case at hand only the axial strain $\varepsilon_{22}^{\omega} \equiv \varepsilon_x^{\omega}$ is non-null. A simple calculation shows that

$$\varepsilon_x^{\omega} = (1/CA)\partial\omega/\partial x. \tag{5.17}$$

Thus, and in view of Eq. (5.11):

$$\varepsilon_x^{\omega} = \frac{\overline{T}(p)}{C} e^{-x\sqrt{(a^2+p^2)}} . \tag{5.18}$$

At this point we note that the inverse transform of the function $f(p)$, where

$$\overline{f}(p) = e^{-x\sqrt{(a^2+p^2)}}, \tag{5.19}$$

is given by Eq. (5.20)

$$f(z) = \delta(z-y) - \frac{ax}{\sqrt{(z^2-x^2)}} J_1\left\{a(z^2-x^2)^{1/2}\right\}, \tag{5.20}$$

where $\delta(\cdot)$ is the Dirac delta function. Thus in the event that $\overline{T} = T^0/p$, as above,

$$\varepsilon_x^\omega = (T_0/C)\left\{H(z-x) - ax\int_x^z \frac{J_1\left\{a(z'^2-x^2)^{1/2}\right\}}{(z'^2-x^2)^{1/2}}dz'\right\}. \tag{5.21}$$

The spatial strain distribution ε_x^ω behind the wave front is given in Fig. 3 for a value of a equal to 10.

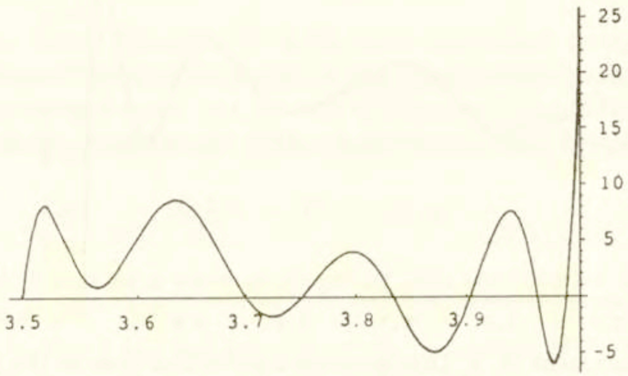

FIG. 3. Strain (arbitrary units) vs. x. Strain band-waves close to the propagating front.

We note that $T(z)$, in the general case, pertains to and is the function of the "local" stress-plastic strain curve of a plastic hardening solid. In this general case, ε_z^ω is given by the convolution integral depicted in Eq. (5.22):

$$C\varepsilon_x^\omega = T(z-x) - ax\int_x^z T(z-z')\frac{J_1\left\{a(z'^2-x^2)^{1/2}\right\}}{(z'^2-x^2)^{1/2}}dz'. \tag{5.22}$$

Discussion. As pointed out earlier, the strain field in D consists of the separate contributions of the several activated mechanisms. To bring out the effects of the wave mechanism, we assume that only the local and wave-mechanisms are active, so that

$$\varepsilon = \varepsilon_p + \varepsilon_\omega\,. \tag{5.23}$$

Prior and up to yielding, the strain field is local, elastic and $z = 0$. When the plate is stretched at and beyond yield, if the material hardens, z increases and the local strain field is given by Eq. (5.24):

$$\boldsymbol{\varepsilon}_p = \boldsymbol{\varepsilon}^{(p)} + \boldsymbol{\varepsilon}^{(e)}, \tag{5.24}$$

where $\boldsymbol{\varepsilon}^{(p)}$ is the local plastic strain while $\boldsymbol{\varepsilon}^{(e)}$ is its elastic counterpart. Note that z is the Euclidean norm of the plastic strain tensor, i.e.,

$$z = \|\boldsymbol{\varepsilon}^{(p)}\| \tag{5.25}$$

so that in simple tension:

$$z = \sqrt{(2/3)}\,\varepsilon_2^{(p)} = \sqrt{(2/3)}\,\varepsilon_x^{(p)} \tag{5.26}$$

assuming plastic incompressibility.

Thus during extension at yield, if the material is elastic-perfectly plastic, or beyond yield if the material is hardening, the strain field will consist of a uniform local field ε_{px}, given by Eq. (5.27):

$$\varepsilon_{px} = (T/E) + \sqrt{(3/2)}\,z, \tag{5.27}$$

where T is the traction and E the elastic Young's modulus, and the internal (wave) field ε_x^{ω} given by Eq. (5.22).

5.3. One final note

The oscillatory nature of the solution has already been pointed out. As the material parameter a increases, the solution becomes more and more oscillatory and its spatial visual representation will be that of numerous bands of increasing thinness. Likewise, the history of the solution will become highly oscillatory. This is illustrated in Fig. 4, where the displacement history is shown at $x = 0$ for $a = 20$.

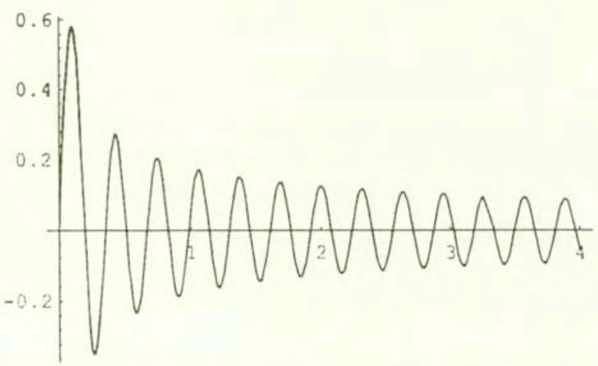

FIG. 4. Displacement (arbitrary units) vs. z. Displacement history at $x = 0$ ($a = 20$).

Appendix

The field equations and boundary conditions, pertaining to the gradient theory of internal variables, have been derived elsewhere [3, 4, 6]. However, because at times the analysis has been too concise and because the physics of the problem needs to be well understood, we give here a detailed and cogent derivation for the sake of completeness. Specifically, the boundary conditions are discussed at some length.

We begin a mathematical derivation, without concern for the physical nature of the internal variables, but discuss the resulting equations and boundary conditions, in the context of the pertaining physics, later in the Appendix. We limit the analysis to small deformation and isothermal conditions. The large deformation case has been discussed elsewhere [6]. Both the Helmholtz and Gibbs formalism are presented.

A.1. Helmholtz formulation

Let the material domain of a body of volume V and surface S be a *closed* thermodynamic system in the sense that it cannot exchange matter with its surroundings, is devoid of sources and sinks (either of the heat or material type), and is free of chemical reactions and electromagnetic fields. Then the following fundamental thermodynamic inequality applies under isothermal conditions:

$$\dot{\Psi} \leq \int_S T_i v_i \, dS + \int_V f_i v_i \, dV, \tag{A.1}$$

where Ψ is the Helmholtz free energy of the domain, T_i the surface tractions, f_i the body forces (including inertia forces) and v_i the velocity field, real or virtual, i.e., due to the set $(T_i; f_i)$ or an external agency.

Let D denote the internal dissipation in the sense that

$$D = \int_V Q_i \dot{q}_i \, dV\,, \tag{A.2}$$

where Q_i are internal dissipation forces, q_i are dual internal displacements, different and independent of the continuum average displacements u_i, and a superposed dot denotes a time derivative. Both u_i and q_i are continuous and twice differentiable in V.

Note that D is the rate of work done by Q_i and is always non-negative for $|\dot{q}_i| \neq 0$, bars denoting the Euclidean norm. Thus:

$$D = \int_V Q_i \dot{q}_i \, dV \geq 0 \tag{A.3}$$

and therefore, applying Eq. (A.2) to infinitesimal volumes,

$$Q_i \dot{q}_i \geq 0 \tag{A.4}$$

the equality in Eq. (A.4) pertaining to the case only when $|\dot{q}_i| = 0$. Inequality (A.1) may now be stated in terms of the dissipation D, i.e.,

$$\dot{\Psi} = \int_S T_i v_i \, dS + \int f_i v_i \, dV - D \tag{A.5}$$

or,

$$\dot{\Psi} = \int_S T_i v_i \, dS + \int f_i v_i \, dV - \int Q_i \dot{q}_i \, dV. \tag{A.6}$$

Quite clearly, if $|\dot{q}_i| = 0$, then the body behaves as non-dissipative and the rate of change of the free energy is equal to the rate of work of the surface tractions and the body forces, as in the case of an elastic body. Equation (A.6) is true for all actual and virtual v_i and $\dot{q}_i$ in V, subject to Ineq. (A.4).

At this point we introduce the free energy density ψ where

$$\Psi = \int_V \psi \, dV \tag{A.7}$$

such that,

$$\psi = \psi(u_{i,j};\ q_{i,j};\ q_i). \tag{A.8}$$

In view of Eqs. (A.7) and (A.8) and upon use of the Green-Gauss theorem, we now have the following identity:

$$\dot{\Psi} = \int_S (\partial\psi/\partial u_{i,j}) n_j v_i \, dS - \int_V (\partial\psi/\partial u_{i,j})_{,j} v_i \, dV + \int_S (\partial\psi/\partial q_{i,j}) n_j \dot{q}_i \, dS - \int_V (\partial\psi/\partial \psi_{i,j})_{,j} \dot{q}_i \, dV + \int_V (\partial\psi/\partial q_i) \dot{q}_i \,. \tag{A.9}$$

REMARK. Quite clearly $\dot{q}_i$ on S are not subject to Ineq. (A.3). They constitute a set of measure zero in so far as the integral $\int_V Q_i \dot{q}_i \, dV$ is concerned, in the sense that their value on S contributes nothing to the integral. However, other physical considerations apply. We have given two different physical interpretations for q_i in the literature: 1) where q_i are material *migratory* vectors, VALANIS [6], and 2) where they are *structural* vectors, i.e., deviations from the average continuum displacements u_i – the averaging volume depending on the heterogeneity of the material *structure*, VALANIS [3].

CASE 1. Here q_i are migratory vectors and their behavior at the surface is determined by the physical nature of the latter. We shall deal with two cases: a) when the surface $S_p \subseteq S$ is totally permeable and b) when $S_i \subseteq S$ is totally impermeable, where $S_p \cap S_i = 0$. On S_p, q_i are generally different from zero but are unknown since they are not measurable. However on S_i $q_i = \dot{q}_i = 0$. This is an inviolated kinematic constraint.

Thus, the virtual integral velocities $\dot{q}_i$ are arbitrary on S_p but zero on S_i.

The more general case where a surface is impermeable in the normal direction but permeable in the transverse directions, is treated in the text.

CASE 2. Here q_i are structural variables. We shall treat the case where u_i are "almost" uniform on the part of the surface $S_u \subseteq S$ where they are prescribed, in the sense of VALANIS [3]. Then on S_u, $q_i = \dot{q}_i = 0$. On the part of the surface $S_T \subseteq S$ where "almost" uniform tractions T_i are prescribed, q_i are not known and generally different from zero.

Thus virtual $\dot{q}_i$ are arbitrary on S_T and zero on S_u.

A.2. Derivation of field equations

Equation (A.6) and Eq. (A.9) combine to give rise to Eq. (A.10):

$$\text{(A.10)} \qquad \int_S \left[\left(\frac{\partial \psi}{\partial u_{i,j}}\right) n_j - T_i\right] v_i\, dS + \int_S \frac{\partial \psi}{\partial q_{i,j}} n_j \dot{q}_i\, dS - \int_V \left[\left(\frac{\partial \psi}{\partial u_{i,j}}\right)_{,j} + f_i\right] v_i\, dV + \int_V \left[\left(\frac{\partial \psi}{\partial q_{i,j}}\right)_{,j} - \frac{\partial \psi}{\partial q_i} - Q_i\right] \dot{q}_i\, dV = 0.$$

Let v_i and $\dot{q}_i$ be virtual and set $\dot{q}_i = 0$ in V and on S. Also set $v_i = 0$ on S. Then since v_i are otherwise arbitrary in V, it follows that

$$\text{(A.11)} \qquad \left(\frac{\partial \psi}{\partial u_{i,j}}\right)_{,j} + f_i = 0.$$

This is the equation of (dynamic) equilibrium. For instance if f_i are inertia forces then according to Newton's law, $f_i = -\varrho_0 \partial^2 u_i / \partial t^2$.

Since $\dot{q}_i$ is zero on S and in V and for all arbitrary v_i on S, it follows from Eq. (A.10) that

$$\text{(A.12)} \qquad \left(\frac{\partial \psi}{\partial u_{i,j}}\right) n_j = T_i$$

which is the boundary condition for the tractions T_i.

In the light of Eq. (A.10), the following identity now holds for all virtual $\dot{q}_i$ subject to Ineq. (A.6):

$$\int\limits_V \left[\left(\frac{\partial \psi}{\partial q_{i,j}} \right)_{,j} - \frac{\partial \psi}{\partial q_i} - Q_i \right] \dot{q}_i \, dV = 0. \tag{A.13}$$

Since Eq. (A.13) must be true for infinitesimal dV, then:

$$\left[\left(\frac{\partial \psi}{\partial q_{i,j}} \right)_{,j} - \frac{\partial \psi}{\partial q_i} - Q_i \right] \dot{q}_i = 0 \tag{A.14}$$

for all virtual $\dot{q}_i$ such that $Q_i \dot{q}_i \geq 0$. Let $\dot{q}_i{}^{(r)}$ be three linearly independent vectors that satisfy the foregoing inequality. The existence of such a triad is geometrically obvious since thay lie in the open half-plane of vectors that make an acute angle with Q_i. Further let

$$\left[\left(\frac{\partial \psi}{\partial q_{i,j}} \right)_{,j} - \frac{\partial \psi}{\partial q_i} - Q_i \right] \dot{q}_i = R_i \,. \tag{A.15}$$

It then follows that

$$\dot{q}_i{}^{(r)} R_i = 0. \tag{A.16}$$

This is a set of three linear and homogeneous algebraic equations in R_i. The determinant of the coefficients is different from zero since $\dot{q}_i{}^{(r)}$ are linearly independent. Thus:

$$R_i = 0 \tag{A.17}$$

and hence:

$$\left(\frac{\partial \psi}{\partial q_{i,j}} \right)_{,j} - \frac{\partial \psi}{\partial q_i} = Q_i \,. \tag{A.18}$$

Equation (A.18) constitutes an internal equilibrium equation for the internal forces Q_i.

A.3. Boundary conditions

In view of Eqs. (A.11), (A.12) and (A.18) and in the light of Eq. (A.10), it follows that

$$\int\limits_S (\partial \psi / \partial q_{i,j}) n_j \dot{q}_i \, dS = 0. \tag{A.19}$$

CASE 1. In view of Eq. (A.19):

(A.20)
$$\int_{S_p} (\partial\psi/\partial q_{i,j}) n_j \dot{q}_i \, dS + \int_{S_i} (\partial\psi/\partial q_{i,j}) n_j \dot{q}_i \, dS = 0.$$

But, $\dot{q}_i = 0$ on S_i. Thus,

(A.21)
$$\int_{S_p} (\partial\psi/\partial q_{i,j}) n_j \dot{q}_i \, dS = 0.$$

However, $\dot{q}_i$ are arbitrary on S_p. Letting $\dot{q}_i$ be non-zero on an infinitesimal part of S_p and zero elsewhere, it follows that:

(A.22)
$$(\partial\psi/\partial q_{i,j}) n_j \dot{q}_i = 0$$

for all $\dot{q}_i$. Now letting $\dot{q}_i$ be equal to (1,0,0), (0,1,0) and (0,0,1), we find in turn that

(A.22′)
$$(\partial\psi/\partial q_{i,j}) n_j = 0 \qquad \text{on} \quad S_p \, .$$

On the other hand, on S_i, per force:

(A.23)
$$\dot{q}_i = q_i = 0.$$

CASE 2. In a similar manner:

(A.24)
$$(\partial\psi/\partial q_{i,j}) n_j = 0 \qquad \text{on} \quad S_T \, ,$$

while:

(A.25)
$$\dot{q}_i = q_i = 0 \qquad \text{on} \quad S_u \, .$$

A.4. Conservation of linear and angular momentum

In the presence of migratory motion and similar types of non-affine deformation one cannot assume *a priori* that the local equations of conservation of linear and angular momentum will be preserved. To arrive at these laws we invoke the following axiom.

AXIOM. $\dot{\Psi} = 0$ and $\dot{q}_i = 0$, whenever $v_i = v_i^*$ are virtual rigid body velocities.

The physics of the Axiom is self-evident and was discussed at length elsewhere, see VALANIS [6]. It follows from Eq. (A.6) that:

(A.26)
$$\int_S T_i v_i^* \, dS + \int_V f_i v_i^* \, dV = 0.$$

Let v_i^* be a translational velocity, constant in V and on S. Then since it is also arbitrary:

$$\int_S T_i\, dS + \int_V f_i\, dV = 0. \tag{A.27}$$

Then by the usual arguments of continuum mechanics,

$$T_i = \sigma_{ji} n_j \qquad \text{on} \quad S, \tag{A.28}$$

$$\sigma_{ij,i} + f_j = 0, \tag{A.29}$$

where σ_{ij} is the stress tensor. On the other hand if v_i^* are the result of rigid body rotation, i.e.,

$$v_i^* = e_{ijk} \Omega_j x_k\,, \tag{A.30}$$

where e_{ijk} is the permutation tensor, x_k are the reference coordinates and Ω_j is the angular velocity vector, then in view of Eqs. (A.26), (A.28) and (A.29):

$$\sigma_{ij} = \sigma_{ji}\,. \tag{A.31}$$

It then follows by straightforward analysis, using Eqs. (A.6), (A.7), (A.8), (A.18), (A.28) and (A.29) that

$$\sigma_{ij} = \partial\psi/\partial u_{i,j} \tag{A.32}$$

and thus,

$$\partial\psi/\partial u_{i,j} = \partial\psi/\partial u_{j,i} \tag{A.33}$$

whereby

$$\psi = \psi(\varepsilon_{ij}, \ldots), \tag{A.34}$$

where $\varepsilon_{ij} = (1/2)(u_{i,j} + u_{j,i})$.

A.5. Gibbs formulation

Let ϕ be the Gibbs free energy density of the domain, where

$$\phi(\sigma_{ij};\ q_{i,j};\ q_i) = \psi - \sigma_{ij}\varepsilon_{ij}\,. \tag{A.35}$$

Upon use of Eqs. (A.6), (A.7) and (A.35) one finds the following expression for $\dot{\phi}$:

$$\int_V \dot{\phi}\, dV = -\int_S u_i \dot{T}_i\, dS - \int_V u_i \dot{f}_i\, dV - \int_V Q_i \dot{q}_i\, dV. \tag{A.36}$$

Equation (A.36) is the variational principle, in terms of ϕ, true for virtual traction rates $\dot{T}_i$, body force rates $\dot{f}_i$ and migration velocities $\dot{q}_i$. Use of the Green – Gauss theorem on the surface integral in Eq. (A.36) gives the following result:

$$\text{(A.37)} \qquad \int_V [\partial\phi/\partial\sigma_{ij} + \varepsilon_{ij}]\,\dot{\sigma}_{ij}\,dV + \int_S (\partial\phi/\partial q_{i,j}) n_j \dot{q}_i\,dS - \int_S [(\partial\phi/\partial q_{i,j})_{,j} - \partial\phi/\partial q_i - Q_i]\,dV = 0.$$

Repeating the standard variational arguments presented earlier, we find the following equations in V:

$$\text{(A.38)} \qquad \varepsilon_{ij} = -\partial\phi/\partial\sigma_{ij}\,,$$

$$\text{(A.39)} \qquad (\partial\phi/\partial q_{i,j})_{,j} - \partial\phi/\partial q_i = Q_i\,.$$

On S:

$$\text{(A.40)} \qquad \partial\phi/\partial q_{i,j} n_j = 0 \quad \text{on} \quad S_p\,, \qquad q_i = \dot{q}_i = 0 \quad \text{on} \quad S_i$$

for migratory q_i. However, for structural q_i:

$$\text{(A.41)} \qquad \partial\phi/\partial q_{i,j} n_j = 0 \quad \text{on} \quad S_T\,, \qquad q_i = \dot{q}_i = 0 \quad \text{on} \quad S_u\,.$$

The analysis is now complete.

References

1. D.J. LUBAHN and R.P. FELGAR, *Plasticity and creep in metals*, John Wiley and Sons, New York 1951.
2. K.C. VALANIS, *Fundamental consequences of a new intrinsic time measure. Plasticity as a limit of the endochronic theory*, Arch. Mech., **32**, 171–191, 1980.
3. K.C. VALANIS, *A gradient theory of internal variables*, Acta Mech., **116**, 1–14, 1996.
4. K.C. VALANIS, *A thermodynamic theory of patterned deformation*, Acta Mech. [in press].
5. K.C. VALANIS, *The concept of physical metric in thermodynamics*, Acta Mech., **113**, 169–184, 1995.
6. K.C. VALANIS, *A gradient theory of finite viscoplasticity*, Arch. Mech., **49**, 589–606, 1997.

ENDOCHRONICS
8605 NW Lakecrest Court, Vancouver WA 98665, USA.

Received May 27, 1997; new version October 24, 1997.