Arch. Mech., **50**, 1, pp. 83–89, Warszawa 1998
FIFTY YEARS OF THE ARCHIVES OF MECHANICS

Control of the mechanical systems by means of sliding modes and the solution to the problem of constraint reactions

S. KOTOWSKI (WARSZAWA)

A NEW METHOD of determination of the constraint reactions in mechanics is introduced in this paper; the methods of investigation of the sliding modes in systems of variable structure are used.

1. Introduction

IN THIS ARTICLE, the methodology of control will be presented aimed at achieving accurate tracking of a class of nonlinear systems in the presence of disturbances and parameter uncertainty, and also applications of this method to the determination of the constraint reactions in analytical mechanics. In an ideal form, the method uses piecewice continuous control laws, resulting in the fact that the trajectory is sliding along the discontinuity or sliding surfaces. Proper selection of the methods leads to a perfect tracking of the required law of motion , but in reality the disturbances cause that the trajectory chatters near the sliding surface, what results in undesirable high-frequency vibration in the state trajectory.

Let us consider a piecewice continuous differential model, with discontinuos right-hand side, on the hypersurface. If the trajectories approach the discontinuity surface and, after reaching this position, stay on this surface, the surface is named the sliding surface. The sliding surface imposes certain type of constraints on the system dynamics. By suitable choice of the nonlinear systems, application of the discontinuous control law and the sliding surface, it is possible to obtain a situation in which the sliding surface accurately describes the dynamics of the systems and their trajectory, using the simplest type of control, the on-off control. In a multidimensional problem, the trajectory lies on the intersection of all the discontinuity surfaces.

2. Systems of variable structure

Systems of variable structure, which are subjected to the sliding motion, are most often described by differential equations with a linear control input. They have the following form:

$$\dot{x} = F(x) + Bu, \qquad s = Kx\,,$$

(1)
$$u = \begin{cases} u^{+} & \text{for } s > 0, \\ u^{-} & \text{for } s < 0, \end{cases}$$

where $F(x)$ – n vector function, B – $n \times m$ matrix, x – n vector, u – m vector, K – $m \times n$ matrix.

The system which is described in such a way has a variable structure which is determined by the control function u, varying on the surface (in our case s is a hyperplane). We shall consider the properties of a motion which takes place on a switching surface, which is called a sliding motion. When the sliding conditions [1] are satisfied, then the sliding motion which is described by the differential equation (1) occurs on the discontinuity surface [1, 3]:

(2)
$$\dot{x} = F(x) - B(KB)^{-1}KF(x).$$

A situation, in which the sliding trajectory lies on the intersection of all of the discontinuity surfaces and the dynamics of the motion on this surfaces is independent of the dynamics outside the surfaces, is analogous to the mechanical systems with constraints.

3. Mechanical systems with constraints

Let us consider the mechanical system described by the first mode Lagrange equations [5]

(3)
$$m_i\, \ddot{\xi}_i = F_i + \sum_{k=1}^{a} \lambda_k \frac{\partial f_k}{\partial \xi_i} + \sum_{c=1}^{b} \mu_c h_{ci} \qquad (i = 1, \ldots, 3n),$$

$$f_k(t, \xi_1, \ldots, \xi_{3n}) = 0, \qquad k = 1, \ldots, a,$$

$$\sum_{i=1}^{3n} h_{ci}\dot{\xi}_i + D_i = 0, \qquad c = 1, \ldots, b.$$

This equation can be transformed into the form of the first order matrix equation by the substitution:

$$\xi_i = x_i\,, \qquad \dot{\xi}_i = x_{3n+i}\,.$$

Then, the equation takes the form

(4)
$$M\dot{x} = F + W\lambda + H\mu\,,$$

$$f_k(t, x_1, \dots, x_{3n}) = 0, \qquad k = 1, \dots, a,$$

$$\sum_{i=1}^{3n} h_{ci} x_{3n+i} + D_i = 0, \qquad c = 1, \dots, b.$$

where the matrices and vectors have the following form:

$$M = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & & & & \vdots \\ \vdots & & 1_{3n} & & & \vdots \\ \vdots & & & m_1 & & \vdots \\ \vdots & & & & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & m_{3n} \end{bmatrix},$$

$$\dot{x}^T = [\dot{x}_1, \dots, \dot{x}_{6n}],$$

$$F^T = [x_1, \dots, x_{3n}, F_1, \dots, F_{3n}],$$

$$W = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & & 0 & \cdots & 0 \\ \vdots & & \vdots & \dfrac{\partial f_k}{\partial x_i} & \\ 0 & \cdots & 0 & & \end{bmatrix} \; - \; 6 \times (3n + a) \;\; \text{matrix},$$

$$\lambda^T = [0, \dots, 0, \lambda_1, \dots, \lambda_a] \; - \; (3n + a) \;\; \text{vector},$$

$$H = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & & 0 & \cdots & 0 \\ \vdots & & \vdots & h_{ci} & \\ 0 & \cdots & 0 & & \end{bmatrix} \; - \; 6 \times (3n + b) \;\; \text{matrix},$$

$$\mu^T = [0, \dots, 0, \mu_1, \dots, \mu_b] \; - \; (3n + b) \;\; \text{vector}.$$

Then, we should construct the abstract variable structure system, which would be similar to our mechanical systems in the meaning that the product which represents the reactions of constraints in a mechanical system is replaced by the matrix with discontinuos control in the system of a variable structure. It realizes the sliding motion on all the constraint surfaces, which replace the discontinuity surfaces in this model. If, in this model, the conditions of existence of the sliding solutions are fulfilled, then the sliding motion is unique on the discontinuity (constraints) surfaces. Hence it corresponds to the motion of mechanical systems with constraints, so that the respective elements of these models should be equal. This fact implies that a part of the equation which represents the equivalent control in the sliding model is equal to the part which represents the reactions of constraints in the mechanical model, which is described by the first mode Lagrange equations. Then by calculating the equivalent control in the sliding mode we could obtain the constraint reactions of the mechanical systems.

The determination of the constraint reactions is one of the most difficult problems of analytical mechanics. The possibility of solving this problem, by its replacement with a less complicated equivalent problem, is crucial to this theory and opens new opportunities for the analysis of mechanical systems.

A variable structure model similar to Eq. (4), has the form

$$M\dot{x} = F + Bu, \tag{5}$$

where B is the $6n \times (a+b)$ matrix, M, F, x are the same as in the mechanical system (4), and u is a vector of discontinuous control in the form:

$$u^T = [u_1, \dots, u_{a+b}],$$
$$u_i = \begin{cases} u_i^+ & \text{for } s_i > 0, \\ u_i^- & \text{for } s_i < 0, \end{cases}$$

$$f_k(t, x_1, \dots, x_{3n}) = 0, \qquad k = 1, \dots, a,$$

$$\sum_{i=1}^{3n} h_{ci} x_{3n+i} + D_i = 0, \qquad c = 1, \dots, b.$$

Then we can determine the matrix K, the rows of which consist of the gradient constraint surfaces s_i:

$$K = \left[\frac{\partial f_k}{\partial x_i}, h_{ci}\right] \quad (a+b) \times 6n \quad \text{matrix.} \tag{6}$$

Once these parameters are known, we can determine the sliding mode equation for this mechanical systems (5):

$$M\dot{x} = F - B(KB)^{-1}KF. \tag{7}$$

Since this equation is unique, it describes the same motion as the Eq. (4) in [4]. It means that the components which represent the equivalent control should be the same as the elements $W\lambda + H\mu$ in the mechanical system equation (4).

Then we get the equation for the unknown reactions of constraints

$$W\lambda + H\mu = -B(KB)^{-1}KF. \tag{8}$$

This equation connects the constraint reactions with the equivalent control in the discontinuous (variable structure) systems and enable the determination of constraint reactions without solving the equations of motion, what simplifies the problem of solution of the equations of motion.

EXAMPLE 1. Let us verify our theory on a simple example taken from the paper [3].

$$\ddot{x} = u, \qquad s = cx + \dot{x}, \qquad u = \mathrm{sgn}[s].$$

This equation can be transformed into the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \tag{9}$$

$$s = cx_1 + x_2 .$$

Gradient of the discontinuity surface takes the form

$$\left[\frac{\partial s}{\partial x_1}, \frac{\partial s}{\partial x_2}\right] = [c, 1].$$

Then we can obtain the equivalent control by scalar multiplication

$$[c, 1] \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [c, 1] \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{eq} = 0,$$

$$u_{eq} = -cx_2 ,$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} cx_2 ,$$

$$\begin{aligned} \dot{x}_1 &= x_2 , \\ \dot{x}_2 &= -cx_2 . \end{aligned}$$

Then $R = cx_2$, and from the equation (9) we obtain

$$\dot{x}_1 = x_2 , \qquad \dot{x}_2 = \lambda_1 .$$

By differentiating the constraint equation we obtain

$$\dot{x}_2 = -cx_2 .$$

Thus the conclusion is: by means of two different methods we obtain the same values of the constraint reactions.

EXAMPLE 2. This example is taken from the book [2], where the nonholonomic case known as Chaplygin's sleighs, is considered.

By introducing the change of variables: $x = x_1$, $y = x_2$, $\varphi = x_3$, we obtain the system:

$$\begin{aligned} \dot{x}_1 &= x_4\,, & \dot{x}_2 &= x_5\,, & \dot{x}_3 &= x_6\,,\\ \dot{x}_4 &= \lambda \sin x_3\,, & \dot{x}_5 &= -\lambda \cos x_3\,, & \dot{x}_6 &= 0, \end{aligned}$$

with the constraint surface:

$$x_4 \sin x_3 - x_5 \cos x_3 = 0.$$

An equivalent discontinuous system can be written in the following form:

$$\begin{bmatrix} \dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\\ \dot{x}_5\\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix} \begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6 \end{bmatrix} + \begin{bmatrix} 0\\ 0\\ 0\\ \sin x_3\\ -\cos x_3\\ 0 \end{bmatrix} u,$$

$$s = x_4 \sin x_3 - x_5 \cos x_3\,.$$

This system has the gradient vector:

$$K = [0, 0, x_4 \cos x_3 + x_5 \sin x_3, \sin x_3, -\cos x_3, 0].$$

Once this system is known, we can determine the equivalent control by scalar multiplication:

$$[0, 0, x_4 \cos x_3 + x_5 \sin x_3,\ \sin x_3,\ -\cos x_3, 0] \begin{bmatrix} 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix} \begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6 \end{bmatrix}$$

$$[0, 0, x_4 \cos x_3 + x_5 \sin x_3,\ \sin x_3,\ -\cos x_3, 0] \begin{bmatrix} 0\\ 0\\ 0\\ \sin x_3\\ -\cos x_3\\ 0 \end{bmatrix} u = 0.$$

From this we obtain:

$$x_6[x_4 \cos x_3 + x_5 \sin x_3] + [\sin^2 x_3 + \cos^2 x_3]u = 0,$$

and then

$$u_{eq} = -x_6[x_4 \cos x_3 + x_5 \sin x_3].$$

By introducing this equivalent control into the discontinuous equation we obtain:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sin x_3 \\ -\cos x_3 \\ 0 \end{bmatrix} (-x_6[x_4 \cos x_3 + x_5 \sin x_3]).$$

Then we get the constraint reaction in the same form as that given in the book [2].

The determination of constraint reactions is usually very complicated and, in many cases, it is an important condition necessary to find the solution to the equations of motion in various mechanical systems. Due to those difficulties, sometimes the second mode Lagrange's equations are used where the reactions do not appear. Then we propose to solve this problem in a different manner, using the sliding modes equations. This method can be used both in the analysis and synthesis of mechanical systems where the effects of constraints are crucial for determining the motion.

References

1. V.I. UTKIN, *Sliding mode and its applications to variable structure systems*, MIR, Moscow 1978.
2. J.I. NEJMARK and N.A. FUFAJEV, *Dynamics of nonholonomics systems* [in Polish], PWN, Warszawa 1971.
3. R.A. DE CARLO, S.H. ŻAK and G.P. MATTHEWS, *Variable structure control in nonlinear multivariable systems. A tutorial*, Proc. of the IEEE, **76**, 3, 1988.
4. S. KOTOWSKI, *Invariance of sliding modes with respect to control matrix*, Arch. Mech, **48**, 1, 1996.
5. R. GUTOWSKI, *Analytical mechanics* [in Polish], PWN, Warszawa 1971.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH
e-mail: skot@ippt.gov.pl

Received March 7, 1997; new version October 14, 1997.