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2-D boundary value problems of thermoelasticity in a multi-wedge – multi-layered region Part 2. Systems of integral equations

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IN THE PAPER, arbitrary 2-D BVP of termoelasticity in a wedge-shaped – layered region are reduced to special systems of singular integral equations with fixed point singularities. For this purpose, the Fourier and Mellin integral transforms of the solutions in the layered and wedge-shaped parts of the domain are "fitted" together along the common interface. This interface is characterized by the conditions of given discontinuities of displacements and tractions. The theory, developed by the author elsewhere, is applied to investigate the systems obtained. The results, concerning existence and properties of the solutions are presented depending on the exterior boundary conditions. The numerical method applied to solve the systems of equations is justified.

1. Introduction

INTEGRAL TRANSFORMS are often applied to solve boundary value problems in infinite domains. So the Fourier transform in layered regions and the Mellin transform in wedge-shaped ones make it possible to find solutions of some problems in closed forms (see [19, 20, 21] and others). In the other cases, integral transforms allow us to reduce problems to integral equations (singular integral equations) which are very effective in solving numerous boundary value problems of the theory of elasticity (see [3, 9]). Thus, as it follows from [3, 9], if the problems under consideration have piecewise smooth boundaries, then singular integral equations with fixed point singularities appear, as a rule.

Previously the idea of using the Fourier and Mellin transforms simultaneously in order to solve arbitrary two-dimensional boundary value problems for the Poisson's equation in combined domains was presented in [10, 12, 13]. All of the problems were reduced to a class of singular equations (systems of the singular equations) on the semi-axis with fixed point singularities in the neighbourhood of zero and at infinity. To this end, it was essential that the composition of sine or cosine Fourier and the Mellin transforms should be represented in the form of a product of the Mellin transform with modified argument and a certain function of the argument. Some integral equations with fixed and moving singularities are considered in [3, 9], but they can not be applied to singular equations obtained in [10, 12, 13].

In [11] the mentioned class of singular integral equations with fixed point singularities on a half-axis are investigated. Conditions of solvability of the integral equations in some Banach spaces with a relevant weight were obtained, and the convergence of projective methods to find their numerical solutions was proved. Corresponding theorems for the systems of integral equations were presented in Appendix [10] without proofs. The results obtained were based on the theory of the integral operators of the Wiener–Hopf type which had been constructed in [4, 5, 7, 18]. As it was indicated in [10, 14], the process of the numerical approximation of the solutions of such equations is very stable.

In [10, 12, 13] it was shown that symbols of the obtained systems of integral equations depend essentially on the type of the interfacial and exterior boundary conditions. In turn, this is a consequence of asymptotic behaviour of functions (matrix-functions) appearing in the kernels of the integral operators.

This paper deals with the boundary value problems of thermoelasticity, and is a continuation of the previous paper [15]. Corresponding formulation of the problems is presented precisely in the second section of [15]. All definitions and notations from [15] are still valid in this part of the paper. We show that the "sweep method" [8] and the method of integral transforms similar to that used in the paper [10] make it possible to reduce all linear boundary value problems in combined domains to systems of singular integral equations with fixed point singularities, for different types of the interior and exterior conditions along the boundaries. Then the estimations made in the previous part of the paper [15] allow us to calculate the symbols of the integral operators, and to regularize those of the integral equations the indices of which are not equal to zero.

In the second section of this paper, fitting of the Fourier and Mellin transformations along common boundaries between the first layer and wedges is drawn. As a result, systems of functional equations are obtained. In the next section, necessary conditions for solvability of some boundary value problems under consideration are discussed from the mechanical and mathematical points of view.

In the fourth section, the mentioned systems of functional equations are reduced to systems of singular integral equations. The process of reduction depends essentially on the types of the exterior and interior boundary conditions. In the fifth section, analysis of the corresponding systems of integral equations is presented, their symbols are calculated depending on the types of the boundary conditions, and the parameters of functional spaces in which these equations are investigated. These parameters determine the behaviour of the solutions of the boundary value problems near singular points of the domain (at infinity and in the neighbourhood of the wedge tip). In the Appendix, some necessary formulas are presented which have complicated forms.

2. Fitting of the Fourier and Mellin transforms along the common boundary Γ_0

First of all we mentally cut the solid under consideration into three (layered – Ω_L and wedge-shaped – $\Omega^{\pm}$) parts, and apply the Fourier and Mellin transforms to Eqs. (2.1) – (2.2) of the paper [15], and to the exterior and interior boundary conditions (2.3) – (2.12) in [15], in the respective domains. Using the sweep method, the relations between the corresponding transformations of unknown vectors of displacements and tractions along the common boundary $\Gamma_0 = \Gamma_0^- \cup \Gamma_{m_+}^+$ between the domains of different geometry have been obtained:

$$\overline{\mathbf{u}}_b^1(\lambda) = \mathbf{M}_\sigma(\lambda)\overline{\boldsymbol{\sigma}}_b^1(\lambda) + \mathbf{m}_\sigma(\lambda), \tag{2.1}$$

$$\widetilde{\mathbf{v}}_t^{m+}(s) = \mathbf{M}_p(s)\widetilde{\mathbf{p}}_t^{m+}(s) + \mathbf{m}_p(s), \tag{2.2}$$

$$\widetilde{\mathbf{w}}_b^1(s) = \mathbf{M}_q(s)\widetilde{\mathbf{q}}_b^1(s) + \mathbf{m}_q(s) \tag{2.3}$$

(see Eqs. (3.25), (4.18) and (4.19) in [15]). Here matrix-functions and vector-functions $\mathbf{M}_\sigma(\lambda)$, $\mathbf{m}_\sigma(\lambda)$ and $\mathbf{M}_p(s)$, $\mathbf{m}_p(s)$, $\mathbf{M}_q(s)$, $\mathbf{m}_q(s)$ calculated in [15] contain all information about the layered part and the wedge-shaped parts of the domain, respectively. They can be effectively calculated and their asymptotics near zero point depend in an essential way on the exterior boundary conditions (see Lemma 1, Lemma 2 in [15]). Besides, *a priori* estimations (2.13) in [15] lead to conditions (3.7) in [15] for unknown vector-functions $\overline{\mathbf{u}}_b^1(\lambda)$, $\overline{\boldsymbol{\sigma}}_b^1(\lambda)$, in particular. On the other hand, the vector-functions $\widetilde{\mathbf{q}}_b^1(s)$, $\widetilde{\mathbf{p}}_t^{m+}(s)$ should be analytic in the strip $-\gamma_0 < \Re s < \gamma_2$ in view of the mentioned *a priori* assumptions, but $\widetilde{\mathbf{v}}_t^{m+}(s)$, $\widetilde{\mathbf{w}}_b^1(s)$ are analytic in $0 < \Re s < \gamma_1$, in general.

Returning to Eqs. (3.16) in [15], let us consider new unknown odd and even vector-functions $\mathbf{z}_-$, $\mathbf{z}_+$ defined by the relation:

$$\boldsymbol{\sigma}_b^1(x_1) = \mathbf{z}_+(x_1) + \mathbf{z}_-(x_1). \tag{2.4}$$

Then, using the second equations of the interfacial conditions (2.7) and (2.8) from [15], relations (2.1) – (2.3) are rewritten in the form:

$$\overline{\mathbf{u}}_b^1(\lambda) = \left[\mathbf{M}_\sigma^+(\lambda) + \mathbf{M}_\sigma^-(\lambda)\right]\left[\overline{\mathbf{z}}_+(\lambda) + \overline{\mathbf{z}}_-(\lambda)\right] + \mathbf{m}_\sigma^+(\lambda) + \mathbf{m}_\sigma^-(\lambda), \tag{2.5}$$

$$\widetilde{\mathbf{v}}_t^{m+}(s) = \mathbf{M}_p(s)\left[\widetilde{\mathbf{z}}_+(s+1) + \widetilde{\mathbf{z}}_-(s+1) - \widetilde{\delta\boldsymbol{\sigma}}_+(s+1)\right] + \mathbf{m}_p(s), \tag{2.6}$$

$$\widetilde{\mathbf{w}}_b^1(s) = \mathbf{M}_q(s)\left[\widetilde{\mathbf{z}}_-(s+1) - \widetilde{\mathbf{z}}_+(s+1) + \widetilde{\delta\boldsymbol{\sigma}}_-(s+1)\right] + \mathbf{m}_q(s), \tag{2.7}$$

where $\mathbf{M}_\sigma^+(\lambda)$, $\mathbf{m}_\sigma^+(\lambda)$, $\mathbf{M}_\sigma^-(\lambda)$, $\mathbf{m}_\sigma^-(\lambda)$ are even and odd components of $\mathbf{M}_\sigma(\lambda)$, $\mathbf{m}_\sigma(\lambda)$.

Now, we have the situation when all equations and interior and exterior boundary conditions of the problems are satisfied, except the first equations of the interfacial conditions (see (2.7) – (2.8) in [15]) which can be rewritten in the form:

$$\mathbf{u}^{(1)}\big|_{\Gamma_{m_+}^+}(x_1) = (\mathbf{v}^{(m_+)} + \delta\mathbf{u}_+)\big|_{\Gamma_{m_+}^+}(r), \qquad x_1 = r, \tag{2.8}$$

$$\mathbf{u}^{(1)}\big|_{\Gamma_0^-}(x_1) = (\mathbf{w}^{(1)} + \delta\mathbf{u}_-)\big|_{\Gamma_0^-}(r), \qquad x_1 = -r. \tag{2.9}$$

Let us represent vector-function $\mathbf{u}^{(1)}$ along boundary Γ_0 in (2.8), (2.9) by the inverse Fourier transform

$$\mathbf{u}^{(1)}(x_1)_{|\Gamma_0} = \mathcal{F}^{-1}[\overline{\mathbf{u}}_b^1](x_1),$$

or taking into account (2.5) and parity of vector-functions $\mathbf{M}_\sigma^\pm(\lambda)$, $\mathbf{m}_\sigma^\pm(\lambda)$:

$$\text{(2.10)}\quad \mathbf{u}^{(1)}_{|\Gamma_0} = 2\mathcal{F}_c^{-1}\left[\mathbf{M}_\sigma^+\overline{\mathbf{z}}_+ + \mathbf{M}_\sigma^-\overline{\mathbf{z}}_- + \mathbf{m}_\sigma^+\right](x_1) - 2i\mathcal{F}_s^{-1}\left[\mathbf{M}_\sigma^+\overline{\mathbf{z}}_- + \mathbf{M}_\sigma^-\overline{\mathbf{z}}_+ + \mathbf{m}_\sigma^-\right](x_1),$$

where

$$\mathcal{F}_c^{-1}[f(\lambda);\ \lambda \to x_1] \equiv \int_0^\infty f(\lambda)\cos(\lambda x_1)d\lambda,$$

$$\mathcal{F}_s^{-1}[f(\lambda);\ \lambda \to x_1] \equiv \int_0^\infty f(\lambda)\sin(\lambda x_1)d\lambda,$$

are the sine and cosine Fourier transforms [19].

Replacing in (2.8), (2.9) argument x_1 by r and $-r$, respectively, and applying the Mellin transform to both sides of the equations, we obtain

$$\text{(2.11)}\quad \begin{aligned} &2\mathcal{M}\mathcal{F}_c^{-1}\left[\mathbf{M}_\sigma^+\overline{\mathbf{z}}_+ + \mathbf{M}_\sigma^-\overline{\mathbf{z}}_- + \mathbf{m}_\sigma^+\right](s) \\ &\qquad - 2i\mathcal{M}\mathcal{F}_s^{-1}\left[\mathbf{M}_\sigma^+\overline{\mathbf{z}}_- + \mathbf{M}_\sigma^-\overline{\mathbf{z}}_+ + \mathbf{m}_\sigma^-\right](s) = \widetilde{\mathbf{v}}_t^{m+}(s) + \widetilde{\delta\mathbf{u}}_+(s), \\ &2\mathcal{M}\mathcal{F}_c^{-1}\left[\mathbf{M}_\sigma^+\overline{\mathbf{z}}_+ + \mathbf{M}_\sigma^-\overline{\mathbf{z}}_- + \mathbf{m}_\sigma^+\right](s) \\ &\qquad + 2i\mathcal{M}\mathcal{F}_s^{-1}\left[\mathbf{M}_\sigma^+\overline{\mathbf{z}}_- + \mathbf{M}_\sigma^-\overline{\mathbf{z}}_+ + \mathbf{m}_\sigma^-\right](s) = \widetilde{\mathbf{w}}_b^1(s) + \widetilde{\delta\mathbf{u}}_-(s). \end{aligned}$$

Substituting $\widetilde{\mathbf{v}}_t^{m+}$, $\widetilde{\mathbf{w}}_b^1$ from (2.2), (2.3) in these equations, they can be rewritten in the form:

$$\text{(2.12)}\quad \begin{aligned} &2\mathcal{M}\mathcal{F}_c^{-1}\left[\mathbf{M}_\sigma^+\overline{\mathbf{z}}_+ + \mathbf{M}_\sigma^-\overline{\mathbf{z}}_- + \mathbf{m}_\sigma^+\right](s) = \mathbf{M}_+(s)\widetilde{\mathbf{z}}_-(s+1) \\ &\qquad + \mathbf{M}_-(s)\widetilde{\mathbf{z}}_+(s+1) + \mathbf{d}_+(s), \\ &2i\mathcal{M}\mathcal{F}_s^{-1}\left[\mathbf{M}_\sigma^+\overline{\mathbf{z}}_- + \mathbf{M}_\sigma^-\overline{\mathbf{z}}_+ + \mathbf{m}_\sigma^-\right](s) = -\mathbf{M}_-(s)\widetilde{\mathbf{z}}_-(s+1) \\ &\qquad - \mathbf{M}_+(s)\widetilde{\mathbf{z}}_+(s+1) + \mathbf{d}_-(s). \end{aligned}$$

Here we denote matrix-functions $\mathbf{M}_\pm(s)$ and vector-functions $\mathbf{d}_\pm(s)$ as follows:

$$\begin{aligned} 2\mathbf{M}_\pm(s) &= \mathbf{M}_p(s) \pm \mathbf{M}_q(s), \\ 2\mathbf{d}_\pm(s) &= \mathbf{M}_q\widetilde{\delta\boldsymbol{\sigma}}_-(s+1) \mp \mathbf{M}_p\widetilde{\delta\boldsymbol{\sigma}}_+(s+1) + \mathbf{m}_q(s) \pm \mathbf{m}_p(s) + \widetilde{\delta\mathbf{u}}_-(s) \pm \widetilde{\delta\mathbf{u}}_+(s). \end{aligned}$$

As it was mentioned above, Eqs. (2.12) hold in the strip $0 < \Re s < \gamma_1$, in general. They constitute the system of two functional equations of vector-functions $\mathbf{z}_\pm(\lambda)$. We can conveniently consider the system in terms of vector-functions $\overline{\mathbf{z}}_\pm(\lambda)$. For this purpose, we proceed as in [12] and represent unknown vector-funtions $\widetilde{\mathbf{z}}_\pm(s+1)$ from the right-hand sides of the equations in the form:

$$\widetilde{\mathbf{z}}_+(s+1) = 2\mathcal{M}\mathcal{F}_c^{-1}[\overline{\mathbf{z}}_+](s+1), \qquad \widetilde{\mathbf{z}}_-(s+1) = -2i\mathcal{M}\mathcal{F}_s^{-1}[\overline{\mathbf{z}}_-](s+1). \tag{2.13}$$

In [12] it is shown that operators $\mathcal{M}\mathcal{F}_c^{-1}$, $\mathcal{M}\mathcal{F}_s^{-1}$ can be represented in the forms of products of one integral operator – the Mellin transform with a modified argument, and certain functions of the argument. Namely, for any $0 < \alpha, \beta < 1$ the identities hold

$$\begin{aligned} \mathcal{M}\mathcal{F}_c^{-1}[f_+](s) &= \Gamma(s)\cos\frac{\pi s}{2}\mathcal{M}[f_+](1-s), \qquad 0 < \Re s < \beta, \\ \mathcal{M}\mathcal{F}_s^{-1}[f_-](s) &= \Gamma(s)\sin\frac{\pi s}{2}\mathcal{M}[f_-](1-s), \qquad -\alpha < \Re s < \beta, \end{aligned} \tag{2.14}$$

where functions $f_\pm$ should be summable on $\mathbb{R}_+$, and satisfy the estimates:

$$\begin{aligned} f_\pm(\lambda) &= o(\lambda^{-1+\beta}), \qquad \lambda \to 0, \\ f_\pm(\lambda) &= o(\lambda^{-1-\alpha}), \qquad \lambda \to \infty. \end{aligned} \tag{2.15}$$

Besides, the first equation of (2.14) can be extended to a wider strip than that mentioned above. Really, it can be seen that the right-hand side of $(2.14)_1$ is an analytic function in the strip $-\alpha < \Re s < \beta$, except maybe one point $s = 0$. At this point it can have a simple pole, connected with the behaviour of the Gamma-function. From (2.15) and properties of the Fourier transform we can obtain

$$\mathcal{F}_c^{-1}[f_+] = \mathrm{Const} + \mathcal{O}(x^\alpha), \qquad x \to 0.$$

This fact makes it possible to extend analytically the left-hand side of equation $(2.14)_1$ to the whole strip $-\alpha < \Re s < \beta$.

As it follows from Corollary 1 of [15], terms $[\mathbf{M}_\sigma^+\overline{\mathbf{z}}_\pm + \mathbf{M}_\sigma^-\overline{\mathbf{z}}_\mp + \mathbf{m}_\sigma^\pm]$ satisfy exactly conditions (2.15) with $\alpha = \gamma_0$ and $\beta = \min\{1, \gamma_1\}$. Consequently, the left-hand sides of Eqs. (2.12) can be reduced to the form:

$$\begin{aligned} &2\mathcal{M}\mathcal{F}_c^{-1}\left[\mathbf{M}_\sigma^+\overline{\mathbf{z}}_+ + \mathbf{M}_\sigma^-\overline{\mathbf{z}}_- + \mathbf{m}_\sigma^+\right](s) \\ &\qquad = 2\Gamma(s)\cos\frac{\pi s}{2}\mathcal{M}\left[\mathbf{M}_\sigma^+\overline{\mathbf{z}}_+ + \mathbf{M}_\sigma^-\overline{\mathbf{z}}_- + \mathbf{m}_\sigma^+\right](1-s), \\ &2i\mathcal{M}\mathcal{F}_s^{-1}\left[\mathbf{M}_\sigma^+\overline{\mathbf{z}}_- + \mathbf{M}_\sigma^-\overline{\mathbf{z}}_+ + \mathbf{m}_\sigma^-\right](s) \\ &\qquad = 2i\Gamma(s)\sin\frac{\pi s}{2}\mathcal{M}\left[\mathbf{M}_\sigma^+\overline{\mathbf{z}}_- + \mathbf{M}_\sigma^-\overline{\mathbf{z}}_+ + \mathbf{m}_\sigma^-\right](1-s), \end{aligned} \tag{2.16}$$

where identities hold in the strip $-\gamma_0 < \Re s < \min\{1, \gamma_1\}$ and it is possible that a simple pole exists at point $s = 0$:

$$(2.17)\qquad 2\mathcal{M}\mathcal{F}_c^{-1}\left[\mathbf{M}_\sigma^+\mathbf{\overline{z}}_+ + \mathbf{M}_\sigma^-\mathbf{\overline{z}}_- + \mathbf{m}_\sigma^+\right](s) = \frac{1}{s}\mathbf{u}_* + \mathcal{O}(1), \qquad s \to 0.$$

Identities (2.14) are not directly adapted to reduce operators $\mathcal{M}\mathcal{F}_c^{-1}$, $\mathcal{M}\mathcal{F}_s^{-1}$ which are in the right-hand sides of (2.12). The reason is that the arguments of the operators are situated in another region, and the conditions as (2.15) are not satisfied for vector-functions $\mathbf{z}_\pm(\lambda)$. However, as it is shown in [12], relations similar to (2.12) hold in this case also. Namely:

$$(2.18)\qquad \begin{aligned} \mathcal{M}\mathcal{F}_c^{-1}[f_+](s+1) &= -\Gamma(s+1)\sin\frac{\pi s}{2}\mathcal{M}[f_+^*](-s) + \frac{\pi}{2}f_+(0)\Gamma(s+1), \\ \mathcal{M}\mathcal{F}_s^{-1}[f_-](s+1) &= \Gamma(s+1)\cos\frac{\pi s}{2}\mathcal{M}[f_-](-s), \end{aligned}$$

in the strip $-\alpha < \Re s < \beta$, when the following estimates are satisfied:

$$\begin{aligned} f_\pm(\lambda),\ \lambda\frac{\partial}{\partial\lambda}f_\pm(\lambda) &= o(\lambda^{-\alpha}), \qquad \lambda \to \infty, \\ f_-(\lambda),\ \lambda\frac{\partial}{\partial\lambda}f_\pm(\lambda) &= o(\lambda^{\beta}), \qquad \lambda \to 0, \\ f_+(\lambda) = f_+(0) &+ o(\lambda^{\beta}), \qquad \lambda \to 0. \end{aligned}$$

Here we choose function $f_+^*(\lambda)$ in the form

$$(2.19)\qquad f_+^*(\lambda) = f_+(\lambda) - f_+(0)(1+\lambda^2)^{-1},$$

so that the following relations are true:

$$f_+^*(\lambda) = \mathcal{O}(\lambda^{\min\{\beta,2\}}), \quad \lambda \to 0, \qquad f_+^*(\lambda) = \mathcal{O}(\lambda^{-\min\{\alpha,2\}}), \quad \lambda \to \infty.$$

It remains now to verify whether the necessary conditions for equality (2.18) are identical with the estimations presented in Corollary 1 [15] for vector-functions $\mathbf{z}_\pm(\lambda)$ with $\alpha = \gamma_0$, $\beta = \gamma_2$. Hence the system of functional equations (2.12) can be reduced to the following form:

$$(2.20)\qquad \begin{aligned} \widehat{\mathbf{Y}}(s) &= \mathbf{\Phi}(s)\widehat{\mathbf{Z}}(s) + \mathbf{F}(s), \qquad -\gamma_0 < \Re s < \gamma_\infty, \\ \mathbf{Y}(\lambda) &= \mathbf{L}(\lambda)\mathbf{Z}(\lambda) + \mathbf{l}(\lambda), \qquad 0 < \lambda < \infty, \end{aligned}$$

where we have introduced the notations:

$$\widehat{f}(s) = \widetilde{f}(-s) \equiv \mathcal{M}[f](-s), \qquad \gamma_\infty = \min\{1, \gamma_1, \gamma_2\},$$

and

$$\mathbf{Z}(\lambda) = \begin{pmatrix} \overline{\mathbf{z}}_+^*(\lambda) \\ i\overline{\mathbf{z}}_-(\lambda) \end{pmatrix}, \qquad \mathbf{F}(s) = \frac{\mu_1}{\Gamma(s)\sin\pi s} \begin{pmatrix} (\mathbf{d}_+(s) + \mathbf{M}_-\mathbf{d}_*(s))\sin\dfrac{\pi s}{2} \\ (\mathbf{d}_-(s) - \mathbf{M}_+\mathbf{d}_*(s))\cos\dfrac{\pi s}{2} \end{pmatrix},$$

$$\mathbf{L}(\lambda) = \mu_1\lambda \left(\begin{array}{c|c} \mathbf{M}_\sigma^+ & -i\mathbf{M}_\sigma^- \\ \hline i\mathbf{M}_\sigma^- & \mathbf{M}_\sigma^+ \end{array}\right), \qquad \mathbf{l}(\lambda) = \mu_1\lambda \begin{pmatrix} \mathbf{M}_\sigma^+\mathbf{z}_*^+(1+\lambda^2)^{-1} + \mathbf{m}_\sigma^+ \\ i\mathbf{M}_\sigma^-\mathbf{z}_*^+(1+\lambda^2)^{-1} + i\mathbf{m}_\sigma^- \end{pmatrix},$$

$$\mathbf{d}_*(s) = \mathbf{z}_*^+\pi\Gamma(s+1), \qquad \overline{\mathbf{z}}_+^*(\lambda) = \overline{\mathbf{z}}_+(\lambda) - \mathbf{z}_*^+(1+\lambda^2)^{-1},$$

$$\mathbf{\Phi}(s) = \mu_1 s \left(\begin{array}{c|c} -\mathbf{M}_-(s)\mathrm{tg}\dfrac{\pi s}{2} & -\mathbf{M}_+(s) \\ \hline \mathbf{M}_+(s) & \mathbf{M}_-(s)\mathrm{ctg}\dfrac{\pi s}{2} \end{array}\right).$$

We normalize the relations by μ_1 so that the vector-functions $\mathbf{Y}(\lambda)$, $\mathbf{Z}(\lambda)$ consisting of four components, and corresponding to Fourier transforms of the vectors of displacements and tractions along the interfacial boundary Γ_0, have similar dimensions. Besides, 4×4-matrix-function $\mathbf{\Phi}(s)$ has no physical dimensions, and consists of four blocks of 2×2-matrix-functions (as well as matrix-function $\mathbf{L}(\lambda)$). Value of the unknown constant vector $\mathbf{z}_*^+ = \overline{\mathbf{z}}_+(0)$ depends on the combination of the boundary conditions, and will be defined later. Note that vector $\mathbf{z}_+(x_1)$ can be easily calculated, and $\mathbf{z}_+(x_1) = \mathbf{z}_+^*(x_1) + \mathbf{z}_*^+\pi\exp(-|x_1|)$.

The form of the first equation in (2.20) makes it possible to consider vector-functions $\mathbf{Y}(\lambda)$, $\mathbf{Z}(\lambda)$ along the half-axis $\mathbb{R}_+$ only. Then the value of these vector-functions for negative magnitudes of λ can be found due to parity (the first two components are even functions, but the last ones are odd functions of λ).

As one can conclude from the *a priori* estimations given in Corollary 1 in [15], $\widehat{\mathbf{Y}}(s)$, $\widehat{\mathbf{Z}}(s)$ should be analytical in the strips $-\gamma_0 < \Re s < \gamma_1$ and $-\gamma_0 < \Re s < \gamma_2$, respectively. From Lemma 1 in [15] and definition (2.20) it follows that vector-function $\mathbf{F}(s)$ and matrix-function $\mathbf{\Phi}(s)$ are analytical in the strip $|\Re s| < 1$, at least, except maybe point $s = 0$, where a second degree pole can appear.

So, once the vector-functions $\mathbf{Z}(\lambda)$ (or $\mathbf{Y}(\lambda)$) will be obtained from systems (2.20), then all vectors of displacements and tractions $\mathbf{u}^{(i)}(x_1, x_2)$, $\mathbf{v}^{(j)}(r, \theta)$, $\mathbf{w}^{(k)}(r, \theta)$, $\boldsymbol{\sigma}_b^{(i)}(\lambda)$, $\mathbf{p}_t^{(j)}(s)$ and $\mathbf{q}_b^{(k)}(s)$ can be calculated by formulas (3.22)–(3.24), (4.13), (4.16) from [15] and the inverse Fourier and Mellin transforms.

3. Satisfaction of equilibrium conditions

The right-hand side of the first equation in (2.20) has, in general, a pole of the second degree at the zero point, if the boundary conditions along the exterior wedge surfaces are not of the first type, i.e. $\mathcal{J}^{\pm} \neq 1$ (see Lemma 2 in [15]), but the left-hand side should be always analytical at this point. Consequently, the corresponding additional conditions (see (4.22), (4.24) from [15]) for the unknown vector $\mathbf{z}_*^+$ should be satisfied. On the other hand, the right-hand side of the second equation of (2.20) has a singularity near the zero point or does not equal zero, depending on the exterior conditions along boundary Γ_n of the last layer ($\mathcal{J} = 2-5$, see Lemma 1 [15]), but the left-hand side tends to zero at $\lambda \to 0$. Then the respective additional conditions $(3.27)_2$ of [15] for vector $\mathbf{z}_*^+$ should be true. In the case, when the value of $\mathbf{z}_*^+$ should satisfy both of the mentioned conditions simultaneously, we have some relations connecting all exterior forces and tractions. They are the usual equilibrium conditions.

We shall not write here all the equilibrium equations depending on the possible combinations of the exterior boundary conditions, and consider only some of them as examples. Let us consider problem $(2,2,1)$ where the displacements are prescribed along the exterior boundary of the layers, but along the exterior wedge surfaces the tractions are given. Then from (4.22) of [15] we have $\mathbf{z}_*^+ = (2\pi)^{-1}\Xi_W$, and no equilibrium conditions are obtained. But if $\mathcal{J} = 5$, $\mathcal{J}^{\pm} = 2$ (the problem where the last layer is a half-plane, and along the exterior wedge surfaces the tractions are prescribed), then $\mathbf{z}_*^+ = (2\pi)^{-1}\Xi_W$, and $\mathbf{z}_*^+ = \Xi_L$ from $(3.27)_2$ in [15]. Consequently, the following equilibrium conditions follow:

$$2\pi\Xi_L - \Xi_W = 0. \tag{3.1}$$

Here $-2\pi\Xi_L$ and Ξ_W defined in Lemma 1 and Lemma 2 of [15] are the principal vectors of all exterior forces and tractions acting on the layered and wedge-shaped parts of the body, respectively. The same equilibrium equations occur in the case of problem $(2,2,2)$. If we consider problem $(3,3,5)$ (or $(3,3,2)$), then $\mathbf{z}_*^+ = \Xi_L$, and these conditions do not appear in (4.22) of [15], in general. However, when wedge-shaped parts $\Omega^{\pm}$ of the body contain the angles: $\theta_0^+ = \theta_{m_-}^- = -\pi/2$ (it corresponds, in particular, to the situation when the crack is perpendicular to the bimaterial interface), the relation $[0,1]\{\mathbf{z}_*^+ - (2\pi)^{-1}\Xi_W\} = 0$ follows from Eq. (4.24) of [15]. Hence the following equilibrium equation is obtained $[0,1]\{2\pi\Xi_L - \Xi_W\} = 0$. For the problem $(4,4,2)$ or $(4,4,5)$, in the case $\theta_0^+ = \theta_{m_-}^- = -\pi/2$, we obtain $[1,0]\{2\pi\Xi_L - \Xi_W\} = 0$. As the last example, we consider the problem $(2,2,3)$. Then from (3.27) and (4.22) in [15] it follows that $[0,1]\{\mathbf{z}_*^+ - \Xi_L\}$ and $\mathbf{z}_*^+ = (2\pi)^{-1}\Xi_W$. Hence the equilibrium equation $[0,1]\{2\pi\Xi_L - \Xi_W\} = 0$ should be true.

Nevertheless, there are combinations of the boundary conditions when constant vector $\mathbf{z}_*^+$ (or one of its components) can not be found (for example problems $(1,1,1)$, $(1,3,1)$ or problem $(3,4,1)$ with the restriction $\theta_0^+ - \theta_{m_-}^- \neq \pi/2$). In such cases we shall use additional conditions (3.29) from [15] for the displacements near the wedge tip to calculate the unknown value of vector $\mathbf{z}_*^+$.

Besides, if the stresses tend to zero at infinity in such a manner that the torque has a sense, (i.e. $\gamma_2 > 1$ in the *a priori* assumption (2.13) of [15] and $(3.27)_2$ of

[15] for the cases $\mathcal{J} = 2, 4$), then the additional torque balance condition

$$2\pi\Big\{\Upsilon_L - [0,1]y_n\Xi_L\Big\} = [1,0]\Big\{\Delta\mathbf{p}_*(1) - \Delta\mathbf{q}_*(1)\Big\} + \Upsilon_p + \Upsilon_q, \tag{3.2}$$

follows for problems $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$; $\mathcal{J}, \mathcal{J}^\pm = 2, 4$ from $(3.27)_2$ and (4.26) of [15].

4. Reducing the problems to systems of integral equations

Let us note that from Lemma 1 of [15] and definition (2.19) for component $\overline{\mathbf{z}}^*_+(\lambda)$, it follows:

$$\mathbf{L}(\lambda) + \mathbf{L}_\infty = \mathcal{O}(\lambda^{-2}), \qquad \lambda \to \infty, \tag{4.1}$$

$$\mathbf{Y}(\lambda) + \mathbf{L}_\infty\mathbf{Z}(\lambda) = \mathcal{O}(\lambda^{-2-\gamma_0}), \qquad \lambda \to \infty. \tag{4.2}$$

Here a 4×4-matrix $\mathbf{L}_\infty$ is the limit value of matrix-function $-\mathbf{L}(\lambda)$ at infinity $(\lambda \to \infty)$, and is constructed by the identity 2×2-matrix $\mathbf{I}$ and 2×2-matrix $\mathbf{E}_2$ defined in Eq. (3.9) of [15]:

$$\mathbf{L}_\infty = \mathbf{A}(1-\nu_1, 1/2-\nu_1), \tag{4.3}$$

$$\mathbf{A}(\xi,\eta) = \left(\begin{array}{c|c} \xi\mathbf{I} & -\eta\mathbf{E}_2 \\ \hline \eta\mathbf{E}_2 & \xi\mathbf{I} \end{array}\right), \qquad \mathbf{B}(\xi,\eta) = \left(\begin{array}{c|c} \xi\mathbf{E}_2 & -\eta\mathbf{I} \\ \hline \eta\mathbf{I} & \xi\mathbf{E}_2 \end{array}\right). \tag{4.4}$$

Let us note here that 4×4 block-matrices $\mathbf{A}(a,b)$ and $\mathbf{B}(c,d)$ construct a commutative algebra for arbitrary values of parameters a, b, c, d:

$$\begin{aligned}
\mathbf{A}(a,b)\mathbf{A}(c,d) &= \mathbf{A}(ac+bd, ad+bc),\\
\mathbf{B}(a,b)\mathbf{B}(c,d) &= \mathbf{A}(-ac-bd, ad+bc),\\
\mathbf{A}(a,b)\mathbf{B}(c,d) &= \mathbf{B}(ac-bd, ad-bc),\\
\mathbf{A}^{-1}(a,b) &= \mathbf{A}\left(\frac{a}{a^2-b^2}, \frac{-b}{a^2-b^2}\right),\\
\mathbf{B}^{-1}(a,b) &= \mathbf{B}\left(\frac{-a}{a^2-b^2}, \frac{b}{a^2-b^2}\right), \qquad a \neq b.
\end{aligned}$$

This fact as well as the matrix $\mathbf{B}$ itself will be used below.

Taking this fact into account, we rewrite Eqs. (2.20) in the form:

$$\begin{gathered}
\Big[\widehat{\mathbf{Y}} + \mathbf{L}_\infty\widehat{\mathbf{Z}}\Big](s) = \mathbf{\Phi}_*(s)\widehat{\mathbf{Z}}(s) + \mathbf{F}(s), \qquad \mathbf{\Phi}_*(s) = \mathbf{L}_\infty + \mathbf{\Phi}(s),\\
\mathbf{Y}(\lambda) = \mathbf{L}(\lambda)\mathbf{Z}(\lambda) + \mathbf{l}(\lambda).
\end{gathered} \tag{4.5}$$

From (4.2) one can conclude that the left-hand side of the first equation of (4.5) is an analytic vector–function in the strip $-2-\gamma_0 < \Re s < \gamma_\infty$, which is wider than the analyticity strips of $\widehat{\mathbf{Y}}(s)$, $\widehat{\mathbf{Z}}(s)$.

For further analysis we need the following estimations of matrix-functions $\mathbf{\Phi}_*$ at infinity which are true for arbitrary problems $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ under consideration:

$$\mathbf{\Phi}_*(s) = \mathbf{N}_1^0 + \mathbf{N}_2^0 \operatorname{tg}\frac{\pi s}{2} + \mathbf{N}_3^0(s), \qquad \mathbf{N}_3^0 = \mathcal{O}(e^{-\varepsilon|\Im s|}), \qquad |\Im s| \to \infty. \tag{4.6}$$

Here $\varepsilon = \min\{\phi_{m_+}^+, \phi_1^-\}$, but values of matrices $\mathbf{N}_1^0$, $\mathbf{N}_2^0$ can be calculated basing on the results of Lemma 2 from the previous paper [15]:

$$\begin{aligned} \mathbf{N}_1^0 &= \mathbf{L}_\infty + \frac{\mu_1}{2}\mathbf{A}(\chi_1^+ + \chi_1^-, -\chi_2^+ - \chi_2^-), \\ \mathbf{N}_2^0 &= \frac{\mu_1}{2}\mathbf{B}(\chi_2^+ - \chi_2^-, \chi_1^+ - \chi_1^-), \end{aligned} \tag{4.7}$$

where the constants were defined as follows:

$$\chi_1^+ = \frac{1-\nu_{m_+}^+}{\mu_{m_+}^+}, \qquad \chi_1^- = \frac{1-\nu_1^-}{\mu_1^-}, \qquad \chi_2^+ = \frac{1-2\nu_{m_+}^+}{2\mu_{m_+}^+}, \qquad \chi_2^- = \frac{1-2\nu_1^-}{2\mu_1^-}.$$

Using the fact that matrices $\mathbf{N}_1^0$, $\mathbf{N}_2^0$ belong to the commutative algebra, we can obtain:

$$\begin{aligned} \mathbf{\Phi}_*^{-1}(s) &= \mathbf{N}_1^\infty + \mathbf{N}_2^\infty \operatorname{tg}\frac{\pi s}{2} + \mathbf{N}_3^\infty(s), \\ \mathbf{N}_3^\infty(s) &= \mathcal{O}(e^{-\varepsilon|\Im s|}), \qquad |\Im s| \to \infty, \end{aligned} \tag{4.8}$$

where

$$\mathbf{N}_1^\infty = \left[(\mathbf{N}_1^0)^2 + (\mathbf{N}_2^0)^2\right]^{-1}\mathbf{N}_1^0, \qquad \mathbf{N}_2^\infty = -\left[(\mathbf{N}_1^0)^2 + (\mathbf{N}_2^0)^2\right]^{-1}\mathbf{N}_2^0. \tag{4.9}$$

Besides, the inverse matrix in these relations exists in view of (4.7).

From the mentioned Lemma 2, we can also find the following asynptotics near the zero point, depending on the external boundary conditions. Thus for problems $(1, 1, \mathcal{J})$, when $\mathcal{J} = 1-5$, we obtain

$$\begin{aligned} \mathbf{\Phi}_*(s) &= \mathbf{L}_\infty + \left(\begin{array}{c|c} \mathbf{0} & -s\mu_1\mathbf{M}_+^0 \\ \hline s\mu_1\mathbf{M}_+^0 & \dfrac{2\mu_1}{\pi}\mathbf{M}_-^0 \end{array}\right) + \mathcal{O}(s^2), \\ \mathbf{F}(s) &= \mathcal{O}(1), \qquad s \to 0, \end{aligned} \tag{4.10}$$

but for all remaining problems $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$, $\mathcal{J}^+, \mathcal{J}^- = 1-4$, $\mathcal{J}^+\mathcal{J}^- > 1$, $\mathcal{J} = 1-5$, the corresponding asymptotic expansions have the form:

$$
\Phi_*(s) = \mathbf{L}_\infty + \frac{1}{s^2}\left(\begin{array}{c|c} -\dfrac{\pi\mu_1 s^2}{2}\mathbf{M}_-^0 & -s\mu_1\mathbf{M}_+^0 \\ \hline s\mu_1\mathbf{M}_+^0 & \dfrac{2\mu_1}{\pi}\mathbf{M}_-^0 \end{array}\right) + \mathcal{O}(s),
$$

(4.11)

$$
\mathbf{F}(s) = \mathcal{O}(s^{-2}), \qquad s \to 0.
$$

Here matrices $\mathbf{M}_\pm^0$ are calculated on the basis of the limiting behaviour of matrix-functions $\mathbf{M}_\pm(s)$ near the zero point.

We shall not present here the exact formulas for asymptotics of matrix-functions $\Phi_*^{-1}(s)$ and vector $\mathbf{F}(s)$ near the zero point (basing on Lemma 1 and Lemma 2 of [15], some results are presented in the Appendix). Let us only note that for any problems under consideration the estimations hold true

$$
\Phi_*^{-1}(s)\mathbf{F}(s) = \mathcal{O}(1), \qquad s \to 0,
$$

in view of the additional conditions determining the value of constant vector $\mathbf{z}_*^+$ (see the third section). When $\mathbf{z}_*^+$ cannot be found from the *a priori* estimations, $\Phi_*^{-1}(s)\mathbf{F}(s)$ is analytical near the zero point for any values of $\mathbf{z}_*^+$.

REMARK 1. By direct verification, it can be concluded that function $\det\Phi_*(s)$ is not equal to zero near point $s = 0$ and $s \to \pm i\infty$ for problems under considerations. Taking into account the fact that vector-functions $\widehat{\mathbf{Y}}(it)$, $\widehat{\mathbf{Z}}(it)$ should be analytical for any $t \in \mathbb{R}$, from the first equation of (2.20) it would be expected that $\det\Phi_*(s)$ has no zero point along the imaginary axis. For special cases of the boundary conditions and for a small number of wedges, this fact can be directly verified. Unfortunately, the author has not succeeded in proving this fact in a general case (under arbitrary geometry and the boundary conditions of the problems). Nevertheless, we shall further assume that

$$
\Delta(it, \mathcal{J}^+, \mathcal{J}^-) = \det\Phi_*(it) \neq 0, \qquad t \in \overline{\mathbb{R}}. \tag{4.12}
$$

The reason to do this is the fact that $\Delta(s, \mathcal{J}^+, \mathcal{J}^-)$ is in turn the transcendental function which determines the eigenvalues of the solutions for the wedge-shaped media [6]. It can be calculated by applying the Mellin transform only to the similar problems under the additional assumption that the layered part of the domain is a homogeneous half-plane. In conclusion let us note that the method enabling effective calculation of zeros of this determinant is presented in [2], and it is based on the "sweep method" exposed in [8].

By $\vartheta_\infty(\mathcal{J}^+, \mathcal{J}^-) \in (0, 1)$ let us denote the real part of this zero of the function $\Delta(s, \mathcal{J}^+, \mathcal{J}^-)$ which is the nearest to the imaginary axis in half-plane $\Re s > 0$.

Using the results of Lemma 2 [15], it can be proved that $\mathbf{\Phi}_*(-s) = \mathbf{\Phi}_*(s)^\top$. Hence, matrix-function $\mathbf{\Phi}_*^{-1}(s)$ is analytical in the strip $|\Re s| < \vartheta_\infty(\mathcal{J}^+, \mathcal{J}^-)$.

Now we can rewrite the first equation of system (4.5) in an equivalent form inside the respective strip $|\Re s| < \vartheta_\infty(\mathcal{J}^+, \mathcal{J}^-)$:

$$\mathbf{\Phi}_*^{-1}(s)\left[\widehat{\mathbf{Y}} + \mathbf{L}_\infty \widehat{\mathbf{Z}}\right](s) - \mathbf{\Phi}_*^{-1}(s)\mathbf{F}(s) = \widehat{\mathbf{Z}}(s). \tag{4.13}$$

Note that the nearest pole of the left-hand side of (4.13) in the half-plane $\Re s < 0$ coincides with the first zero of the function $\Delta(s, \mathcal{J}^+, \mathcal{J}^-)$. This is because the vector-function $[\widehat{\mathbf{Y}} + \mathbf{L}_\infty \widehat{\mathbf{Z}}](s)$ is analytical in the strip $-2-\gamma_0 < \Re s < \gamma_\infty$ in view of the *a priori* estimations and (4.2), but $\mathbf{F}(s)$ is analytical in $|\Re s| < 1$, at least, except maybe the zero point (let us remind that vector-function $\mathbf{\Phi}_*^{-1}(s)\mathbf{F}(s)$ has no pole at this point). On the other hand, the vector-function $\widehat{\mathbf{Z}}(s)$ should be analytical in the strips $-\gamma_0 < \Re s < \gamma_\infty$. Consequently, we can conclude that the exponent determining the stress singularity is defined as follows:

$$\gamma_0 = \vartheta_\infty(\mathcal{J}^+, \mathcal{J}^-), \tag{4.14}$$

in the *a priori* estimations (2.13) in [15]. When this zero is simple and real, the principal asymptotic term of the solution of system (4.13) is of the form:

$$\mathbf{Z}(\lambda) = \mathbf{\Lambda}_\infty \lambda^{-\vartheta_\infty} + \mathcal{O}(\lambda^{-\vartheta_\infty^*}), \qquad \lambda \to \infty. \tag{4.15}$$

Here $\vartheta_\infty^* > \vartheta_\infty$ is the real part of the next zero of function $\Delta(s, \mathcal{J}^+, \mathcal{J}^-)$, but vector $\mathbf{\Lambda}_\infty$ can be calculated as *an integral measure* of solution $\mathbf{Z}(\lambda)$ from the relation:

$$\mathbf{\Lambda}_\infty = \lim_{s \to -\vartheta_\infty} (s + \vartheta_\infty)\mathbf{\Phi}_*^{-1}(s)\left\{[\widehat{\mathbf{Y}} + \mathbf{L}_\infty \widehat{\mathbf{Z}}](-\vartheta_\infty) - \mathbf{F}(-\vartheta_\infty)\right\}. \tag{4.16}$$

This fact is very important making it possible to calculate the constants in the principal term of the stress asymptotics near the corner tip.

Further on, we rewrite the system (4.13) taking into account the behaviour of matrix-function $\mathbf{\Phi}_*^{-1}(s)$ at infinity (4.8):

$$\mathbf{N}_3^\infty(s)\left[\widehat{\mathbf{Y}} + \mathbf{L}_\infty \widehat{\mathbf{Z}}\right](s) + \left[\mathbf{N}_1^\infty \widehat{\mathbf{Y}} + (\mathbf{N}_1^\infty \mathbf{L}_\infty - \mathbf{I})\widehat{\mathbf{Z}}\right](s) + \mathbf{N}_2^\infty \operatorname{tg}\frac{\pi s}{2}\left[\widehat{\mathbf{Y}} + \mathbf{L}_\infty \widehat{\mathbf{Z}}\right](s) = \mathbf{\Phi}_*^{-1}(s)\mathbf{F}(s).$$

Applying to this equation the inverse Mellin transform, we obtain

$$[\mathbf{N}_1^\infty \mathbf{Y} + (\mathbf{N}_1^\infty \mathbf{L}_\infty - \mathbf{I})\mathbf{Z}](\lambda) + \int_0^\infty \mathbf{\Psi}(\lambda, \xi)[\mathbf{Y} + \mathbf{L}_\infty \mathbf{Z}](\xi)d\xi - \frac{2}{\pi}\mathbf{N}_2^\infty \int_0^\infty [\mathbf{Y} + \mathbf{L}_\infty \mathbf{Z}](\xi)\frac{\lambda d\xi}{\lambda^2 - \xi^2} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \lambda^s \mathbf{\Phi}_*^{-1}(s)\mathbf{F}(s)\, ds, \tag{4.17}$$

where homogeneous matrix-function $\boldsymbol{\Psi}(\lambda,\xi)$ of degree -1 is defined from the relation:

$$\boldsymbol{\Psi}(\lambda,\xi) = \frac{1}{2\pi i\xi}\int\limits_{-i\infty}^{i\infty} \mathbf{N}_3^{\infty}(s)\left(\frac{\lambda}{\xi}\right)^s ds.$$

4.1. Systems of integral equations for problems $(\mathcal{J}^+,\mathcal{J}^-,1)$ and $(\mathcal{J}^+,\mathcal{J}^-,5)$ $(\mathcal{J}^\pm = 1-4)$

To analyze these problems, we can directly use the second equation of (2.20) to eliminate the vector-function $\mathbf{Y}(\lambda)$, because the matrix-function $\mathbf{L}(\lambda)$ and the vector-function $\mathbf{l}(\lambda)$ can be estimated in the following manner:

$$\begin{aligned} &\mathcal{J}=1: \quad \mathbf{L}(\lambda)=\mathcal{O}(\lambda), \quad \mathbf{l}(\lambda)=\mathcal{O}(\lambda), \quad \lambda\to 0,\\ &\mathcal{J}=5: \quad \mathbf{L}(\lambda)=\mathcal{O}(1), \quad \mathbf{l}(\lambda)=\mathcal{O}(\lambda), \quad \lambda\to 0. \end{aligned} \tag{4.18}$$

Here in the case of $\mathcal{J}=5$, we take into account the fact that the value of the unknown constant vector $\mathbf{z}_+^*$ appearing in $\mathbf{l}(\lambda)$ has been defined by (3.27) of [15].

Substituting $\mathbf{Y} = \mathbf{LZ} + \mathbf{l}$ in Eq. (4.17), we obtain a system of four integral equations:

$$\mathcal{Q}_Z(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})\mathbf{Z} = \mathcal{G}_Z, \qquad \mathcal{J}=1,5, \tag{4.19}$$

where

$$[\mathcal{Q}_Z\mathbf{u}](\lambda) = \mathbf{u}(\lambda) + \int\limits_0^\infty \mathbf{K}(\lambda)\boldsymbol{\Psi}(\lambda,\xi)[\mathbf{L}(\xi)+\mathbf{L}_\infty]\mathbf{u}(\xi)d\xi - \frac{2}{\pi}\int\limits_0^\infty \mathbf{K}(\lambda)\mathbf{N}_2^\infty[\mathbf{L}(\xi)+\mathbf{L}_\infty]\mathbf{u}(\xi)\frac{\lambda d\xi}{\lambda^2-\xi^2},$$

but matrix-function $\mathbf{K}(\lambda)$ and vector-function $\mathcal{G}_Z(\lambda)$ are calculated from the relations:

$$\mathbf{K}(\lambda) = \{\mathbf{N}_1^\infty[\mathbf{L}(\lambda)+\mathbf{L}_\infty]-\mathbf{I}\}^{-1},$$

$$\mathcal{G}_Z = \mathbf{K}(\lambda)\left\{\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty}\lambda^s\boldsymbol{\Phi}_*^{-1}(s)\mathbf{F}(s)\,ds - \mathbf{N}_1^\infty\mathbf{l}(\lambda) - \int\limits_0^\infty \boldsymbol{\Psi}(\lambda,\xi)\mathbf{l}(\xi)d\xi + \frac{2}{\pi}\int\limits_0^\infty \mathbf{N}_2^\infty\mathbf{l}(\xi)\frac{\lambda d\xi}{\lambda^2-\xi^2}\right\}.$$

4.2. Systems of integral equations for problems $(\mathcal{J}^+,\mathcal{J}^-,2)$ and $(\mathcal{J}^+,\mathcal{J}^-,5)$ $(\mathcal{J}^\pm = 1-4)$

Problems $(\mathcal{J}^+,\mathcal{J}^-,5)$ have been reduced to systems of integral equations (4.19), nevertheless we can do it in a different way together with problems $(\mathcal{J}^+,\mathcal{J}^-,2)$. Namely, let us rewrite the second equation of (2.20) in an equivalent form:

$$\mathbf{Z}(\lambda) = \mathbf{L}^{-1}(\lambda)\mathbf{Y}(\lambda) - \mathbf{n}(\lambda), \qquad \mathbf{n}(\lambda) = \mathbf{L}^{-1}(\lambda)\mathbf{l}(\lambda); \tag{4.20}$$

then the following estimations can be proved:

$$\begin{aligned} \mathcal{J} = 2: &\qquad \mathbf{L}^{-1}(\lambda) = \mathcal{O}(\lambda), &\qquad \mathbf{n}(\lambda) = \mathcal{O}(\lambda), &\qquad \lambda \to 0,\\ \mathcal{J} = 5: &\qquad \mathbf{L}^{-1}(\lambda) = \mathcal{O}(1), &\qquad \mathbf{n}(\lambda) = \mathcal{O}(\lambda), &\qquad \lambda \to 0, \end{aligned} \tag{4.21}$$

where the unknown constant vector $\mathbf{z}_+^*$ is defined in Eq. (3.27) of [15].

Substituting (4.20) into Eq. (4.17), we obtain a system of four integral equations:

$$\mathcal{Q}_Y(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})\mathbf{Y} = \mathcal{G}_Y, \qquad \mathcal{J} = 2, 5, \tag{4.22}$$

where

$$[\mathcal{Q}_Y\mathbf{u}](\lambda) = \mathbf{u}(\lambda) + \int\limits_0^\infty \mathbf{L}(\lambda)\mathbf{K}(\lambda)\mathbf{\Psi}(\lambda,\xi)[\mathbf{I} + \mathbf{L}_\infty\mathbf{L}^{-1}(\xi)]\mathbf{u}(\xi)d\xi \\ - \frac{2}{\pi}\int\limits_0^\infty \mathbf{L}(\lambda)\mathbf{K}(\lambda)\mathbf{N}_2^\infty[\mathbf{I} + \mathbf{L}_\infty\mathbf{L}^{-1}(\xi)]\mathbf{u}(\xi)\frac{\lambda d\xi}{\lambda^2 - \xi^2},$$

and the vector-function $\mathcal{G}_Y(\lambda)$ is calculated from the relations:

$$\mathcal{G}_Y = \mathbf{L}(\lambda)\mathbf{K}(\lambda) \times \left\{ -[\mathbf{N}_1^\infty\mathbf{L}_\infty - \mathbf{I}]\mathbf{n}(\lambda) + \frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty} \lambda^s\mathbf{\Phi}_*^{-1}(s)\mathbf{F}(s)\,ds \right. \\ \left. - \int\limits_0^\infty \mathbf{\Psi}(\lambda,\xi)\mathbf{L}_\infty\mathbf{n}(\xi)d\xi + \frac{2}{\pi}\int\limits_0^\infty \mathbf{N}_2^\infty\mathbf{L}_\infty\mathbf{n}(\xi)\frac{\lambda d\xi}{\lambda^2 - \xi^2} \right\}.$$

Unfortunately, systems (4.19) and (4.22) cannot be applied to solve problems $(\mathcal{J}^+,\mathcal{J}^-,3)$ and $(\mathcal{J}^+,\mathcal{J}^-,4)$. This is because the matrix-function $\mathbf{L}(\lambda)$ and vector-function $\mathbf{l}(\lambda)$ are not bounded at zero point in these cases (see Lemma 1 of [15] and definition (2.20)). Moreover, $\mathbf{L}(\lambda)$ is degenerate near that point so that $\mathbf{L}^{-1}(\lambda)$ is not bounded, as well as the vector-function $\mathbf{L}^{-1}(\lambda)\mathbf{l}(\lambda)$. Consequently, the corresponding integral operators will be not bounded.

4.3. Systems of integral equations for arbitrary problems

To investigate arbitrary problems let us introduce new auxiliary vector-function as a linear combination of vector-functions $\mathbf{Y}(\lambda)$ and $\mathbf{Z}(\lambda)$:

$$\mathbf{V}(\lambda) = \mathbf{N}_1^{\infty}\mathbf{Y}(\lambda) + [\mathbf{N}_1^{\infty}\mathbf{L}_{\infty} - \mathbf{I}]\mathbf{Z}(\lambda). \tag{4.23}$$

Taking into account the fact that all coefficients in (4.23) are constant matrices, all the *a priori* estimations (3.27) of [15] are true for matrix-function $\mathbf{V}(\lambda)$ as well. Then the vector-functions $\mathbf{Z}(\lambda)$, $\mathbf{Y}(\lambda)$ which should be found, are calculated from the relations:

$$\begin{aligned} \mathbf{Z}(\lambda) &= \mathbf{K}(\lambda)\Big\{\mathbf{V}(\lambda) - \mathbf{N}_1^{\infty}\mathbf{l}(\lambda)\Big\}, \\ \mathbf{Y}(\lambda) &= \mathbf{L}(\lambda)\mathbf{K}(\lambda)\Big\{\mathbf{V}(\lambda) - [\mathbf{N}_1^{\infty}\mathbf{L}_{\infty} - \mathbf{I}]\mathbf{l}(\lambda)\Big\}, \\ \mathbf{Y}(\lambda) + \mathbf{L}_{\infty}\mathbf{Z}(\lambda) &= \mathbf{K}(\lambda)\Big\{[\mathbf{L}(\lambda) + \mathbf{L}_{\infty}]\mathbf{V}(\lambda) - \mathbf{l}(\lambda)\Big\}, \end{aligned} \tag{4.24}$$

where matrix-function $\mathbf{K}(\lambda)$ has been defined above in (4.19). Here we use the fact that matrices $\mathbf{N}_1^{\infty}$, $\mathbf{L}(\lambda)$ and $\mathbf{L}_{\infty}$ belong to a commutative algebra. This is because they are symmetrical and have nonzero components along two main diagonals only (they are of the block form as $\mathbf{N}_1^{\infty}$, $\mathbf{L}_{\infty}$, see (4.3) and (4.9)).

It can be proved that for any problems $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ $(\mathcal{J}^{\pm} = 1-4,\ \mathcal{J} = 1-5)$:

$$\mathbf{K}(\lambda) = \mathcal{O}(1), \qquad \mathbf{L}(\lambda)\mathbf{K}(\lambda) = \mathcal{O}(1), \qquad \mathbf{K}(\lambda)\mathbf{l}(\lambda) = \mathcal{O}(\lambda), \qquad \lambda \to 0. \tag{4.25}$$

Then substituting (4.23) in Eq. (4.17), we obtain a system of four integral equations:

$$\mathcal{Q}_V(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})\mathbf{V} = \mathcal{G}_V, \qquad \mathcal{J} = 1-5, \tag{4.26}$$

where

$$\begin{aligned} [\mathcal{Q}_V\mathbf{u}](\lambda) = \mathbf{u}(\lambda) &+ \int_0^{\infty} \mathbf{\Psi}(\lambda,\xi)\mathbf{K}(\xi)[\mathbf{L}_{\infty} + \mathbf{L}(\xi)]\mathbf{u}(\xi)d\xi \\ &- \frac{2}{\pi}\int_0^{\infty} \mathbf{N}_2^{\infty}\mathbf{K}(\xi)[\mathbf{L}_{\infty} + \mathbf{L}(\xi)]\mathbf{u}(\xi)\frac{\lambda d\xi}{\lambda^2 - \xi^2}, \end{aligned}$$

and the vector-function $\mathcal{G}_V(\lambda)$ is calculated in the following manner:

$$\begin{aligned} \mathcal{G}_V(\lambda) = \Bigg\{ &\mathbf{K}(\lambda)\mathbf{l}(\lambda) + \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \lambda^s \mathbf{\Phi}_*^{-1}(s)\mathbf{F}(s)\,ds \\ &+ \int_0^{\infty} \mathbf{\Psi}(\lambda,\xi)\mathbf{K}(\xi)\mathbf{l}(\xi)d\xi - \frac{2}{\pi}\int_0^{\infty} \mathbf{N}_2^{\infty}\mathbf{K}(\xi)\mathbf{l}(\xi)\frac{\lambda d\xi}{\lambda^2 - \xi^2} \Bigg\}. \end{aligned}$$

Let us note in conclusion that the integral operators $\mathcal{Q}_Z$, $\mathcal{Q}_Y$ and $\mathcal{Q}_V$ from systems (4.19), (4.22) and (4.26) include not only fixed point singularities at zero and infinity points, but also the usual moving singularity with the kernel of the same type as $(\lambda-\xi)^{-1}$.

5. Analysis of the systems of integral equations

In this section, we investigate systems of singular integral equations (4.19), (4.22) and (4.26) obtained in the previous section. For this purpose, the results from [10, 11] are used without details.

Let $L^{p,\alpha,\beta}(\mathbb{R}_+)$ and $W^{p,\alpha,\beta}_{(l)}(\mathbb{R}_+)$ ($p \geq 1$, $\alpha,\beta \in \mathbb{R}$) be the Banach spaces of summable functions with the weight

$$\varrho_{\alpha,\beta}(\lambda) = \begin{cases} \lambda^{\alpha}, & 0<\lambda\leq 1, \\ \lambda^{\beta}, & 1<\lambda<\infty, \end{cases}$$

and norms of these spaces are defined as follows (see [11]):

$$\|u\|_{L^{p,\alpha,\beta}} = \left(\int\limits_0^{\infty} |u(\xi)|^p \varrho^p_{\alpha,\beta}(\xi)\frac{d\xi}{\xi}\right)^{1/p}, \qquad \|u\|_{W^{p,\alpha,\beta}_{(l)}} = \sum_{j=0}^{m} \|u^{(j)}\|_{L^{p,\alpha+j,\beta+j}}.$$

Here derivatives $u^{(j)}(\xi)$ are of the distributional sense.

By $\mathbf{L}^{p,\alpha,\beta}(\mathbb{R}_+) = \left[L^{p,\alpha,\beta}(\mathbb{R}_+)\right]^4$ $\left(\mathbf{W}^{p,\alpha,\beta}_{(l)}(\mathbb{R}_+) = [W^{p,\alpha,\beta}_{(m)}(\mathbb{R}_+)]^4\right)$ we denote Banach spaces of vector-functions with any standard matrix-norm [5].

Taking into account the results of Lemma 2 from [15], one can conclude that the inclusions hold true:

$$\Psi_{ij}(\cdot,1) \in W^{1,-\vartheta_\infty+\varepsilon,\vartheta_\infty-\varepsilon}_{(m)}(\mathbb{R}_+), \tag{5.1}$$

for any $\varepsilon>0$, $1\leq p<\infty$, $m\in\mathbb{N}$. Here $\mathbf{\Psi}$ is the matrix-function belonging to the kernels of the integral operators of systems (4.19), (4.22) and (4.26).

Let us note that the *a priori* estimates (3.27) of the paper [15] for solutions $\mathbf{Z}$, $\mathbf{Y}$ of the systems of integral equations under consideration can be rewritten in terms of the functional spaces in the following manner:

$$\mathbf{Z},\mathbf{Y},\mathbf{V} \in \mathbf{W}^{1,-\gamma_\infty+\varepsilon,\gamma_0-\varepsilon}_{(1)}(\mathbb{R}_+), \tag{5.2}$$

for an arbitrary $\varepsilon>0$. Here $\gamma_0 = \vartheta_\infty(\mathcal{J}^+,\mathcal{J}^-)$, but $\gamma_\infty = \gamma_\infty(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})$ is the unknown constant. The inclusion for the vector-function $\mathbf{V}$ follows immediately from (4.23).

In [11] it is shown that $\mathbf{L}^{p,\alpha,\beta}(\mathbb{R}_+)$ is a natural space in which solutions of such systems can be sought. Taking this fact into account, we shall assume a weaker condition in comparison with that in (5.2)

$$\mathbf{Z},\mathbf{Y},\mathbf{V}\in\mathbf{L}^{p,-\gamma_\infty+\varepsilon,\vartheta_\infty-\varepsilon}(\mathbb{R}_+). \tag{5.3}$$

REMARK 2. If systems of integral equations (4.19), (4.22) and (4.26) have solutions from the spaces (5.3), then by investigating smoothness of all matrix-functions from the kernels of the corresponding integral operators $\mathcal{Q}_Z$, $\mathcal{Q}_Y$ and $\mathcal{Q}_V$, and using similar line of reasoning as in Corollary 2 from [11], we can obtain inclusion (5.2). Therefore conditions (5.2) and (5.3) are equivalent in our cases.

5.1. Symbols of operators $\mathcal{Q}_Z$

As in the previous section, let us consider problems $(\mathcal{J}^+,\mathcal{J}^-,1)$ and $(\mathcal{J}^+,\mathcal{J}^-,5)$ in the cases $\mathcal{J}^\pm=1-4$. Then basing on Lemma 1 from the paper [15] it can be easily seen that the components of the matrix-functions $\mathbf{K}(\lambda)$, $\mathbf{L}(\lambda)+\mathbf{L}_\infty$ from the kernels of integral operators $\mathcal{Q}_Z$ belong to space $C^\infty(\mathbb{R}_+)$. Besides, the following estimates can be verified:

$$\begin{aligned}
&\mathbf{K}(\lambda)=\mathbf{K}_Z+\mathcal{O}(\lambda), &&\mathbf{L}(\lambda)+\mathbf{L}_\infty=\mathbf{L}_Z+\mathcal{O}(\lambda), &&\lambda\to 0,\\
&\mathbf{K}(\lambda)=-\mathbf{I}+\mathcal{O}(\lambda^{-2}), &&\mathbf{L}(\lambda)+\mathbf{L}_\infty=\mathcal{O}(\lambda^{-2}), &&\lambda\to\infty.
\end{aligned}$$

Here the values of the matrices $\mathbf{K}_Z=\mathbf{K}(0)$, $\mathbf{L}_Z=\mathbf{L}(0)+\mathbf{L}_\infty$ depend on the type of the boundary conditions along the exterior boundary of the layered part of the domain $(\mathcal{J}=1,5)$:

$$\begin{aligned}
&\mathcal{J}=1: &&\mathbf{L}_Z=\mathbf{L}_\infty, &&\mathbf{K}_Z=\left(\mathbf{N}_1^\infty\mathbf{L}_\infty-\mathbf{I}\right)^{-1},\\
&\mathcal{J}=5: &&\mathbf{L}_Z=\mathbf{L}_0+\mathbf{L}_\infty, &&\mathbf{K}_Z=\left(\mathbf{N}_1^\infty\mathbf{L}_Z-\mathbf{I}\right)^{-1},
\end{aligned} \tag{5.4}$$

where we have introduced the notation

$$\mathbf{L}_0=-\frac{\mu_1}{\mu_{n+1}}\mathbf{A}(1-\nu_{n+1},1/2-\nu_{n+1}), \tag{5.5}$$

with the matrix $\mathbf{A}(\xi,\eta)$ defined in (4.4). Basing on these estimations and the results from [10, 11], we can formulate the following

THEOREM 1.
Let $1\le p<\infty$, $\beta<\vartheta_\infty(\mathcal{J}^+,\mathcal{J}^-)$, $-\min\{1,\vartheta_\infty(\mathcal{J}^+,\mathcal{J}^-)\}<\alpha$, $\beta-\alpha\ge 0$, *then operators* $\mathcal{Q}_Z:\mathbf{L}^{p,\alpha,\beta}(\mathbb{R}_+)\to\mathbf{L}^{p,\alpha,\beta}(\mathbb{R}_+)$ *are bounded and its presymbols are calculated by the relations*:

$$\mathrm{Symb}_{\mathcal{Q}_Z}(t,\theta)=\{\mathbf{I}+\mathbf{K}_Z\mathbf{\Psi}_*(\alpha-it)\mathbf{L}_Z\}\,\frac{1+\theta}{2}+\mathbf{I}\frac{1-\theta}{2}\,, \tag{5.6}$$

where the matrix-functions $\mathbf{\Psi}_*(s)$ *are of the form:*

(5.7) $$\mathbf{\Psi}_*(s) = \widetilde{\mathbf{\Psi}}(s,1) + \mathbf{N}_2^\infty \mathrm{tg}\pi s/2 \equiv \mathbf{\Phi}_*^{-1}(s) - \mathbf{N}_1^\infty .$$

REMARK 3. Taking into account the form of (5.6), we can only investigate the presymbols for the value of $\theta = 1$. The corresponding matrix-functions will be denoted by

$$\mathbf{Q}_Z(t) = \mathrm{Symb}_{\mathcal{Q}_Z}(t,1) = \mathbf{I} + \mathbf{K}_Z \mathbf{\Psi}_*(\alpha - it)\mathbf{L}_Z .$$

They are not symbols of the operators because their limited values at the infinity point do not coincide ($\mathbf{Q}_Z(i\infty) \neq \mathbf{Q}_Z(-i\infty)$), in general.

REMARK 4. There exist cases when the presymbols represent the usual symbols of the operators. Such situations appear only if the elasticity parameters of two wedges $\Omega_{m_+}^+$, Ω_1^- (which are in contact with the layered part of domain Ω_L, see [15]) are similar:

$$\mu_{m_+}^+ = \mu_1^- , \qquad \nu_{m_+}^+ = \nu_1^- .$$

Then the matrices $\mathbf{N}_2^0$ (and consequently $\mathbf{N}_2^\infty$) from (4.7), (4.9) are equal to zero. In these cases, all integral operators $\mathcal{Q}_Z$, $\mathcal{Q}_Y$ and $\mathcal{Q}_V$ are singular operators with fixed point singularities only and do not contain the Cauchy-type singularities. What is interesting to note is that if the relations $\mu_1 = \mu_1^-$, $\nu_1 = \nu_1^-$ are valid additionally to those mentioned above in this Remark, then the following identities can be easily verified $\mathbf{N}_1^0 = 2(1-\nu_1)\mathbf{I}$, ($\mathbf{N}_2^0 = \mathbf{0}$, $\mathbf{N}_1^\infty = [2(1-\nu_1)]^{-1}\mathbf{I}$, $\mathbf{N}_2^\infty = \mathbf{0}$) in (4.7), (4.9).

REMARK 5. Let us note that the matrix $\mathbf{L}_Z$ is equivalent to zero if the additional conditions $\nu_1 = \nu_{n+1}$, $\mu_1 = \mu_{n+1}$ for problems $\mathcal{J} = 5$, $\mathcal{J}^\pm = 1-4$ are satisfied (the last layer is a half-space having the same elasticity parameters as the first layer). It means that in these cases the symbols of the corresponding operators are equal to identical matrix. Consequently, these operators are of Fredholm type (equal to the identical operator with an accuracy to compact ones). Hence, the corresponding systems of equations (4.19) have unique solutions in spaces $\mathbf{L}^{p,\alpha,\beta}(\mathbb{R}_+)$ for any $p \geq 1$, $\beta < \vartheta_\infty(\mathcal{J}^+,\mathcal{J}^-)$, $\beta - \alpha \geq 0$, $-\min\{1, \vartheta_\infty(\mathcal{J}^+,\mathcal{J}^-)\} < \alpha < 0$, and they can be calculated by projectional numerical methods, for example. Here we use the fact that the corresponding boundary value problems which are equivalent to systems of Eqs. (4.19), have unique solutions for such values of the parameters of spaces (see Remark 1 of [15]). Moreover, these solutions belong to spaces $\mathbf{W}_{(1)}^{p,\alpha,\beta}(\mathbb{R}_+)$. This fact follows from the differential properties of vector-functions $\mathbf{K}(\lambda)$, $\mathbf{L}(\lambda) + \mathbf{L}_\infty$ and their estimations near zero and infinity points (see Corollary 3 from [11]).

Let us note that the matrix $\mathbf{L}_Z$ is degenerate if and only if the assumptions of Remark 5 hold true. For the remaining cases, one can obtain:

(5.8) $$\mathbf{Q}_Z(t) = \mathbf{K}_Z\{\mathbf{\Phi}_*^{-1}(\alpha - it) - \mathbf{L}_Z^{-1}\}\mathbf{L}_Z ,$$

taking into account the fact that the matrix-functions $\mathbf{N}_1^\infty$, $\mathbf{L}(\lambda)$ and, consequently, $\mathbf{K}(\lambda)$ belong to the commutative algebra. Moreover, the matrix $\mathbf{K}_Z$ is nondegenerate for all the problems under consideration.

5.2. Symbols of operators $\mathcal{Q}_Y$

Now we consider problems $(\mathcal{J}^+, \mathcal{J}^-, 2)$ and $(\mathcal{J}^+, \mathcal{J}^-, 5)$ in the cases $\mathcal{J}^\pm = 1-4$. Then the components of the matrix-functions $\mathbf{L}(\lambda)\mathbf{K}(\lambda)$, $\mathbf{I}+\mathbf{L}_\infty\mathbf{L}^{-1}(\lambda)$ from the kernels of integral operators $\mathcal{Q}_Y$ belong to space $C^\infty(\mathbb{R}_+)$, and the following estimates can be verified:

$$\begin{aligned} &\mathbf{L}(\lambda)\mathbf{K}(\lambda) = \mathbf{K}_Y + \mathcal{O}(\lambda), && \mathbf{I}+\mathbf{L}_\infty\mathbf{L}^{-1}(\lambda) = \mathbf{L}_Y + \mathcal{O}(\lambda), && \lambda \to 0,\\ &\mathbf{L}(\lambda)\mathbf{K}(\lambda) = \mathbf{L}_\infty + \mathcal{O}(\lambda^{-2}), && \mathbf{I}+\mathbf{L}_\infty\mathbf{L}^{-1}(\lambda) = \mathcal{O}(\lambda^{-2}), && \lambda \to \infty. \end{aligned}$$

Here the values of the matrices $\mathbf{K}_Y$, $\mathbf{L}_Y$ depend on the type of the boundary conditions along the exterior boundary of the layered part of the domain ($\mathcal{J} = 2, 5$):

$$\begin{aligned} &\mathcal{J} = 2: \quad \mathbf{L}_Y = \mathbf{I}, && \mathbf{K}_Y = (\mathbf{N}_1^\infty)^{-1},\\ &\mathcal{J} = 5: \quad \mathbf{L}_Y = \mathbf{L}_0^{-1}\mathbf{L}_Z, && \mathbf{K}_Y = \mathbf{L}_0\mathbf{K}_Z\,, \end{aligned} \tag{5.9}$$

where the matrices $\mathbf{L}_Z$, $\mathbf{K}_Z$ and $\mathbf{L}_0$ have been defined in (5.4), (5.5).

As above, Theorem 1 holds true for operators $\mathcal{Q}_Y$, and their presymbols are of the form:

$$\operatorname{Symb}_{\mathcal{Q}_Y}(t,\theta) = \{\mathbf{I} + \mathbf{K}_Y\mathbf{\Psi}_*(\alpha - it)\mathbf{L}_Y\}\frac{1+\theta}{2} + \mathbf{I}\frac{1-\theta}{2}\,. \tag{5.10}$$

For the cases mentioned in Remark 5, operators $\mathcal{Q}_Y$ are equal to the identical ones with an accuracy to compacts operators, and all conclusions of this Remark are valid.

For the remaining cases ($\mathbf{L}_Y$ is nondegenerate), we can obtain:

$$\mathbf{Q}_Y(t) = \mathbf{K}_Y\{\mathbf{\Phi}_*^{-1}(\alpha - it) - \mathbf{L}_*\}\mathbf{L}_Y\,, \tag{5.11}$$

using a similar line of the reasoning as that used in (5.8). Here and in the sequel, matrix $\mathbf{L}_*$ will assume the limiting value:

$$\mathbf{L}_* = \lim_{\lambda\to 0}\,[\mathbf{L}(\lambda) + \mathbf{L}_\infty]^{-1}. \tag{5.12}$$

5.3. Symbols of operators $\mathcal{Q}_V$

Now we consider problems $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ in the cases $(\mathcal{J}^\pm = 1-4, \mathcal{J} = 1-5)$. Components of the matrix-function $\mathbf{K}(\lambda)[\mathbf{L}_\infty + \mathbf{L}(\lambda)]$ from the kernels of integral operators $\mathcal{Q}_V$ belong also to space $C^\infty(\mathbb{R}_+)$, and the following estimates can be verified:

$$\mathbf{K}(\lambda)[\mathbf{L}_\infty + \mathbf{L}(\lambda)] = \mathbf{K}_V + \mathcal{O}(\lambda), \quad \lambda \to 0,$$
$$\mathbf{K}(\lambda)[\mathbf{L}_\infty + \mathbf{L}(\lambda)] = \mathcal{O}(\lambda^{-2}), \qquad \lambda \to \infty.$$

Here the values of the matrix $\mathbf{K}_V$ depend on the type of the boundary conditions along the exterior boundary of the layered part of the domain ($\mathcal{J} = 1 - 5$) and are calculated by the formula:

(5.13) $$\mathbf{K}_V = \left(\mathbf{N}_1^\infty - \mathbf{L}_*\right)^{-1},$$

except the cases mentioned in the Remark 5, when $\mathbf{K}_V = \mathbf{0}$. Here tle matrix $\mathbf{L}_*$ has been defined by Eq. (5.12) and can be calculated from the relatons:

(5.14) $$\mathbf{L}_* = \begin{cases} \mathbf{L}_\infty^{-1}, & \mathcal{J} = 1, \\ \mathbf{0}, & \mathcal{J} = 2, \\ \mathbf{L}_3, & \mathcal{J} = 3, \\ \mathbf{L}_4, & \mathcal{J} = 4, \\ \mathbf{L}_Z^{-1}, & \mathcal{J} = 5, \end{cases}$$

where the matrices $\mathbf{L}_3, \mathbf{L}_4$ are calculated in the following manner:

$$\mathbf{L}_3 = \frac{1}{1-\nu_1}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{L}_4 = \frac{1}{1-\nu_1}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

As above, Theorem 1 holds true for operators $\mathcal{Q}_V$, and their presynbols are of the form:

(5.15) $$\mathrm{Symb}_{\mathcal{Q}_V}(t,\theta) = \{\mathbf{I} + \mathbf{\Psi}_*(\alpha - it)\mathbf{K}_V\}\frac{1+\theta}{2} + \mathbf{I}\frac{1-\theta}{2},$$

or, for all cases except those mentioned in Remark 5, we have:

(5.16) $$\mathbf{Q}_V(t) = \{\mathbf{\Phi}_*^{-1}(\alpha - it) - \mathbf{L}_*\}\mathbf{K}_V.$$

Let us note that all symbols $\mathbf{Q}_Z$, $\mathbf{Q}_Y$, $\mathbf{Q}_V$ contain the common matrix-function of the form:

(5.17) $$\mathbf{Q}(s, \mathcal{J}^+, \mathcal{J}^-, \mathcal{J}) = \mathbf{\Phi}_*^{-1}(s) - \mathbf{L}_*,$$

where the first term of this sum $\mathbf{\Phi}_*^{-1}(s) = \mathbf{\Phi}_*^{-1}(s, \mathcal{J}^+, \mathcal{J}^-)$ depends on the external boundary conditions along the wedge surfaces (defined by the values of

$\mathcal{J}^+, \mathcal{J}^- = 1 - 4$), but the second term $\mathbf{L}_* = \mathbf{L}_*(\mathcal{J})$ depends on the external boundary conditions along the last layer ($\mathcal{J} = 1 - 5$) and has been given by Eq. (5.14). Asymptotics of the matrix-functions $\mathbf{\Phi}_*^{-1}(s)$ are presented in (4.8) (in the neighbourhood of the infinity point), and in the Appendix (near the zero point). Besides, the following identities can be verified:

$$\begin{aligned}
\mathbf{Q}(s, \mathcal{J}^+, \mathcal{J}^-, 1) &= -\mathbf{\Phi}_*^{-1}(s)\mathbf{\Phi}(s)\mathbf{L}_*,\\
\mathbf{Q}(s, \mathcal{J}^+, \mathcal{J}^-, 2) &= \mathbf{\Phi}_*^{-1}(s),\\
\mathbf{Q}(s, \mathcal{J}^+, \mathcal{J}^-, \mathcal{J}) &= \mathbf{\Phi}_*^{-1}(s)[\mathbf{I} - \mathbf{\Phi}(s)\mathbf{L}_*], \qquad \mathcal{J} = 3, 4,\\
\mathbf{Q}(s, \mathcal{J}^+, \mathcal{J}^-, 5) &= \mathbf{\Phi}_*^{-1}(s)[\mathbf{L}(0) - \mathbf{\Phi}(s)]\mathbf{L}_*.
\end{aligned}$$

They will be useful during the investigation of the symbols. Moreover, the additional relations hold true for problems $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ when $\mathcal{J} = 3, 4$:

$$\det \mathbf{Q}(s, \mathcal{J}^+, \mathcal{J}^-, k+2) = \det \mathbf{\Phi}_*^{-1}(s) \frac{\mu_1^2 s^2}{(1-\nu_1)^2} \mathbf{m}_{pkk}(s)\mathbf{m}_{qkk}(s), \qquad k = 1, 2,$$

where $\mathbf{m}_{pkk}(s)$, $\mathbf{m}_{qkk}(s)$ are the diagonal elements of the 2×2 matrix-functions $\mathbf{M}_p(s)$, $\mathbf{M}_q(s)$ defined in Lemma 2 [15].

Let us note that for the right-hand sides of systems (4.19), (4.22) and (4.26), the following inclusions hold true:

$$\mathcal{G}_Z, \mathcal{G}_Y, \mathcal{G}_V \in \mathbf{W}_{(2)}^{p,\alpha,\vartheta_\infty - \varepsilon}(\mathbb{R}_+) \tag{5.19}$$

for any $\varepsilon > 0$, $1 \le p < \infty$, $\alpha > -\min\{1, \vartheta_\infty(\mathcal{J}^+, \mathcal{J}^-\}$.

Taking into account the volume of the paper, we can not present here a complete analysis of all the problems under consideration, because there exist fifty different combinations of the external boundary conditions. Nevertheless, the results presented above make it possible to investigate arbitrary boundary conditions.

Thus, let us now outline only the main points of such analysis.

5.4. Investigation of the symbols of the operators

First of all let us remind that in the case $\mathcal{J} = 5$, $\nu_1 = \nu_{n+1}$, $\mu_1 = \mu_{n+1}$ all systems of integral equations (4.19), (4.22) and (4.26) are of the Fredholm type with compact operators, and have unique solutions (see Remark 5). Below we do not consider these situations.

All remaining problems can be divided into two groups, depending on whether there exist zeros of the functions

$$\mathbf{q}(s, \mathcal{J}^+, \mathcal{J}^-, \mathcal{J}) = \det \mathbf{Q}(s, \mathcal{J}^+, \mathcal{J}^-, \mathcal{J}) \tag{5.20}$$

on the imaginary axis or not. We note here that these functions do not degenerate and are bounded at the infinity point ($s \to \pm\infty$). Moreover, in spite of the fact mentioned above that the limiting values of matrix-functions $\mathbf{Q}(s)$ are different for $s \to i\infty$ and $s \to -i\infty$ (Remark 3), it is easily proved that $\mathbf{q}(s)$ exhibits similar behaviour at the infinity point. This fact is a consequence of the structure of matrices $\mathbf{N}_1^\infty$, $\mathbf{N}_2^\infty$ and $\mathbf{L}_*$ (see (4.9) and (5.14)).

$$\lim_{s\to\pm i\infty} \mathbf{q}(s,\mathcal{J}^+,\mathcal{J}^-,\mathcal{J}) = \mathrm{Const}(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J}) \in \mathbb{R}.$$

Moreover, one can prove that this constant is not equal to zero. On the other hand, basing on Remark 1 and the structures of the symbols presented above, it can be proved that zeros of the symbols can appear at the point $s = 0$ only.

Let us consider such problems $(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})$ for which the corresponding functions $\mathbf{q}(s,\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})$ have no zeros on the imaginary axis.

PROPOSITION 1. There exist such values $\vartheta_0(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J}) > 0$ that

$$\operatorname{ind} \mathbf{q}(\alpha - it,\mathcal{J}^+,\mathcal{J}^-,\mathcal{J}) = 0, \qquad |\alpha| < \vartheta_0(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J}).$$

It is evident that $\vartheta_0(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})$ are the real parts of zeros (or poles) of functions $\mathbf{q}(s,\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})$ which are the nearest to the imaginary axis. Then, in order to prove this Proposition it is sufficient to note that the matrix-functions $\mathbf{Q}(s,\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})$ are Hermitian ones.

THEOREM 2.

Let $-\min\{1,\ \vartheta_0(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J}),\ \vartheta_\infty(\mathcal{J}^+,\mathcal{J}^-)\} < \alpha < \vartheta_0(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})$, $1 \le p < \infty$, $\beta < \vartheta_\infty(\mathcal{J}^+,\mathcal{J}^-)$, $\beta - \alpha \ge 0$, *then*

1. *Operators* $\mathcal{Q}_Z$, $\mathcal{Q}_Y$ *and* $\mathcal{Q}_V$ *in the spaces* $\mathbf{L}^{p,\alpha,\beta}(\mathbb{R}_+)$ *are normally solvable with the indices and partial indices equal to zero* ($\kappa = 0$, $\kappa_j = 0$, $j = 1,..,4$).

2. *Systems of equations* (4.19), (4.22) *and* (4.26) *have unique solutions from* $\mathbf{W}^{p,\alpha,\beta}_{(1)}(\mathbb{R}_+) \subset \mathbf{L}^{p,\alpha,\beta}(\mathbb{R}_+)$.

3. *Galerkin method for the systems of the equations with respect to the set of vector-functions* $\Theta_j^{\alpha,\beta} = \Theta_j\varrho^{-1}_{\alpha,\beta}$:

$$\Theta_j(\lambda) = \begin{cases} \sqrt{2}\,\lambda\,\Lambda_j(-2\ln\lambda), & 0<\lambda<1, \qquad j = 0,1,2,\dots, \\ 0, & 1<\lambda<\infty, \end{cases} \tag{5.21}$$
$$\Theta_j(\lambda) = -\Theta_{-j-1}(\lambda^{-1}), \qquad j = 0,1,2,\dots,$$

is valid in the Hilbert space $\mathbf{L}^{2,\alpha,\beta}(\mathbb{R}_+)$. *Here* $\Lambda_j(t)$, $(j = 0,1,2,...)$ *are normed Laguerre polynomials with vector-valued constants*, [4].

4. *The solutions of the systems have asymptotic expansion in the neighbourhood of zero in the form:* $\mathbf{U}(\lambda) = \mathcal{O}(\lambda^{\gamma_\infty})$, $\lambda \to 0$, $\gamma_\infty = \min\{1, \vartheta_0(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})\}$, *and at infinity point the asymptotics are defined by the relations* (4.15), (4.16).

The remaining parameters $\mathbf{u}_*$, $\gamma_\infty(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})$ in the definition of the class $\mathbf{LW}(\Omega)$ (see the second section of the paper [15]) can be obtained from Theorem 2. Solving numerically the corresponding systems of the equations, we can find the approximate solutions of the corresponding boundary value problems $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ and the asymptotics of their solutions in the neighbourhood of zero and infinity points. Note that the constants $\mathbf{u}_*$ Λ_∞, which play an important role in applications, can be calculated as *integral measure* of the approximate solutions of the systems (4.19), (4.22) and (4.26). Integral formulae for the constants in the asymptotics 4 similar to those in (4.15) and (4.16) can be obtained from functional equations (2.20) by passing to the limit $s \to \vartheta_0$.

Problems $(\mathcal{J}^+, \mathcal{J}^-, 5)$ for arbitrary $\mathcal{J}^\pm = 1-4$, $(1,1,2)$ and $(2,2,1)$ belong to this group of the problems under consideration. Let us note that the unknown vectors $\mathbf{z}_*^+$ are calculated by the relations $(3.27)_2$ and (4.22) in [15], so that the right-hand sides of the systems of the equations have been defined.

Now we consider situations when there exists a zero $s = 0$ of $2l$ multiplicity of the functions $\mathbf{q}(s, \mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ and investigate the corresponding group of problems $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$. Parity of the multiplicity of this zero follows immediately from the fact that the matrix-functions $\mathbf{Q}(s, \mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ are Hermitian ones.

PROPOSITION 2. There exist such values $\vartheta_0(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J}) > 0$ that

$$\kappa = -\mathrm{ind}\ \mathbf{q}(\alpha - it, \mathcal{J}^+, \mathcal{J}^-, \mathcal{J}) = \begin{cases} -l, & \vartheta_0 < \alpha < 0, \\ l, & 0 < \alpha < \vartheta_0. \end{cases}$$

It is evident that $\vartheta_0(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ are the real parts of the zeros (or poles) of functions $\mathbf{q}(s, \mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ nearest to the imaginary axis. Moreover, partial indices $(\kappa_j,\ j = 1-4)$ of the matrix-functions $\mathbf{Q}(\alpha - it, \mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ satisfy the relations: $-1 \le \kappa_j \le 0$ when $-\vartheta_0 < \alpha < 0$; and $0 \le \kappa_j \le 1$ when $0 < \alpha < \vartheta_0$.

For problems $(1,1,1)$, $(2,2,2)$, $(2,3,2)$, $(2,4,2)$, and for problems $(3,4,1)$, $(3,3,1)$, $(4,4,1)$ $(3,4,2)$, $(3,3,2)$, $(4,4,2)$ it can be shown that $l = 2$, except the last six problems when l can be equal to one under special assumptions on the geometry of the domain (see the corresponding formulas in the Appendix and Eq. (4.24) in [15]). For all remaining cases we have $l = 1$.

THEOREM 3.

Let $1 \le p < \infty$, $-\vartheta_0 < \alpha < 0$, $\beta < \vartheta_\infty$, $\beta - \alpha \ge 0$, *then:*

1. *Operators* $\mathcal{Q}_Z$, $\mathcal{Q}_Y$, $\mathcal{Q}_V$ *in spaces* $\mathbf{L}^{p,\alpha,\beta}(\mathbb{R}_+)$ *are normally solvable with the index* $\kappa = -l$.

2. *There are unique solutions of systems of the equations* (4.19), (4.22) *and* (4.26) *from* $\mathbf{W}^{p,\alpha,\beta}_{(1)}(\mathbb{R}_+)$.

3. *Asymptotic behaviour of the solutions near the zero point is* $\mathbf{U}(\lambda) = \mathcal{O}(\lambda^{\vartheta_0})$, $\lambda \to 0$, *but at the infinity point asymptotics has been defined by relation* (4.15).

Let us note that certain additional conditions are necessary for the solvability of the systems for some problems $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$. Such conditions are presented in $(3.27)_2$, (3.29), (4.22), (4.24) of [15] depending on the problem under consideration. For some cases they represent equilibrium conditions and are discussed in

Sec. 3. For example, for problem (2, 2, 2) the unknown vector $\mathbf{z}_*^+$ defined in (2.20) can be calculated from $(3.27)_2$ as well as from (4.22) in [15]. It leads us to the equilibrium equations (3.1).

For the other cases, when $\mathbf{z}_*^+$ can not be defined from the mentioned relations (for example (1,1,1)), the additional conditions follow from (3.29) of [15]. Thus the right-hand sides of the corresponding systems of equations can be represented in the form: $\mathcal{G}(\lambda) = \mathcal{G}_1(\lambda) + z_{*1}^+ \ \mathcal{G}_2(\lambda) + z_{*2}^+ \ \mathcal{G}_3(\lambda)$, where z_{*j}^+ $(j = 1, 2)$ are the components of the vector $\mathbf{z}_*^+$, but vector-functions $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ have no singularity in the neighbourhood of the zero point. Of course, this representation is true for all the problems, but the respective vector-functions are not bounded near point $\lambda = 0$.

The systems of equations under the conditions of Theorem 3 can not be directly solved as it is shown in point 3 of Theorem 2, and a regularization of the systems is necessary (see [4]). For this purpose, a method of factorization of matrix-functions of special forms proposed in [1] could be useful.

Thus, if the value of $\mathbf{z}_*^+$ is known and the corresponding equilibrium conditions are satisfied, then Theorem 2 holds true for the corresponding regularized systems of the equations. In the opposite cases, when the value of $\mathbf{z}_*^+$ (or one of the components) can not be calculated from relations $(3.27)_2$, (4.22), (4.24) in [15], the unique solutions of regularized systems of integral equations with the right-hand sides $\mathcal{G}_1(\lambda)$, $\mathcal{G}_2(\lambda)$ and $\mathcal{G}_3(\lambda)$ can be found. Then the value of the vector $\mathbf{z}_*^+$ is calculated from relations (3.29) in [15].

REMARK 6. As it has been shown in Sec. 3, the additional torque balance condition (3.2) should be true for problems $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ $(\mathcal{J}^\pm, \mathcal{J} = 2, 4)$. However, what is interesting to note is that this condition is not necessary for solvability of the corresponding systems of integral equations. It only plays an important role when the tractions and displacements along internal boundaries Γ_j $(j = 1, 2..., n)$ and $\Gamma_j^\pm$ $(j = 1, 2..., m_\pm)$ (between the layers and the wedges, respectively) are calculated by the recurrent relations (3.23), (3.24) and (4.13), (4.16) shown in the previous paper [15]. Namely, if the mentioned condition is not satisfied then the tangential component of the displacements is not bounded at infinity.

THEOREM 4.

Let $1 \le p < \infty$, $0 < \alpha < \vartheta_0$, $\beta < \vartheta_\infty$, $\beta - \alpha \ge 0, m \in \mathbb{N}$; *then*

1. *Operators* $\mathcal{Q}_Z$, $\mathcal{Q}_Y$, $\mathcal{Q}_V$ *in spaces* $\mathbf{L}^{p,\alpha,\beta}(\mathbb{R}_+)$ *are normally solvable with the index* $\kappa = l$.

2. *Homogeneous systems of equations* $\mathcal{Q}_{Z(Y,V)}\mathbf{U} = 0$ *have exactly* l *nontrivial solutions* $\mathbf{U}_j \in \mathbf{L}^{p,\alpha,\beta}(\mathbb{R}_+)$ $(j = 1, ..., l)$ *belonging to all spaces* $\mathbf{U}_j \in \bigcap \mathbf{W}^{p,\alpha,\beta}_{(m)}(\mathbb{R}_+)$.

3. *The asymptotic expansions of the solutions in the neighbourhood of zero are:*

$$\mathbf{U}_j(\lambda) = \mathbf{A}_j \ln\lambda + \mathbf{B}_j + \mathcal{O}(\lambda^{\vartheta_0}), \qquad \lambda \to 0,$$

where $\mathbf{A}_j$, $\mathbf{B}_j$ *(*$j = 1, ..., l$*) are certain vectors, but the relations* (4.15) *are satisfied at the infinity point.*

Let us note that nontrivial solutions of the respective homogeneous boundary problems play an important role in asymptotic methods [17]. However, in order to obtain them by the nontrivial solutions of the homogeneous integral equations, it is necessary to use generalized integral transforms and to justify all the obtained relations as it has been done in [16].

5.5. Analysis of the systems in the case of symmetrical domain

Now we consider such situations when the domain under consideration is symmetrical with respect to the axis $0X_2$. Besides, we assume that all mechanical parameters of the wedge parts of the domain have similar values to the symmetrical wedges. It is evident that in such cases the strain-stress state of the domain can be represented by symmetrical and antisymmetrical ones (Mode I and Mode II, respectively).

From Corollary 3 in [15] it follows that matrix-functions $\mathbf{\Phi}_*^{-1}(s)$ in the symbols of the integral operators have structures similar to those for the matrix $\mathbf{A}$ (see (4.3)) because of:

$$\mathbf{\Phi}(s) = \mu_1 s \begin{pmatrix} -f_2(s)\mathrm{tg}\dfrac{\pi s}{2} & 0 & 0 & -f_1(s) \\ 0 & -f_3(s)\mathrm{tg}\dfrac{\pi s}{2} & f_1(s) & 0 \\ 0 & f_1(s) & f_2(s)\mathrm{ctg}\dfrac{\pi s}{2} & 0 \\ -f_1(s) & 0 & 0 & f_3(s)\mathrm{ctg}\dfrac{\pi s}{2} \end{pmatrix},$$

but elements on the main diagonal are not identical. Moreover, in this case the conclusions of Remark 4 are true, so that $\mathbf{N}_2^0 = \mathbf{0}$, $\mathbf{N}_2^\infty = \mathbf{0}$ ($\chi_j^{\pm} = \chi_j^{\mp}$ in Eqs. (4.6), (4.8)), and consequently, the corresponding systems of integral equations contain only fixed point singularities.

For any 4×4 matrix $\mathbf{M}$ and vector $\mathbf{G}$ appearing in systems of the integral equations (4.19), (4.22) and (4.26), we introduce the following notations:

$$\begin{gathered} \mathbf{M}^{(1)} = \begin{pmatrix} m_{11} & m_{14} \\ m_{41} & m_{44} \end{pmatrix}, \quad \mathbf{M}^{(2)} = \begin{pmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{pmatrix}, \\ \mathbf{G}^{(1)} = \begin{pmatrix} g_1 \\ g_4 \end{pmatrix}, \qquad \mathbf{G}^{(2)} = \begin{pmatrix} g_2 \\ g_3 \end{pmatrix}. \end{gathered} \tag{5.22}$$

Then each of the systems of 4×4 integral equations corresponding to problems $(\mathcal{J}^+, \mathcal{J}^+, \mathcal{J})$ ($\mathcal{J}^+ = 1-4$, $\mathcal{J} = 1-5$) is divided into two systems of 2×2 integral equations of a similar form:

$$\mathcal{Q}^{(j)}_{Z(Y,V)}(\mathcal{J}^+,\mathcal{J}^+,\mathcal{J})\mathbf{U}^{(j)} = \mathcal{G}^{(j)}_{Z(Y,V)}, \tag{5.23}$$
$$\mathcal{J}^+ = 1-4, \qquad \mathcal{J} = 1-5, \qquad j = 1,2.$$

Symbols of the operators $\mathcal{Q}^{(j)}_{Z(Y,V)}$ in spaces $\mathbf{L}^{p,\alpha,\beta}(\mathbb{R}_+)$ of two-component vector-functions are defined by the corresponding 2×2 matrix-functions $\mathbf{Q}^{(j)}(s,\mathcal{J}^+,\mathcal{J}) = \mathbf{Q}^{(j)}(s,\mathcal{J}^+,\mathcal{J}^+,\mathcal{J})$ which are given by (5.17) and (5.22). Here superscript j is equal to 1 and 2 for the Mode I and Mode II, respectively.

Thus, we have forty different problems $(\mathcal{J}^+,\mathcal{J}^+,\mathcal{J})_j$ depending on the combinations of the external boundary problems (the values of $\mathcal{J}^+ = 1-4$, $\mathcal{J} = 1-5$) as well as the strain-stress state ($j = 1,2$ for Mode I and Mode II). As before, let us denote by $\vartheta^{(j)}_\infty(\mathcal{J}^+) > 0$ the real part of zero of the determinant of the 2×2 matrix-function $\mathbf{\Phi}^{(j)}_*(s)$ which is the nearest to the imaginary axis. But by $\vartheta^{(j)}_0(\mathcal{J}^+,\mathcal{J}) > 0$ we denote the real part of zero (pole) of the determinant of the 2×2 matrix-function $\mathbf{Q}^{(j)}(s)$ which is the nearest to the imaginary axis. Let us note that $\vartheta_0(\mathcal{J}^+,\mathcal{J}^+,\mathcal{J}) = \min\{\vartheta^{(j)}_0(\mathcal{J}^+,\mathcal{J})\}$, and $\vartheta_\infty(\mathcal{J}^+,\mathcal{J}^+) = \min\{\vartheta^{(j)}_\infty(\mathcal{J}^+)\}$.

All symmetrical problems under consideration are divided into two groups. For the first one, the indices of the determinants of matrix-functions $\mathbf{Q}^{(j)}(s)$ are equal to zero:

$$\kappa = -\operatorname{det}\mathbf{Q}^{(j)}(\alpha - it,\mathcal{J}^+,\mathcal{J}) = 0, \qquad |\alpha| < \vartheta^{(j)}_0(\mathcal{J}^+,\mathcal{J}).$$

For the second group, the relations hold true:

$$\kappa = -\operatorname{det}\mathbf{Q}^{(j)}(\alpha - it,\mathcal{J}^+,\mathcal{J}) = \begin{cases} 1, & 0<\alpha<\vartheta^{(j)}_0(\mathcal{J}^+,\mathcal{J}), \\ -1, & -\vartheta^{(j)}_0(\mathcal{J}^+,\mathcal{J}) < \alpha < 0. \end{cases}$$

The following problems are rated to the first group:

$$(\mathcal{J}^+,\mathcal{J}^+,5)_j,\ (2,2,1)_j,\ (1,1,2)_j, \qquad \mathcal{J}^+ = 1-4, \quad j=1,2;$$

$$(1,1,3)_2,\ (1,1,4)_1,\ (2,2,3)_2,\ (2,2,4)_1,\ (3,3,3)_2,\ (3,3,4)_1,\ (4,4,3)_2,\ (4,4,4)_1.$$

The second group consists of the problems:

$$(1,1,1)_j,\ (2,2,2)_j,\quad j=1,2; \qquad (1,1,3)_1,\ (1,1,4)_2,\ (2,2,3)_1,\ (2,2,4)_2,$$
$$(3,3,1)_2,\ (3,3,2)_2,\ (3,3,4)_2,\ (4,4,1)_1,\ (4,4,2)_1,\ (4,4,3)_1.$$

Finally, the following problems have symbols with nonzero indices ($|\kappa| = 1$), in general, except the special case of a crack terminating normally to the interface, when these problems are of zero indices (see (A.6) in the Appendix):

$$(3,3,1)_1,\ (3,3,2)_1,\ (3,3,3)_1,\ (4,4,1)_2,\ (4,4,2)_2,\ (4,4,4)_2.$$

Theorems formulated before are true for the symmetrical problems under consideration depending on the indices of the respective symbols. Moreover, for some problems for which the indices are not equal to zero in a general case, one of the two systems of integral equations can have a zero symbol! Hence, the corresponding systems of the integral equations (for Mode I or Mode II strain-stress state), should not be regularized.

6. Conclusions

We have considered all possible boundary value problems for different geometries of the domain as well as arbitrary combinations of the external boundary conditions. The corresponding problems have been reduced to systems of singular integral equations. Symbols of the corresponding operators have been presented and indices of the operators have been calculated.

Let us remember that in this paper we assume that the interfacial conditions between the first layer and the two nearest wedges (see (2.7), (2.8) in [15]) are characterized by given discontinuities of the displacements and tractions. Nevertheless, these conditions can be generalized, so that the tractions will be proportional to the jump of the displacements. Such problems can also be solved by the presented method. However, as it has been shown in [13] for the case of the Poisson equation in a similar domain, such investigation is a little different from that presented above, and the symbols of the corresponding integral operators are degenerate, in general.

Appendix

Here we present some estimations for matrix-functions $\mathbf{\Phi}_*^{-1}(s,\mathcal{J}^+,\mathcal{J}^-)$ depending on the values of $\mathcal{J}^\pm = 1-4$.

For problems $(1,1,\mathcal{J})$, $(\mathcal{J}=1-5)$ the following relations can be verified basing on the results obtained from Lemma 2 in [15]:

$$\mathbf{\Phi}_*^{-1}(s) = \mathbf{L}_\infty^{-1} + \left(\begin{array}{c|c} \mathbf{E}_2\mathbf{X} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{X}\end{array}\right)\left(\begin{array}{c|c} -\frac{b_\infty^2}{a_\infty}\mathbf{I} & b_\infty\mathbf{I} \\ \hline -b_\infty\mathbf{I} & a_\infty\mathbf{I}\end{array}\right)\left(\begin{array}{c|c} \mathbf{E}_2 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I}\end{array}\right) + \mathcal{O}(s), \qquad s\to 0, \tag{A.1}$$

$$\mathbf{X} = \frac{1}{b_\infty^2 - a_\infty^2}\mathbf{I} + \frac{\pi}{2\mu_1 a_\infty}\left[\frac{\pi(a_\infty^2+b_\infty^2)}{2\mu_1 a_\infty}\mathbf{I} + \mathbf{M}_-^0\right]^{-1},$$

$$a_\infty = 1-\nu_1, \qquad b_\infty = 1/2 - \nu_1.$$

For the remaining problems $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ $(\mathcal{J}^+\mathcal{J}^- > 1,\ \mathcal{J} = 1-5)$, under the additional assumption $\det \mathbf{M}_+^0 \neq 0$, we can obtain:

$$\mathbf{\Phi}_*^{-1}(s, \mathcal{J}^+, \mathcal{J}^-) = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & s\mathbf{I} \end{pmatrix} \left[\begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} + \mathcal{O}(s) \right] \times \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & s\mathbf{I} \end{pmatrix}, \qquad s \to 0,$$

$$\text{(A.2)} \qquad \mathbf{A}_{11} = \left[a_\infty \mathbf{I} - \frac{\mu_1 \pi}{2} \mathbf{M}_+^0 + \frac{\mu_1 \pi}{2} \mathbf{M}_-^0 (\mathbf{M}_+^0)^{-1} \mathbf{M}_-^0 \right]^{-1},$$

$$\mathbf{A}_{12} = \frac{\pi}{2} \mathbf{A}_{11} \mathbf{M}_-^0 (\mathbf{M}_+^0)^{-1}, \qquad \mathbf{A}_{21} = -\frac{\pi}{2} (\mathbf{M}_+^0)^{-1} \mathbf{M}_-^0 \mathbf{A}_{11},$$

$$\mathbf{A}_{22} = \frac{\pi}{2\mu_1} \left[(\mathbf{M}_+^0)^{-1} - \frac{\mu_1 \pi}{2} (\mathbf{M}_+^0)^{-1} \mathbf{M}_-^0 \mathbf{A}_{11} \mathbf{M}_-^0 (\mathbf{M}_+^0)^{-1} \right].$$

In some cases these relations can be simplified. Namely, for problems $(1, 2, \mathcal{J})$, $(\mathcal{J} = 1-5)$ the identities $\mathbf{M}_-^0 = \mathbf{M}_+^0$ hold true, and the matrices are still nondegenerate. Hence we obtain:

$$\mathbf{A}_{11} = \frac{1}{a_\infty} \mathbf{I}, \qquad \mathbf{A}_{12} = -\mathbf{A}_{21} = \frac{\pi}{2a_\infty} \mathbf{I},$$

$$\mathbf{A}_{22} = \frac{\pi}{2\mu_1 a_\infty} \left[a_\infty (\mathbf{M}_+^0)^{-1} - \frac{\mu_1 \pi}{2} \mathbf{I} \right].$$

Besides, for problems $(\mathcal{J}^+, \mathcal{J}^+, \mathcal{J})$ $(\mathcal{J}^+ = 3, 4)$ when $\theta_0^+ \neq \theta_{m_-}^-$, and for problems $(3, 4, \mathcal{J})$ when $\theta_0^+ - \theta_{m_-}^- \neq \pi/2$, the following equation can be verified: $(\mathbf{M}_+^0)^{-1}\mathbf{M}_-^0 = \mathbf{M}_-^0(\mathbf{M}_+^0)^{-1}$.

Finally, let us present the relations when matrices $\mathbf{M}_\pm^0$ are degenerate. Such situations appear for the problems:

a) $(1, 3, \mathcal{J})$, $(1, 4, \mathcal{J})$ $(\mathcal{J} = 1-5)$, where $\mathbf{M}_-^0 = \mathbf{M}_+^0 = \mathbf{B}_0$;

b) $(3, 3, \mathcal{J})$, $(4, 4, \mathcal{J})$ $(\mathcal{J} = 1-5)$ under the additional geometrical conditions $\theta_0^+ = \theta_{m_-}^-$, what leads to the relations: $\mathbf{M}_+^0 = y\mathbf{B}_0$, $\mathbf{M}_-^0 = \mathbf{B}_0$;

c) $(3, 4, \mathcal{J})$ $(\mathcal{J} = 1-5)$ under conditions $\theta_0^+ = \theta_{m_-}^- + \pi/2$, when the identities hold true: $\mathbf{M}_-^0 = y\mathbf{B}_0$, $\mathbf{M}_+^0 = \mathbf{B}_0$.

Here certain constants y $(|y| \leq 1)$ and the corresponding matrices $\mathbf{B}_0$ $(\mathbf{B}_0 \neq \mathbf{0}$, $\det \mathbf{B}_0 = 0)$ are calculated basing on the results of Lemma 2, so that

$$\mathbf{B}_0 = \begin{pmatrix} b_{11}^2 & b_{11} b_{12} \\ b_{11} b_{12} & b_{12}^2 \end{pmatrix}.$$

Then for the mentioned three cases, asymptotic behaviour of matrix-functions $\mathbf{\Phi}_*^{-1}(s)$ is of the form:

$$\mathbf{\Phi}_*^{-1}(s) = d\begin{pmatrix} 4a_\infty^2\xi[b_{11}^2+b_{12}^2]\mathbf{I}+(c-4b_\infty^2\xi)\mathbf{C} & 4a_\infty b_\infty\xi\mathbf{D} \\ 4a_\infty b_\infty\xi\mathbf{D}^\top & (c+4a_\infty^2\xi)\mathbf{C}\end{pmatrix} + \mathcal{O}(s), \qquad s\to 0, \tag{A.3}$$

$$\mathbf{C} = \begin{pmatrix} b_{12}^2 & -b_{11}b_{12} \\ -b_{11}b_{12} & b_{11}^2\end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} b_{11}b_{12} & -b_{11}^2 \\ b_{12}^2 & -b_{11}b_{12}\end{pmatrix},$$

where

$$c = a_\infty\mu_1\pi(b_{11}^2+b_{12}^2)(\eta^2-\xi^2), \qquad d = a_\infty^{-1}(b_{11}^2+b_{12}^2)^{-1}[4\xi(a_\infty^2-b_\infty^2)+c]^{-1}.$$

In these relations the values of parameters ξ, η are defined depending on the three situations mentioned above:

a) $\leftrightarrow$ $\xi=\eta=1$; **b)** $\leftrightarrow$ $\xi=y,\ \eta=1$; **c)** $\leftrightarrow$ $\xi=1,\ \eta=y$.

A.1. The cases when the domain is symmetrical with respect to the $0X_2$ axis

In this part of the Appendix the respective relations are presented for the situations investigated in the last subsection of Sec. 5.

Thus for problems $(1,1,\mathcal{J})_j$, the matrix $\mathbf{M}_+^0$ (and, consequently, the matrix $\mathbf{X}$) are diagonal matrices with elements m_1, m_2; then we can calculate from (A.1) and Lemma 2 of [15] that:

$$\left[\mathbf{\Phi}_*^{(j)}(s,1,1)\right]^{-1} = \frac{1}{a_\infty}\begin{pmatrix} 1+y_{3-j}b_\infty^2 & (-1)^j a_\infty b_\infty y_{3-j} \\ (-1)^j a_\infty b_\infty y_{3-j} & y_{3-j}a_\infty^2\end{pmatrix},$$
$$y_j = \frac{\pi}{\pi(a_\infty^2+b_\infty^2)+2\mu_1 a_\infty m_j}, \qquad m_1,\ m_2>0. \tag{A.4}$$

For the remaining problems $(\mathcal{J}^+,\mathcal{J}^+,\mathcal{J})_j$ $(\mathcal{J}^+=2,3,4,\ \mathcal{J}=1-5,\ j=1,2)$ for which Eqs. (A.2) are valid (except the two cases considered in (A.3)), it can be shown that the matrices $\mathbf{M}_\pm^0$ are of the form:

$$\mathbf{M}_+^0 = \begin{pmatrix} 0 & m_1 \\ -m_1 & 0\end{pmatrix}, \qquad \mathbf{M}_-^0 = \begin{pmatrix} m_2 & 0 \\ 0 & m_3\end{pmatrix}, \qquad m_2, m_3<0.$$

Then

$$\left[\mathbf{\Phi}_*^{(1)}(s)\right]^{-1} = \begin{pmatrix}1&0\\0&s\end{pmatrix}\left[\begin{pmatrix} A & \dfrac{\pi m_1}{2m_3}A \\ \dfrac{\pi m_1}{2m_3}A & \dfrac{1}{m_3}+\dfrac{\mu_1\pi m_1^2}{2m_3^2}A\end{pmatrix}+\mathcal{O}(s)\right]\begin{pmatrix}1&0\\0&s\end{pmatrix},$$
$$\left[\mathbf{\Phi}_*^{(2)}(s)\right]^{-1} = \begin{pmatrix}1&0\\0&s\end{pmatrix}\left[\begin{pmatrix} B & -\dfrac{\pi m_1}{2m_2}B \\ -\dfrac{\pi m_1}{2m_2}B & \dfrac{1}{m_2}+\dfrac{\mu_1\pi m_1^2}{2m_2^2}B\end{pmatrix}+\mathcal{O}(s)\right]\begin{pmatrix}1&0\\0&s\end{pmatrix}. \tag{A.5}$$

$$A^{-1} = a_\infty - \frac{\mu_1 \pi}{m_3}(m_2 m_3 + m_1^2), \qquad B^{-1} = a_\infty - \frac{\mu_1 \pi}{m_2}(m_2 m_3 + m_1^2).$$

Besides, for problems $(3, 3, \mathcal{J})$ and $(4, 4, \mathcal{J})$ when $\theta_0^+ \neq \theta_{m_-}^-$ it can be proved that $m_2 m_3 + m_1^2 = 0$.

Finally, only one of the three last cases, when matrices $\mathbf{M}_0^\pm$ are degenerate, can be realized. Namely, for problems $(3,3,\mathcal{J})$ and $(4,4,\mathcal{J})$ when $\theta_0^+ = \theta_{m_-}^- = -\pi/2$, it can be found that $b_{12} = 0$ and $b_{11} = 0$, respectively, and then

(A.6)
$$\begin{aligned}
\left[\mathbf{\Phi}_*^{(1)}(s,3,3)\right]^{-1} &= d b_{11}^2 \begin{pmatrix} 4a_\infty^2 & -4a_\infty b_\infty y \\ -4a_\infty b_\infty y & c + 4a_\infty^2 y \end{pmatrix} + \mathcal{O}(s), \qquad s \to 0,\\
\left[\mathbf{\Phi}_*^{(2)}(s,3,3)\right]^{-1} &= d b_{11}^2 \begin{pmatrix} 4a_\infty^2 + c - 4b_\infty^2 y & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{O}(s), \qquad s \to 0,\\
\left[\mathbf{\Phi}_*^{(1)}(s,4,4)\right]^{-1} &= d b_{12}^2 \begin{pmatrix} 4a_\infty^2 + c - 4b_\infty^2 y & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{O}(s), \qquad s \to 0,\\
\left[\mathbf{\Phi}_*^{(2)}(s,4,4)\right]^{-1} &= d b_{12}^2 \begin{pmatrix} 4a_\infty^2 & 4a_\infty b_\infty y \\ 4a_\infty b_\infty y & c + 4a_\infty^2 y \end{pmatrix} + \mathcal{O}(s), \qquad s \to 0.
\end{aligned}$$

In conclusion let us note that values of the matrices $\mathbf{M}_\pm^0$ in asymptotics (4.10), (4.11) as well as all other constants used in the Appendix are calculated basing on the results of Lemma 2 in [15].

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