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Treating a singular case for a motion of rigid body in a Newtonian field of force

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THIS PAPER presents a rotational motion of a rigid body about a fixed point in a Newtonian force field for a singular value of the natural frequency $\omega = 3$. Such a singularity appears in [3] and has been never studied in full generality. Poincaré's small parameter method [4] is applied to investigate analytical periodic solutions, with non-zero basic amplitudes, for equations of motion of the body. A geometric interpretation of motion is given using Euler's angles to describe the orientation of the body at any instant of time.

1. Introduction

CONSIDER A RIGID BODY of mass (M), with one fixed point; its ellipsoid of inertia is arbitrary, and its center of mass does not necessarily coincide with the fixed point. Assume that $0x$, $0y$ and $0z$ represent the principal axes of the ellipsoid of inertia (fixed frame of the body), and $0X$, $0Y$ and $0Z$ represent the fixed frame in space. Assume A, B and C to be the principal moments of inertia, x_0, y_0 and z_0 to be the coordinates of the center of mass in the moving coordinate system, γ, γ' and γ'' to be the direction cosines of the vertical Z-axis, directed downwards and p, q and r to be the projections of the angular velocity vector of the body on the principal axes of inertia. It is taken into consideration that at the initial instant of time, the body rotates about z-axis with a high angular velocity r_0 and that this axis makes an angle $\theta_0 \neq m\pi/2$ $(m = 0, 1, 2, \ldots)$ with Z-axis. The six nonlinear differential equations of motion and their three first integrals are reduced to the following system of two degrees of freedom and one first integral [3]

$$\begin{aligned} \ddot{p}_2 + 9p_2 &= \mu^2 F(p_2, \dot{p}_2, \gamma_2, \dot{\gamma}_2, \mu), \\ \ddot{\gamma}_2 + \gamma_2 &= \mu^2 \Phi(p_2, \dot{p}_2, \gamma_2, \dot{\gamma}_2, \mu); \end{aligned} \tag{1.1}$$

$$\begin{aligned} \gamma_0''^{-2} - 1 = \gamma_2^2 + \dot{\gamma}_2^2 + 2\mu(\nu p_2\gamma_2 + \nu_2\dot{p}_2\dot{\gamma}_2 + s_{21}) \\ + \mu^2 \left[\nu_2^2\dot{p}_2^2 - 2\dot{\gamma}_2\left(e_2A_1^{-1}\dot{\gamma}_2 + A_1^{-1}\dot{p}_2 s_{21} + \frac{1}{2}\dot{\gamma}_2 s_{11} - y_0' a^{-1}A_1^{-1}\right)\right. \\ \left. + \nu^2 p_2^2 + s_{21}^2 + 2\left(s_{22} - \frac{1}{2}s_{11}\right)\right] + \mu^3(\ldots), \end{aligned} \tag{1.2}$$

where

$$
\begin{aligned}
&F = F_2 + \mu F_3 + \ldots, && \Phi = \Phi_2 + \mu\Phi_3 + \ldots,\\
&F_2 = f_2 + 8\nu e_1 p_2, && \Phi_2 = \phi_2 - 8\nu(e + e_1\gamma_2),\\
&F_3 = f_3 - e_1\phi_2 + 8\nu e_1(e + e_1\gamma_2), && \Phi_3 = \phi_3 - \nu f_2 - 8\nu^2 e_1 p_2,
\end{aligned}
$$

$$
\begin{aligned}
f_2 &= A_1 b^{-1} x_0' s_{21} - 9p_2 s_{11} + C_1 A_1^{-1} p_2 \dot{p}_2^2 - y_0' a^{-1} p_2 \dot{\gamma}_2\\
&\quad - y_0' A_1^{-1}(A_1 + a^{-1})\gamma_2 \dot{p}_2 + x_0' \dot{p}_2 \dot{\gamma}_2 - z_0' a^{-1} p_2\\
&\quad - k\left[(1 - C_1)\gamma_2 \dot{p}_2 \dot{\gamma}_2 + A_1(1 + B_1)\gamma_2 s_{21} - A_1 p_2(1 - \dot{\gamma}_2^2)\right],\\
\phi_2 &= -\gamma_2 s_{11} + (1 + B_1) p_2 s_{21} - (1 - C_1) A_1^{-1} p_2 \dot{p}_2 \dot{\gamma}_2 + x_0' \dot{\gamma}_2^2 - y_0' \gamma_2 \dot{\gamma}_2\\
&\quad - z_0' b^{-1}\gamma_2 + x_0' b^{-1} - A_1^{-2}\gamma_2 \dot{p}_2^2 + k(C_1 \dot{\gamma}_2^2 - B_1)\gamma_2,\\
f_3 &= C_1 A_1^{-1} \dot{p}_2 \left[e\dot{p}_2 + e_1\gamma_2 \dot{p}_2 - 2p_2(y_0' a^{-1} - e_2\dot{\gamma}_2)\right]\\
&\quad - 9(e s_{11} + e_1\gamma_2 s_{11} + 2p_2 s_{12}) + A_1 b^{-1} x_0' s_{22}\\
&\quad + x_0'\left[\nu_2 \dot{p}_2^2 - \dot{\gamma}_2(y_0' a^{-1} - e_2\dot{\gamma}_2)\right] - y_0' a^{-1}\left[\dot{\gamma}_2(e + e_1\gamma_2) + \nu_2 p_2 \dot{p}_2\right]\\
&\quad + y_0'(1 + A_1^{-1} a^{-1})\left[\gamma_2(y_0' a^{-1} - e_2\dot{\gamma}_2) - \nu p_2 \dot{p}_2\right]\\
&\quad + \frac{1}{2} z_0'(a^{-1} - A_1 b^{-1})\gamma_2 s_{11} - z_0' a^{-1}(e + e_1\gamma_2 + p_2 s_{21})\\
&\quad + k\Big[(1 - C_1)(y_0' a^{-1} - e_2\dot{\gamma}_2)\gamma_2\dot{\gamma}_2 - \nu(1 - C_1) p_2 \dot{p}_2 \dot{\gamma}_2\\
&\quad - 2\nu_2 A_1 p_2 \dot{p}_2 \dot{\gamma}_2 - \nu_2(1 - C_1)\gamma_2 \dot{p}_2^2 - \nu A_1(1 + B_1) p_2 s_{21} + 2A_1 p_2 s_{21}\\
&\quad + (9 - A_1)\gamma_2 s_{22} + A_1(e + e_1\gamma_2)(1 - \dot{\gamma}_2^2)\Big],\\
\phi_3 &= 2x_0' \nu_2 \dot{p}_2 \dot{\gamma}_2 - 2\gamma_2 s_{12} - \nu p_2 s_{11} + (1 + B_1)[p_2 s_{22} + (e + e_1\gamma_2) s_{21}]\\
&\quad + (1 - C_1) A_1^{-1}\left[p_2\dot{\gamma}_2(y_0' a^{-1} - e_2\dot{\gamma}_2) - \nu_2 p_2 \dot{p}_2^2 - (e + e_1\gamma_2)\dot{p}_2\dot{\gamma}_2\right]\\
&\quad - z_0' b^{-1}(\nu p_2 + \gamma_2 s_{21}) + 2x_0' b^{-1} s_{21} + A_1^{-2}\left[2\gamma_2\dot{p}_2(y_0' a^{-1} - e_2\dot{\gamma}_2) - \nu p_2 \dot{p}_2^2\right]\\
&\quad - y_0'(\nu p_2 \dot{\gamma}_2 + \nu_2\gamma_2\dot{p}_2) + k\left[\nu p_2(C_1\dot{\gamma}_2^2 - B_1) + 2\gamma_2(\nu_2 C_1 \dot{p}_2\dot{\gamma}_2 - B_1 s_{21})\right];
\end{aligned}
\tag{1.3}
$$

$$
\begin{aligned}
&p_2 = p_1 - \mu e - \mu e_1\gamma_2, \qquad \gamma_2 = \gamma_1 - \mu\nu p_2,\\
&q_1 = -A_1^{-1}\dot{p}_2 + \mu A_1^{-1}(y_0' a^{-1} - e_2\dot{\gamma}_2) + \mu^2\Big[(aA_1)^{-1} y_0' s_{21} + \frac{1}{2}A_1^{-1}\dot{p}_2 s_{11}\\
&\quad + k\dot{\gamma}_2 s_{21} - \nu_2\dot{p}_2(a^{-1}A_1^{-1} z_0' - k)\Big] + \mu^3\Big[(aA_1)^{-1} y_0' s_{22} + \frac{1}{2}A_1^{-1} e_1\dot{\gamma}_2 s_{11}\\
&\quad + A_1^{-1}\dot{p}_2 s_{12} + a^{-1}A_1^{-2} e_1 z_0'\dot{\gamma}_2 + a^{-1}A_1^{-1} s_{11}(z_0'\dot{\gamma}_2 - y_0')\\
&\quad + (k - a^{-1}A_1^{-1} z_0')(a^{-1}A_1^{-1} y_0' - a^{-1}A_1^{-1} z_0'\dot{\gamma}_2 + k\dot{\gamma}_2) + a^{-1}A_1^{-2} z_0'\dot{p}_2 s_{21}\\
&\quad + k(\nu\dot{p}_2 s_{21} + \dot{\gamma}_2 s_{22} - A_1^{-1} e_1\dot{\gamma}_2 - \frac{3}{2}\dot{\gamma}_2 s_{11} - 2A_1^{-1}\dot{p}_2 s_{21})\Big] + \ldots,
\end{aligned}
\tag{1.4}
$$

(1.4) [cont.]

$$r_1 = 1 + \frac{1}{2}\mu^2 s_{11} + \mu^3 s_{12} + \dots,$$

$$\gamma_1' = \dot\gamma_2 + \mu\nu_2\dot p_2 + \mu^2\left[(aA_1)^{-1}y_0' - A_1^{-1}(e_2\dot\gamma_2 + \dot p_2 s_{21}) - \frac{1}{2}\dot\gamma_2 s_{11}\right]$$
$$+ \mu^3\Big[-A_1^{-1}(e_1\dot\gamma_2 s_{21} + \dot p_2 s_{22}) + \frac{1}{2}(3A_1^{-1} - \nu)\dot p_2 s_{11} - \dot\gamma_2 s_{12}$$
$$+ \nu_2(k - a^{-1}A_1^{-1}z_0')\dot p_2 + 2a^{-1}A_1^{-1}y_0' s_{21} + (2k - a^{-1}A_1^{-1}z_0')\dot\gamma_2 s_{21}\Big] + \dots,$$

$$\gamma_1'' = 1 + \mu s_{21} + \mu^2\left(s_{22} - \frac{1}{2}s_{11}\right) - \mu^3\left(s_{12} + \frac{1}{2}s_{11}s_{21}\right) + \dots;$$

(1.5)
$$p_1 = p/c\sqrt{\gamma_0''}, \quad q_1 = q/c\sqrt{\gamma_0''}, \quad r_1 = r/r_0, \quad \gamma_1 = \gamma/\gamma_0'',$$
$$\gamma_1' = \gamma'/\gamma_0'', \quad \gamma_1'' = \gamma''/\gamma_0'', \quad r = r_0 t, \quad (\cdot \equiv d/d\tau);$$

(1.6)
$$s_{11} = a(p_{20}^2 - p_2^2) + b(\dot p_{20}^2 - \dot p_2^2)/A_1^2 - 2\left[x_0'(\gamma_{20} - \gamma_2) + y_0'(\dot\gamma_{20} - \dot\gamma_2)\right]$$
$$+ k\left[a(\gamma_{20}^2 - \gamma_2^2) + b(\dot\gamma_{20}^2 - \dot\gamma_2^2)\right],$$
$$s_{12} = a\left[e(p_{20} - p_2) + e_1(p_{20}\gamma_{20} - p_2\gamma_2)\right]$$
$$- bA_1^{-2}\left[y_0' a^{-1}(\dot p_{20} - \dot p_2) - e_2(\dot p_{20}\dot\gamma_{20} - \dot p_2\dot\gamma_2)\right]$$
$$- \nu x_0'(p_{20} - p_2) - \nu_2 y_0'(\dot p_{20} - \dot p_2) + (z_0' - k)s_{21}$$
$$+ k\left[\nu a(p_{20}\gamma_{20} - p_2\gamma_2) + \nu_2 b(\dot p_{20}\dot\gamma_{20} - \dot p_2\dot\gamma_2)\right],$$
$$s_{21} = a(p_{20}\gamma_{20} - p_2\gamma_2) - bA_1^{-1}(\dot p_{20}\dot\gamma_{20} - \dot p_2\dot\gamma_2),$$
$$s_{22} = a\left[\nu(p_{20}^2 - p_2^2) + e(\gamma_{20} - \gamma_2) + e_1(\gamma_{20}^2 - \gamma_2^2)\right]$$
$$+ bA_1^{-1}\left[-\nu_2(\dot p_{20}^2 - \dot p_2^2) + a^{-1}y_0'(\dot\gamma_{20} - \dot\gamma_2) - e_2(\dot\gamma_{20}^2 - \dot\gamma_2^2)\right];$$

(1.7)
$$A_1 = \frac{C - B}{A}, \quad B_1 = \frac{A - C}{B}, \quad C_1 = \frac{B - A}{C}, \quad \gamma_0 \ge 0, \quad 0 < \gamma_0'' < 1,$$
$$a = \frac{A}{C}, \quad b = \frac{B}{C}, \quad c^2 = \frac{Mg\ell}{C},$$
$$\mu = \frac{c\sqrt{\gamma_0''}}{r_0}, \quad x_0 = \ell x_0', \quad y_0 = \ell y_0',$$
$$z_0 = \ell z_0', \quad \ell^2 = x_0^2 + y_0^2 + z_0^2, \quad A_1 B_1 = -9, \quad e = \frac{1}{9}x_0' A_1 b^{-1},$$
$$e_1 = \frac{1}{8}\left[k(9 - A_1) + z_0'(a^{-1} - A_1 b^{-1})\right],$$
$$\nu = -\frac{1}{8}(1 + B_1), \quad e_2 = e_1 + a^{-1}z_0' - kA_1,$$
$$\nu_2 = \nu - A_1^{-1}, \quad k = N\gamma_0''/c^2, \quad N = 3g/R, \quad g = \lambda/R^2.$$

Here R is the distance from the fixed point to the attracting center; λ is the coefficient of attraction of such a center; p_0, q_0, r_0, γ_0, γ_0' and γ_0'' are the initial values of the corresponding variables. Since r_0 is very large, then μ is considered as a small parameter.

2. Proposed method

In this section, Poincaré's small parameter method is applied to satisfy periodic solutions, with non-zero basic amplitudes, of system (1.1). For such a considered system, the following generating system ($\mu = 0$) is obtained

$$\ddot{p}_2^{(0)} + 9p_2^{(0)} = 0, \qquad \ddot{\gamma}_2^{(0)} + \gamma_2^{(0)} = 0, \tag{2.1}$$

which gives periodic solutions in the forms

$$p_2^{(0)} = M_1 \cos 3\tau + M_2 \sin 3\tau, \qquad \gamma_2^{(0)} = M_3 \cos \tau, \tag{2.2}$$

with the period $T_0 = 2\pi$, and M_1, M_2 and M_3 which are constants. Consider the following initial condition

$$\dot{\gamma}_2(0, \mu) = 0, \tag{2.3}$$

which does not affect the generality of the required solutions [4].

The periodic solutions for system (1.1) are expressed by the following forms [5]

$$\begin{aligned} p_2(\tau, \mu) &= \widetilde{M}_1 \cos 3\tau + \widetilde{M}_2 \sin 3\tau + \sum_{k=2}^{\infty} \mu^k G_k(\tau), \\ \gamma_2(\tau, \mu) &= \widetilde{M}_3 \cos \tau + \sum_{k=2}^{\infty} \mu^k H_k(\tau), \end{aligned} \tag{2.4}$$

where

$$\widetilde{M}_i = M_i + \beta_i \qquad (i = 1, 2, 3), \tag{2.5}$$

$$\begin{aligned} U = u &+ \frac{\partial u}{\partial M_1}\beta_1 + \frac{\partial u}{\partial M_2}\beta_2 + \frac{\partial u}{\partial M_3}\beta_3 \\ &+ \frac{1}{2}\frac{\partial^2 u}{\partial M_1^2}\beta_1^2 + \ldots, \qquad \left\{ \begin{matrix} U = G_k, & H_k, \\ u = g_k, & h_k \end{matrix} \right\}, \end{aligned} \tag{2.6}$$

the quantities β_1, $3\beta_2$ and β_3 representing the deviations of the initial values of p_2, $\dot{p}_2$ and γ_2 of system (1.1) from their initial values of system (2.1); these deviations are functions of μ and satisfy the condition $\beta_i(0) = 0$. These functions $g_k(\tau)$ and $h_k(\tau)$ take the forms [1]

$$
\begin{aligned}
g_k(\tau) &= \frac{1}{3}\int\limits_0^{\tau} F_k^{(0)}(t_1)\sin 3(\tau - t_1)\,dt_1\,,\\
h_k(\tau) &= \int_0^{\tau} \Phi_k^{(0)}(t_1)\sin(\tau - t_1)\,dt_1 \qquad (k = 2, 3).
\end{aligned}
\tag{2.7}
$$

The solutions (2.4) have the period $T = T_0 + \alpha(\mu)$ which reduces to T_0 at $\mu = 0$, that is $\alpha(0) = 0$. The initial condition (2.3) can be rewritten using the following relations:

$$
\begin{aligned}
p_2(0,\mu) &= \widetilde{M}_1, \qquad & \dot{p}_2(0,\mu) &= 3\widetilde{M}_2,\\
\gamma_2(0,\mu) &= \widetilde{M}_3, \qquad & \dot{\gamma}_2(0,\mu) &= 0.
\end{aligned}
\tag{2.8}
$$

The solutions (2.2) are rewritten in the following forms

$$
p_2^{(0)} = E\cos(3\tau - \varepsilon), \qquad \gamma_2^{(0)} = M_3\cos\tau,
\tag{2.9}
$$

where $E = \sqrt{M_1^2 + M_2^2}$ and $\varepsilon = \tan^{-1} M_2/M_1$. Making use of (2.9) and (1.4), one gets

$$
s_{ij}^{(0)} = s_{ij}^{(0)}\left(p_2^{(0)}, \dot{p}_2^{(0)}, \gamma_2^{(0)}, \dot{\gamma}_2^{(0)}\right) \qquad (i, j = 1, 2).
\tag{2.10}
$$

The functions $F_k^{(0)}$ and $\Phi_k^{(0)}$ are obtained from (2.9), (2.10) and (1.3). Then making use of (2.7) one obtains $g_k(2\pi)$, $h_k(2\pi)$, $\dot{g}_k(2\pi)$ and $\dot{h}_k(2\pi)$. The quantity $\widetilde{M}_3$ is determined by means of (2.8) into the first integral (1.2), with $\tau = 0$, and can be written in the form

$$
\begin{aligned}
\widetilde{M}_3 = \sqrt{1 - \gamma_0''/\gamma_0''} - \mu\nu\widetilde{M}_1 - 9\mu^2\nu_2^2\widetilde{M}_2^2/2M_3&\\
- 3\mu^3 y_0'\nu_2\widetilde{M}_2/aA_1M_3 + \ldots\,.&
\end{aligned}
\tag{2.11}
$$

The independent periodicity conditions [2] of the solutions $p_2(\tau,\mu)$, $\dot{p}_2(\tau,\mu)$, $\gamma_2(\tau,\mu)$ and $\dot{\gamma}_2(\tau,\mu)$ take the following forms:

$$
L_{310} + \mu(\ldots) = 0, \qquad (\widetilde{L}_{21} - 9\widetilde{N}_{21}) + \mu(\ldots) = 0;
\tag{2.12}
$$

$$
\alpha(\mu) = \mu^2\widetilde{M}_3^{-1}\left[\dot{H}_2(2\pi) + \mu\dot{H}_3(2\pi) + \ldots\right],
\tag{2.13}
$$

where

$$L_{310} = \frac{1}{4}k\Big[8e_1(a-b) - \frac{1}{2}z_0'(a^{-1} - A_1b^{-1})(a-b) + e_2(1-C_1) + e_1A_1 + (A_1 - 9)(ae_1 + be_2A_1^{-1})\Big],$$

(2.14)
$$\widetilde{L}_{21} - 9\widetilde{N}_{21} = a_1(\widetilde{M}_1^2 + \widetilde{M}_2^2) - [a_2 + 9kb(2M_3\beta_3 + \beta_3^2)],$$
$$a_1 = (a-1)(a+b-2)/2b,$$
$$a_2 = z_0'(ab)^{-1}[3(a+b) - 2(2ab+1)] + 18k\left[1 - (a+b) + \frac{1}{2}bM_3^2\right].$$

Equations of the basic amplitudes of (2.12) give

(2.15)
$$M_i = \pm\left[a_2a_1^{-1} - M_j^2\right]^{1/2} \qquad (i = 1,2, \quad j = 2,1).$$

The functions β_1 and β_2 are assumed in the forms

(2.16)
$$\beta_1 = \sum_{k=1}^{3}\mu^k\ell_k + O(\mu^4), \qquad \beta_2 = \sum_{k=1}^{3}\mu^k m_k + O(\mu^4).$$

Making use of (2.16), (2.12) and (2.5), one obtains

(2.17)
$$\ell_1 = -a_1^{-1}M_1^{-1}[a_1M_2m_1 + 9bk\nu M_1M_3],$$
$$\ell_2 = \frac{1}{2}a_1^{-1}M_1^{-1}[9bk\nu^2M_1^2 - 18bkM_3(\nu\ell_1 + 9\nu_2^2M_2^2/2M_3) - a_1(m_1^2 + \ell_1^2 + 2M_2m_2)],$$
$$\ell_3 = \frac{1}{2}a_1^{-1}M_1^{-1}[9b\nu kM_1(2\nu\ell_1 + 9\nu_2^2M_2^2M_3^{-1}) - 54kb\nu_2M_2y_0'a^{-1}A_1^{-1} - 2a_1(m_1m_2 + M_2m_3 + \ell_1\ell_2)].$$

Having Eqs. (2.15) and (2.17), we get a family of arbitrary solutions for the constants M_1 and M_2, and the quantities β_1 and β_2. Equations (2.6) and (2.7) give the functions $G_k(\tau)$ and $H_k(\tau)$; then, the periodic solutions (2.4) are constructed up to the third power of μ. Making use of (1.5) and (1.6), the following periodic solutions are obtained:

(2.18)
$$p = c\sqrt{\gamma_0''}\Big\{M_1\cos 3\tau + M_2\sin 3\tau + \mu(e + \ell_1\cos 3\tau + m_1\sin 3\tau + e_1M_3\cos\tau) + \mu^2\sum_{i=0}^{9}(Q_{1i}\cos i\tau + Q_{1i}'\sin i\tau) + \mu^3\sum_{j=0}^{9}(Q_{2j}\cos j\tau + Q_{2j}'\sin j\tau)\Big\} + \ldots, \qquad i \neq 6,7,8, \quad j \neq 8,$$

(2.18) [cont.]

$$q = c\sqrt{\gamma_0''}\Big\{A_1^{-1}(3M_1\sin 3\tau - 3M_2\cos 3\tau) + \mu A_1^{-1}(y_0'a^{-1} + e_2M_3\sin\tau + 3\ell_1\sin 3\tau - 3m_1\cos 3\tau) + \mu^2\sum_{i=0}^{9}(Q_{10i}\cos i\tau + Q_{10i}'\sin i\tau) + \mu^3\sum_{j=0}^{9}(Q_{11j}\cos j\tau + Q_{11j}'\sin j\tau)\Big\} + \ldots, \qquad i\neq 6,7,8, \quad j\neq 8,$$

$$r = r_0\Big\{1 + \frac{1}{2}\mu^2\Big\{E^2\Big[a\cos^2\varepsilon - \frac{1}{2} + 9bA_1^{-2}\Big(\sin^2\varepsilon - \frac{1}{2}\Big)\Big] - 2M_3[x_0'(1-\cos\tau) + y_0'\sin\tau] - \frac{1}{2}kC_1M_3^2(1-\cos 2\tau) + \frac{1}{2}E^2(9bA_1^{-2} - a)(\sin 2\varepsilon\sin 6\tau + \cos 2\varepsilon\cos 6\tau)\Big\} + \mu^3\sum_{i=0}^{6}(Q_{5i}\cos i\tau + Q_{5i}'\sin i\tau\Big\} + \ldots, \qquad i\neq 5,$$

$$\gamma = \gamma_0''\Big\{M_3\cos\tau + \mu\nu[M_1(\cos 3\tau - \cos\tau) + M_2\sin 3\tau] + \mu^2\sum_{i=0}^{7}(Q_{3i}\cos i\tau + Q_{3i}'\sin i\tau) + \mu^3\sum_{j=0}^{9}(Q_{4j}\cos j\tau + Q_{4j}'\sin j\tau)\Big\} + \ldots, \qquad i\neq 4,6, \quad j\neq 6,8,$$

$$\gamma' = \gamma_0''\Big\{-M_3\sin\tau + \mu[\nu M_1\sin\tau + 3\nu_2(M_2\cos 3\tau - M_1\sin 3\tau)] + \mu^2\sum_{i=0}^{7}(Q_{8i}\cos i\tau + Q_{8i}'\sin i\tau) + \mu^3\sum_{j=0}^{9}(Q_{9j}\cos j\tau + Q_{9j}'\sin j\tau)\Big\} + \ldots, \qquad i\neq 4,6, \quad j\neq 6,8,$$

$$\gamma'' = \gamma_0''\Big\{1 + \mu M_3E\Big[a\cos\varepsilon + \frac{1}{2}(3bA_1^{-1} - a)(\cos\varepsilon\cos 2\tau + \sin\varepsilon\sin 2\tau) - \frac{1}{2}(3bA_1^{-1} + a)(\cos\varepsilon\cos 4\tau + \sin\varepsilon\sin 4\tau)\Big] + \mu^2\sum_{i=0}^{6}(Q_{6i}\cos i\tau + Q_{6i}'\sin i\tau) + \mu^3\sum_{j=0}^{10}(Q_{7j}\cos 7\tau + Q_{7j}'\sin j\tau)\Big\} + \ldots, \qquad i\neq 3,5, \quad j\neq 7,9,$$

and the correction of the period $\alpha(\mu)$ becomes

$$\alpha(\mu) = \pi\mu^2N_{21} + \mu^3\pi(N_{21}^* + N_{31} + N_{38}M_1M_3) + \ldots, \tag{2.19}$$

where the constants N, Q and Q' are determined in terms of the rigid body motion parameters and occupy about twenty pages. The symbols $(\ldots)$ mean terms of order higher than $O(\mu^3)$.

3. Geometric interpretation of motion

Analyzing the obtained motion of the rigid body about a fixed point, using the Eulerian angles θ, ψ and ϕ, the following relations are obtained [6],

$$\begin{aligned} &\cos\theta = \gamma'', && \frac{d\psi}{dt} = \frac{p\gamma + q\gamma'}{1-\gamma''^2}, \\ &\tan\phi_0 = \frac{\gamma_0}{\gamma_0'}, && \frac{d\phi}{dt} = r - \frac{d\psi}{dt}\cos\theta. \end{aligned} \tag{3.1}$$

Since the initial system (1.1) is autonomous then the periodic solutions remain periodic if t is replaced by $t + t_0$, where t_0 is an arbitrary constant. Taking into consideration that the initial instant of time corresponds to the instant $t = t_0$, substituting the solutions (2.18) into Eqs. (3.1), the following angles are deduced:

$$\begin{aligned} &\phi_0 = \frac{\pi}{2} + r_0 t_0 + \ldots, \qquad \theta_0 = \tan^{-1} M_3, \\ &\theta = \theta_0 - \mu\cot\theta_0\{\theta_1(t+t_0) - \theta_1(t_0) + \mu[\theta_2(t+t_0) - \theta_2(t_0)] \\ &\qquad + \mu^2[\theta_3(t+t_0) - \theta_3(t_0)]\} + \ldots, \\ &\psi = \psi_0 + (Mg\ell\alpha_{10}C^{-1}r_0^{-}\cot^2\theta_0)t + \frac{1}{4}\mu\,\mathrm{cosec}\,\theta_0[\psi_1(t+t_0) - \psi_1(t_0)] \\ &\qquad + \mu^2\cot\theta_0\,\mathrm{cosec}\,\theta_0[\psi_2(t+t_0) - \psi_2(t_0)] + \ldots, \\ &\phi = \phi_0 + A_1^* t - \frac{1}{2}\mu\cot\theta_0[\phi_1(t+t_0) - \phi_1(t_0)] \\ &\qquad + \mu^2[\phi_2(t+t_0) - \phi_2(t_0)] + \ldots, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} &\theta_1(t) = \frac{1}{2}M_3E[(3bA_1^{-1} - a)(\cos\varepsilon\cos 2r_0t + \sin\varepsilon\sin 2r_0t) \\ &\qquad - (3bA_1^{-1} + a)(\cos\varepsilon\cos 4r_0t + (\sin\varepsilon\sin 4r_0t)], \\ &\theta_2(t) = \sum_{i=1}^{6}(Q_{6i}\cos ir_0t + Q'_{6i}\sin ir_0t), \qquad i \neq 3,5, \\ &\theta_3(t) = \sum_{j=1}^{10}(Q_{7j}\cos jr_0t + Q'_{7j}\sin jr_0t), \qquad j \neq 7,9, \\ &\psi_1(t) = (1 - 3A_1^{-1})(M_1\sin 2r_0t - M_2\cos 2r_0t) \\ &\qquad + \frac{1}{2}(1 + 3A_1^{-1})(M_1\sin 4r_0t - M_2\cos 4r_0t), \end{aligned} \tag{3.3}$$

(3.3) [cont.]

$$\psi_2(t) = \sum_{i=1}^{8} \frac{1}{i}(\alpha_{1i} \sin ir_0t - \alpha'_{1i} \cos ir_0t), \qquad i \neq 3,5,7,$$

$$\phi_1(t) = \frac{1}{2}(1 - 3A_1^{-1})(M_1 \sin 2r_0t - M_2 \cos 2r_0t) + \frac{1}{4}(1 + 3A_1^{-1})(M_1 \sin 4r_0t - M_2 \cos 4r_0t),$$

$$\phi_2(t) = \sum_{j=1}^{8}(\beta_{1j} \cos jr_0t + \beta'_{1j} \sin jr_0t), \qquad j \neq 3,5,7,$$

the formulae for constants A_1^*, α, α', β and β' occupying about three pages. We note that the expressions for the Eulerian angles θ, ψ and ϕ depend on four arbitrary constants θ_0, ψ_0, ϕ_0 and r_0 (r_0 is large).

4. Discussion of the solutions

In this section, we give a qualitative analysis of the results obtained, and several diagrams, explanations and examples.

The motion considered in this paper is investigated by introducing Euler's angles of nutation θ, precession ψ and pure rotation ϕ, see expressions (3.2). We note that θ is the angle between $0Z$ and $0z$; ψ is the angle between $0X$ and the line $0j$ of intersection of the fixed plane $0XY$ and the moving one $0xy$; and ϕ is the angle between the line $0j$ and the moving axis $0x$, see Fig. 1.

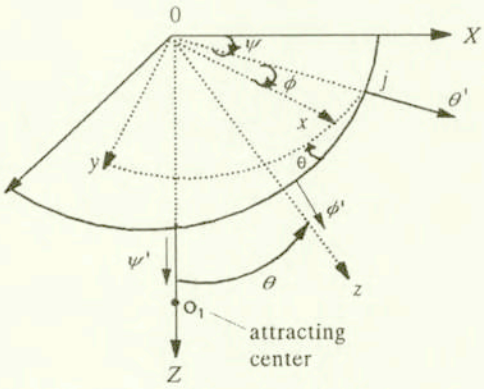

FIG. 1. Representation of Euler's angles.

The zero order approximation of the expressions (3.2) can be formulated as:

(4.1)
$$\theta = \theta_0, \quad \psi = \psi_0 + A^*t, \quad \phi = \phi_0 + A_1^*t,$$
$$\theta' = 0, \quad \psi' = A^* = \text{const}, \quad \phi' = A_1^* = \text{const}, \qquad ' = \frac{d}{dt},$$

which represents regular precession with spin A_1^* about $0z$-axis and precession A^* about the fixed axis $0Z$, see for example Fig. 2.

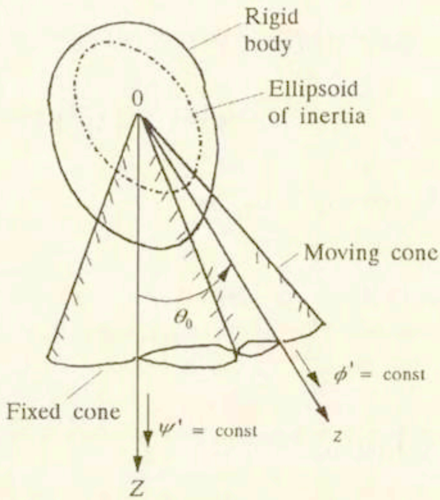

FIG. 2. Representation of zero-order approximation; regular precession.

The first approximation is formulated by

$$
\begin{aligned}
&\theta = \theta_0 + \mu f_1^*(t), \qquad \psi = \psi_0 + A^* t + \mu f_2^*(t),\\
&\phi = \phi_0 + A_1^* t + \mu f_3^*(t),\\
&\theta' = \mu f_1^{*'}(t), \qquad \psi' = A^* + \mu f_2^{*'}(t),\\
&\phi' = A_1^* + \mu f_3^{*'}(t),
\end{aligned}
\tag{4.2}
$$

where f_i^* and $f_i^{*'}$ are periodic functions with periods proportional to $1/r_0$ which means that they are of a fast character. The formulas (4.2) indicate, up to the first approximation, a perturbed pseudo-regular precession, see for example Fig. 3.

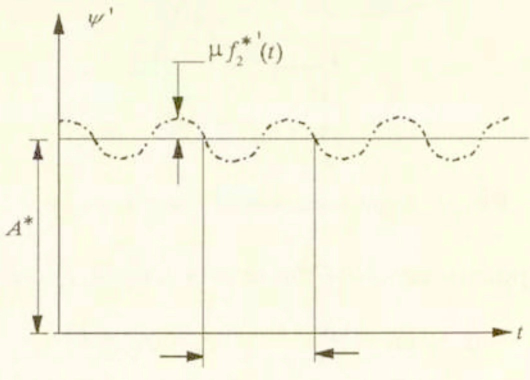

FIG. 3. A perturbed pseudo-regular precession for the first-order approximation.

The second and the third approximations represent the perturbations of the pseudo-regular precession and improves, qualitatively, the geometric interpretation of motion.

5. Conclusions

The periodic solutions with non-zero basic amplitudes for the system of the equations of motion, of the singular case $\omega = 3$, are investigated using the small parameter method of Poincaré. This problem deals with rigid bodies being classified according to the moments of inertia as follows:

1. $C > A > B, \quad B < \frac{1}{4}C, \quad A > \frac{1}{4}C,$
2. $C > B > A, \quad A < \frac{1}{4}C, \quad B > \frac{1}{4}C.$

The obtained solutions are considered as a generalization of the corresponding ones in the uniform gravity field ($k = 0$). Such solutions contain the solutions for the special cases of the basic amplitudes ($M_1 = M_2 = 0$; $M_1 \neq 0$, $M_2 = 0$ and $M_1 = 0$, $M_2 \neq 0$). The geometric interpretation of motion (using Euler's angles) is obtained to show the orientation of the body at any instant of time.

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