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Fabric tensor and constitutive equations for a class of plastic and locking orthotropic materials

S. JEMIOŁO and J.J. TELEGA (WARSZAWA)

THE AIM of the paper is to provide a general and common mathematical structure for a class of orthotropic materials undergoing plastic deformations or exhibiting locking behaviour. The orthotropy is included by using a fabric tensor. The tensorial constitutive relationships are studied from the point of view of tensor functions representations. Specific cases are also discussed.

1. Introduction

IN THE PAPER [18] the tensorial structure of the constitutive relationships for isotropic, perfectly locking materials was examined in detail. It was revealed that the general structure of equations is similar to those of isotropic, perfect plasticity, since they are time-independent.

The main aim of the present paper is to include orthotropy into such a general framework. This has been achieved by using a fabric tensor **H**, which is a particular case of structural tensors characterising anisotropic materials. It was introduced by COWIN [8–10] as a symmetric, positive definite tensor, a square root of the inverse of the mean intercept length tensor **M**. By using the classical spectral theorem, the interrelations between those tensors have been examined in Sec. 2.

In Sec. 3 the general structure of constitutive relationships involving two symmetric tensors and a scalar, common to plastic and locking materials, has been introduced. A different interpretation of the tensor **C** appearing in such a general relationships has been provided. For instance, **C** may be the tensor of plastic deformation or the locking stress tensor. In this manner, plastic hardening and/or softening and non-perfectly locking behaviour may be taken into account. Perfectly plastic and perfectly locking orthotropic materials are characterized by $\mathbf{C} = \mathbf{H}$.

In Sec. 4 the tensorial constitutive relationship introduced in Sec. 3 has been discussed from the point of view of tensor functions representations. Both polynomial and nonpolynomial representations have been investigated.

Our considerations of perfectly plastic and perfectly locking materials have deliberately been restricted to the orthotropy since then the available representation theorems are well developed. The same cannot be said about the case where anisotropy is described by a fourth-order tensor. Specific cases of constitutive relationships, yield and locking conditions have been studied in Secs. 5 and 6.

Throughout this paper the summation convention applies to repeated indices, unless otherwise stated.

2. The fabric tensor

Some materials such as wood, granular materials, bones and plastics exhibit elastic, plastic and locking behaviour under compressive stresses. The stress-deformation curves are then strongly influenced by the density of a material, cf. Figs. 10.3 and 11.5 in [12].

The aim of the present paper is not a study of such particular materials, which should be performed within a framework of elastic-plastic-locking behaviour. In the papers [18, 19] we have noticed a formal similarity between isotropic perfectly plastic and perfectly locking materials. In the present contribution we shall provide a general framework for perfectly plastic and perfectly locking *orthotropic* materials, provided that structural anisotropy is described by a second-order tensor, called the fabric tensor, cf. [1, 8, 13, 14, 20, 25, 42].

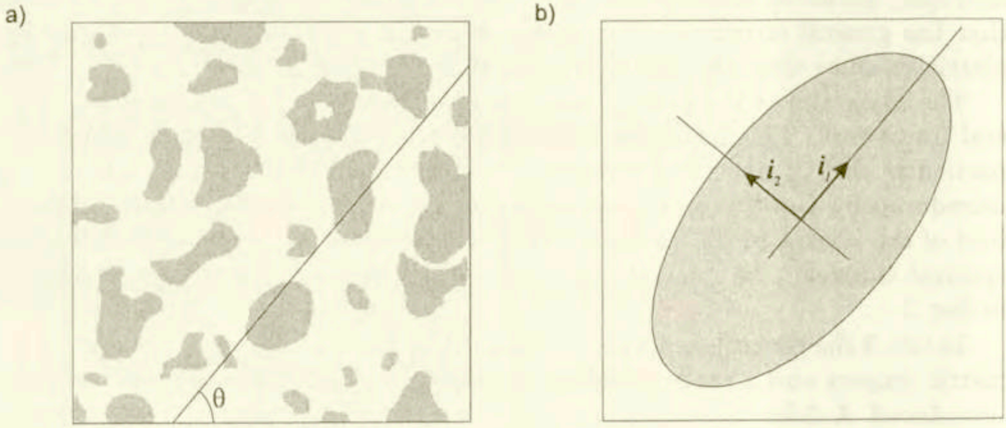

FIG. 1. The mean intercept length ellipse and its construction: a) test lines superimposed on a cellular material specimen. The test lines are oriented at angle Θ, which is varied to obtain the mean intercept length $L(\Theta)$, b) the ellipse constructed according to Eq. (2.1).

Let us introduce this tensor. Firstly, however, following WHITEHOUSE [40] we recall the notion of the mean intercept length L. This author measured L in cancellous bone as a function of direction on polished plane sections. Then L is the distance between two bone/marrow interfaces measured along a line. The value of L is a function of the slope Θ of the line along which the measurement is made. WHITEHOUSE [40] showed that when $L(\Theta)$ is plotted in the polar coordinates then the polar diagram produces ellipses, cf. Fig. 1. If the test lines are rotated through several values of Θ and the corresponding values of $L(\Theta)$ are measured, the data is found to fit the following equation of an ellipse:

$$\frac{1}{L(\Theta)} = M_{11}\cos^2\Theta + M_{22}\sin^2\Theta + 2M_{12}\sin\Theta\cos\Theta\,, \tag{2.1}$$

where M_{11}, M_{22} and M_{12} are constants, provided that the reference line from which the angle Θ is measured is constant.

HARRIGAN and MANN [14] extended Whitehouse's approach to the three-dimensional case and showed that $L(\mathbf{n})$, as a function of a direction $\mathbf{n}$, would be represented by ellipsoids and would therefore be equivalent to a positive definite second-order tensor $\mathbf{M}$ defined by

$$\frac{1}{L(\Theta)} = M_{ij}n_i n_j\,, \tag{2.2}$$

where $\mathbf{n}$ is a unit vector in the direction of the test line.

COWIN [8–10] defined a fabric tensor of cancellous bone to be the inverse square root of the mean intercept length tensor $\mathbf{M}$:

$$\mathbf{H} = \frac{1}{\sqrt{\mathbf{M}}}\,. \tag{2.3}$$

Obviously, $\mathbf{H}$ is well defined because $\mathbf{M}$ is a positive definite and symmetric tensor, cf. MARSDEN and and HUGES [21], pp. 52–55. The components of $\mathbf{M}$ or the mean intercept ellipsoid can be measured by using the techniques described by HARRIGAN and MAN [14] for a cubic specimen.

A specific form of the fabric tensor $\mathbf{H}$ is not required for our subsequent developments. The only assumption is that $\mathbf{H}$ should be a positive definite and symmetric second order tensor.

An alternative approach to the fabric tensor has been discussed by ZYSSET and CURNIER [42]. These authors decompose the fabric tensor $\mathbf{H}$ as follows: $\mathbf{H} = g\mathbf{I} + \mathbf{G}$.

An elementary microstructural description is contained in a single scalar property such as relative density, while material anisotropy requires fabric tensors of higher even rank [20]. KANATANI'S [20] approach can be applied to a class of materials with strictly positive morphological properties that are radially symmetric. In these situations we can use a scalar-valued orientation distribution function $h(\mathbf{N}) > 0$, where $\mathbf{N} = \mathbf{n}\otimes\mathbf{n}$ is the tensor product of the unit vector $\mathbf{n}$ specifying the orientation. Assuming the function to be square integrable it can be expanded in a convergent Fourier series:

$$h(\mathbf{N}) = g(\mathbf{N})1 + \mathbf{G}\cdot\mathbf{F}(\mathbf{N}) + \mathbb{G}\cdot\mathbb{F}(\mathbf{N}) + \ldots\,, \tag{2.4}$$

where 1, $\mathbf{F}(\mathbf{N})$ and $\mathbb{F}(\mathbf{N})$ are even rank tensorial basis functions and g, $\mathbf{G}$ and $\mathbb{G}$ – the corresponding even rank fabric tensors [20]. In bone mechanics we can use an approximation based on a scalar and a symmetric, traceless second rank fabric tensor. Then the first tensorial basis function is: $\mathbf{F}-\frac{1}{3}\mathbf{I}$, while the tensorial coefficients are calculated by

$$g=\frac{1}{4\pi}\int_S h(\mathbf{N})\,dS, \qquad \mathbf{G}=\frac{15}{8\pi}\int_S h(\mathbf{N})\,\mathbf{F}(\mathbf{N})\,dS, \tag{2.5}$$

where S is the surface of the unit sphere. For the particular case of an ellipsoidal distribution function we have

$$h(\mathbf{N})=\frac{1}{\sqrt{\mathbf{N}\cdot\mathbf{M}}}\,. \tag{2.6}$$

GOULET *et al.* [13] applied the concept of the mean intercept length to investigate the relationships between the structural parameters for cancellous bone, to determine their correlation to the mechanical properties, and to evaluate which parameters are important for maintaining bone strength and integrity.

The fabric tensor $\mathbf{H}$, as defined by (2.3), is an isotropic tensor function of $\mathbf{M}$, say $\widehat{\mathbf{H}}(\mathbf{M})$, cf. TING [36]. It means that

$$\forall\,\mathbf{Q}\in O(3) \qquad \mathbf{Q}\widehat{\mathbf{H}}(\mathbf{M})\mathbf{Q}^T=\widehat{\mathbf{H}}\left(\mathbf{Q}\mathbf{M}\mathbf{Q}^T\right)=\mathbf{Q}\frac{1}{\sqrt{\mathbf{M}}}\mathbf{Q}^T. \tag{2.7}$$

Here $O(3)$ stands for the full orthogonal group:

$$O(3)\equiv\{\mathbf{Q}:\quad \mathbf{Q}\mathbf{Q}^T=\mathbf{Q}^T\mathbf{Q}=\mathbf{I}\}, \tag{2.8}$$

where $\mathbf{I}$ is the identity tensor while $\mathbf{Q}^T$ is the transpose of $\mathbf{Q}$.

Let us pass to the determination of the function

$$\mathbf{H}=\widehat{\mathbf{H}}(\mathbf{M})=\frac{1}{\sqrt{\mathbf{M}}}\,. \tag{2.9}$$

Since $\mathbf{M}$ is a symmetric, positive definite tensor, therefore by applying the spectral theorem we may write

$$\mathbf{M}=M_1\mathbf{i}_1\otimes\mathbf{i}_1+M_2\mathbf{i}_2\otimes\mathbf{i}_2+M_3\mathbf{i}_3\otimes\mathbf{i}_3\,, \tag{2.10}$$

where M_j $(j=1,2,3)$ are eigenvalues of the tensor $\mathbf{M}$, and $\mathbf{i}_j$ its eigenvectors. It is assumed that $M_1\geq M_2\geq M_3$, where

$$M_i=\frac{1}{3}I_M+\frac{2}{3}\sqrt{I_M^2-3II_M}\,\cos\left[\frac{2}{3}\pi(i-1)-\varphi\right], \qquad i=1,2,3 \tag{2.11}$$

and

$$\cos 3\varphi=\frac{2I_M^3-9I_MII_M+27III_M}{\sqrt{2(I_M^2-3II_M)^3}}\,. \tag{2.12}$$

The basic invariants of $\mathbf{M}$ are given by

$$(2.13)\qquad \begin{gathered} I_M = \mathrm{tr}\mathbf{M}, \qquad II_M = \frac{1}{2}\left(\mathrm{tr}^2\mathbf{M} - \mathrm{tr}\mathbf{M}^2\right), \\ III_M = \det\mathbf{M} = \frac{1}{6}\left(\mathrm{tr}^3\mathbf{M} - 3\mathrm{tr}\mathbf{M}\mathrm{tr}\mathbf{M}^2 + 2\mathrm{tr}\mathbf{M}^3\right), \end{gathered}$$

where $\mathrm{tr}\mathbf{M}$ is the trace of $\mathbf{M}$. In an orthonormal basis $\{\mathbf{e}_i\}$ $(i = 1, 2, 3)$ we have: $\mathbf{M} = M_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, $\mathrm{tr}\mathbf{M} = M_{ii}$, $(\mathbf{M}^2)_{ij} = (\mathbf{M}\mathbf{M})_{ij} = M_{ik}M_{kj}$, etc.

Note that if

$$(2.14)\qquad d = 4II_M^3 - I_M^2 II_M^2 + 4I_M^3 III_M - 18 I_M II_M II_M + 27 II_M^2 < 0,$$

then M_i in (2.11) are different; for $d = 0$ two of the eigenvalues are equal; in other words, the tensor $\mathbf{M}$ is then two-dimensional. Finally, for

$$(2.15)\qquad I_M^2 = 3II_M\,,$$

$\mathbf{M}$ is a spherical tensor.

In the case of three different eigenvalues, the eigentensors $\mathbf{i}_j \otimes \mathbf{i}_j$ (no summation over j) can be determined in a unique fashion:

$$(2.16)\qquad \mathbf{i}_j \otimes \mathbf{i}_j = \frac{1}{m_j}\left[\mathbf{M}^2 - (I_M - M_j)\mathbf{M} + III_M M_j^{-1}\mathbf{I}\right] \quad \text{(no summation over } j\text{)},$$

where

$$(2.17)\qquad m_j = 2M_j^2 - I_M M_j + III_M M_j^{-1}\,.$$

Consequently the fabric tensor (2.3) satisfying (2.7) can be represented in the following form

$$(2.18)\qquad \mathbf{H} = H_1 \mathbf{i}_1 \otimes \mathbf{i}_1 + H_2 \mathbf{i}_2 \otimes \mathbf{i}_2 + H_3 \mathbf{i}_3 \otimes \mathbf{i}_3\,,$$

where

$$(2.19)\qquad H_i = \frac{1}{\sqrt{M_i}}, \qquad i = 1, 2, 3.$$

In the case of multiple eigenvalues of $\mathbf{M}$, the eigentensors (2.16) are not determined uniquely, cf. JEMIOŁO [17], MORMAN [23], TING [36]. As could be expected, for two (three) identical eigenvalues, (2.16) reduces to the representation of plane (spherical) tensors.

REMARK 1. If $d = 0$ and, for instance $M_1 \neq M_2 = M_3$, then instead of (2.11) one can calculate the eigenvalues similarly as for plane tensors $\overline{\mathbf{M}}$, i.e.:

$$(2.20)\qquad M_{1,2} = \overline{M}_{1,2} = \frac{1}{2}\left(\overline{I}_M \pm \sqrt{\overline{I}_M - 4\overline{II}_M}\right).$$

Consequently, (2.18) is to be replaced by the two-dimensional representation given by

$$\overline{\mathbf{H}} = \frac{1}{\sqrt{\overline{I}_M + 2\overline{II}_M}} \left[\left(1 + \frac{\overline{I}_M}{\sqrt{\overline{II}_M}} \right) \overline{\mathbf{I}} - \frac{1}{\sqrt{\overline{II}_M}} \overline{\mathbf{M}} \right], \tag{2.21}$$

where

$$\overline{I}_M = \mathrm{tr}\,\overline{\mathbf{M}}, \qquad \overline{II}_M = \det \overline{\mathbf{M}} = \frac{1}{2}\left(\mathrm{tr}^2\overline{\mathbf{M}} - \mathrm{tr}\,\overline{\mathbf{M}}^2\right). \tag{2.22}$$

Finally, if (2.15) is satisfied, then $M_1 = M_2 = M_3 = M$ and

$$\overline{\mathbf{H}} = \frac{1}{\sqrt{M}} \overline{\mathbf{I}}. \tag{2.23}$$

3. Plastic or perfectly locking behaviour: common general structure of constitutive relationships

Constitutive relationships describing perfectly plastic and perfectly locking materials exhibit a common feature, as being rate-independent, cf. JEMIOŁO and TELEGA [18, 19]. Consequently, the general structure of constitutive relationships for both classes of materials is similar. MURAKAMI and SAWCZUK [24] extended the approach proposed in [29] to plastic materials with hardening, though softening is not precluded. It means that our subsequent developments can also be generalised to non-perfectly locking materials.

The considerations which follow are restricted to a class of materials obeying the constitutive relationship

$$\mathbf{A} = \mathbf{F}(\varrho, \dot{\mathbf{B}}, \mathbf{C}), \tag{3.1}$$

subject to

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{B}}} \cdot \dot{\mathbf{B}} = \mathbf{0} \qquad \text{if} \qquad \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{B}}} \neq \widetilde{\mathbf{0}}. \tag{3.2}$$

Physical interpretation of the tensors $\mathbf{A}$, $\mathbf{C}$ and $\dot{\mathbf{B}}$ will be given later on. It will also be assumed that

$$D = \phi(\varrho, \dot{\mathbf{B}}, \mathbf{C}) \equiv \mathbf{A} \cdot \dot{\mathbf{B}} = \mathrm{tr}\,\mathbf{A}\dot{\mathbf{B}} \geq 0. \tag{3.3}$$

Here $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are second-order symmetric tensors while $\dot{\mathbf{B}}$ is an objective time derivative of $\mathbf{B}$; ϱ is a scalar parameter, for instance the density. More generally, ϱ may be an internal scalar parameter, which is an isotropic function of $\mathbf{B}$. By $\mathbf{0}$

and $\widetilde{\mathbf{0}}$ we denote null tensors of the second and fourth order, respectively. In an orthonormal basis the relationship $(3.2)_1$ is obviously given by

$$\frac{\partial F_{ij}}{\partial \dot{B}_{kl}} \dot{B}_{kl} = 0. \tag{3.4}$$

From (3.2) and (3.3) we conclude that

$$\frac{\partial \phi}{\partial \dot{\mathbf{B}}} \cdot \dot{\mathbf{B}} = \phi. \tag{3.5}$$

According to the principle of isotropy of the physical space [28], the tensor function (3.1) has to be isotropic:

$$\forall\, \mathbf{Q} \in O(3) \qquad \mathbf{Q}\mathbf{F}(\varrho, \dot{\mathbf{B}}, \mathbf{C})\mathbf{Q}^T = \mathbf{F}(\varrho, \mathbf{Q}\dot{\mathbf{B}}\mathbf{Q}^T, \mathbf{Q}\mathbf{C}\mathbf{Q}^T), \tag{3.6}$$

where the full orthogonal group $O(3)$ is defined by (2.8). Note that the function $\mathbf{F}$ is an anisotropic function with respect to $\dot{\mathbf{B}}$. An anisotropy group S is given by

$$S \equiv \{\mathbf{Q} \in O(3): \quad \mathbf{Q}\mathbf{C}\mathbf{Q}^T = \mathbf{C}\}. \tag{3.7}$$

Consequently, the material anisotropy is determined by the structural (fabric) tensor $\mathbf{C} = \mathbf{H}$. We have

$$\forall\, \mathbf{Q} \in S \qquad \mathbf{Q}\mathbf{F}(\varrho, \dot{\mathbf{B}}, \mathbf{H})\mathbf{Q}^T = \mathbf{F}(\varrho, \mathbf{Q}\dot{\mathbf{B}}\mathbf{Q}^T, \mathbf{H}). \tag{3.8}$$

The tensor $\mathbf{H}$, being a symmetric second-order tensor, enables one to determine the following three cases of anisotropy:

(i) If $H_1 \neq H_2 \neq H_3 \neq H_1$ then S stands for the orthotropy group; more precisely, according to Schoenfliess' notation we then write $S = D_{4h}$, cf. [41]. In this case one has

$$S = S_1 \cap S_2 \cap S_3\,, \tag{3.9}$$

where

$$S_i \equiv \{\mathbf{Q} \in O(3): \quad \mathbf{Q}(\mathbf{i}_i \otimes \mathbf{i}_i)\mathbf{Q}^T = \mathbf{i}_i \otimes \mathbf{i}_i\} \qquad \text{(no summation over } i\text{)}. \tag{3.10}$$

(ii) If two eigenvalues coincide, say $H_1 \neq H_2 = H_3$, then S denotes the transverse isotropy group:

$$S = S_1 = D_{\infty h}\,. \tag{3.11}$$

(iii) If $H_1 = H_2 = H_3$ then $S = O(3)$ is the isotropy group.

The scalar function ϕ defined by (3.3) is an orthotropic function of $\dot{\mathbf{B}}$ provided that $\mathbf{C} = \mathbf{H}$:

$$\forall\, \mathbf{Q} \in S \qquad \phi(\varrho, \dot{\mathbf{B}}, \mathbf{H}) = \phi(\varrho, \mathbf{Q}\dot{\mathbf{B}}\mathbf{Q}^T, \mathbf{H}). \tag{3.12}$$

Let us investigate some important consequences implied by the homogeneity condition (3.2). Firstly, following SAWCZUK and STUTZ [28] we conclude that there exists a scalar function

$$f(\varrho, \mathbf{A}, \mathbf{C}) = 0, \tag{3.13}$$

satisfying the condition

$$\forall\, \mathbf{Q} \in S \qquad f(\varrho, \mathbf{A}, \mathbf{C}) = f(\varrho, \mathbf{Q}\mathbf{A}\mathbf{Q}^T, \mathbf{C}). \tag{3.14}$$

Secondly, one has

$$\det\left(\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{B}}}\right) = 0. \tag{3.15}$$

Thus the constitutive relationships (3.1) is not invertible. However, one can find a semi-invertible relation, now given by

$$\dot{\mathbf{B}} = \lambda \widetilde{\mathbf{F}}(\varrho, \mathbf{A}, \mathbf{C}), \qquad \lambda \geq 0, \tag{3.16}$$

where

$$\lambda = \eta(\varrho, \dot{\mathbf{B}}, \mathbf{C}). \tag{3.17}$$

Note that the notion of semi-invertibility was introduced by TRUESDELL and MOON [37], where a symmetric second-order tensor function of a symmetric, second order tensor was analysed.

REMARK 2. One can consider the case when ϱ and $\mathbf{C}$ are isotropic functions of $\mathbf{B}$: $\varrho = \widetilde{\varrho}(\mathbf{B})$ and $\mathbf{C} = \widetilde{\mathbf{C}}(\mathbf{B})$. The isotropy means that

$$\forall\, \mathbf{Q} \in O(3) \qquad \varrho = \widetilde{\varrho}(\mathbf{B}) = \widetilde{\varrho}(\mathbf{Q}\mathbf{B}\mathbf{Q}^T), \tag{3.18}$$

$$\forall\, \mathbf{Q} \in O(3) \qquad \mathbf{Q}\widetilde{\mathbf{C}}(\mathbf{B})\mathbf{Q}^T = \widetilde{\mathbf{C}}(\mathbf{Q}\mathbf{B}\mathbf{Q}^T). \tag{3.19}$$

To make such a theory complete, evolution laws $\dot{\varrho} = \widehat{\varrho}(\mathbf{B})$ and $\dot{\mathbf{C}} = \widehat{\mathbf{C}}(\mathbf{B})$ must additionally be specified. ◁

In general, the function ϕ is not a potential for $\mathbf{A}$. Below it will be shown that under the following condition, cf. [18, 19],

$$\dot{\mathbf{B}} \cdot \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{B}}} = \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{B}}} \cdot \dot{\mathbf{B}}, \tag{3.20}$$

one has

$$\mathbf{A} = \frac{\partial \phi}{\partial \dot{\mathbf{B}}}. \tag{3.21}$$

In general, the constitutive relationship (3.16) is also not associated with the scalar condition (3.13). The associated rule has the form

$$\dot{\mathbf{B}} = \lambda \frac{\partial f}{\partial \mathbf{A}}, \qquad \lambda \geq 0. \tag{3.22}$$

REMARK 3. Suppose that the set $C(\varrho, \mathbf{C})$ defined by

$$C(\varrho, \mathbf{C}) \equiv \{\mathbf{T} \in T^s \mid f(\varrho, \mathbf{T}, \mathbf{C}) \leq 0\}, \tag{3.23}$$

is convex and closed. Here T^s stands for the space of symmetric, second-order tensors. The indicator function of this set is given by, cf. ROCKAFELLAR [27]

$$\mathbf{I}_{C(\varrho,\mathbf{C})}(\mathbf{T}) = \begin{cases} 0 & \text{if } \mathbf{T} \in C(\varrho, \mathbf{C}), \\ \infty & \text{otherwise.} \end{cases} \tag{3.24}$$

The subdifferential (associated) constitutive relationship has the following form:

$$\dot{\mathbf{B}} \in \partial I_{C(\varrho,\mathbf{C})}(\mathbf{A}). \tag{3.25}$$

In the case when f is differentiable with respect to the second argument, the last law is equivalent to (3.22).

The support function of $C(\varrho, \mathbf{C})$ is a particular case of (3.3) and is calculated as follows, cf. [27]:

$$\phi_1(\varrho, \dot{\mathbf{B}}, \mathbf{C}) = \sup_{\mathbf{T} \in T^s} \left\{\dot{\mathbf{B}} \cdot \mathbf{T} - I_{C(\varrho,\mathbf{C})}(\mathbf{T})\right\} = \sup \left\{\dot{\mathbf{B}} \cdot \mathbf{T} \mid \mathbf{T} \in C(\varrho, \mathbf{C})\right\}. \tag{3.26}$$

The function $\phi_1(\varrho, \cdot, \mathbf{C})$ is convex and subdifferentiable.

The constitutive relationship inverse to (3.25), and equivalent to it, is given by

$$\mathbf{A} \in \partial_2 \phi_1(\varrho, \dot{\mathbf{B}}, \mathbf{C}), \tag{3.27}$$

where $\partial_2 \phi_1(\varrho, \dot{\mathbf{B}}, \mathbf{C})$ stands for the subdifferential of the function $\phi_1(\varrho, \cdot, \mathbf{C})$ at a point $\dot{\mathbf{B}}$. If the function $\phi_1(\varrho, \cdot, \mathbf{C})$ is differentiable then the law (3.21) is recovered and ϕ_1 coincides with ϕ. $\triangleleft$

Let us now specify two particular classes of materials described by the introduced general relations within the theory of small deformations, cf. Fig. 2.

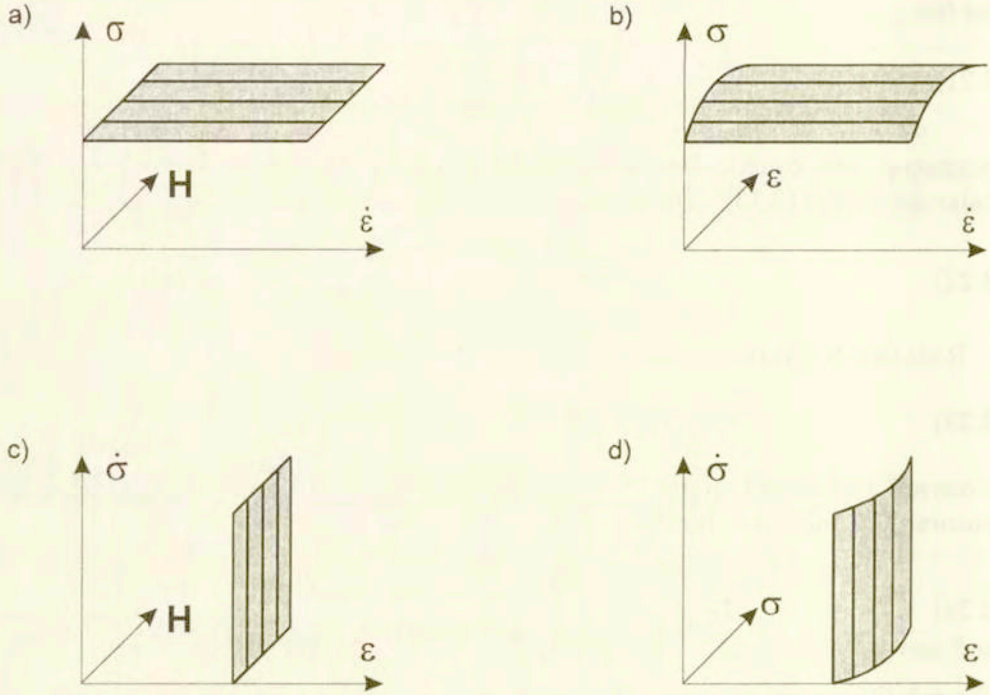

FIG. 2. Schematic representation of the constitutive relationships allowed by Eqs. (3.1) and (3.2): a) $\mathbf{C} = \mathbf{H}$, $\mathbf{A} = \boldsymbol{\sigma}$, $\dot{\mathbf{B}} = \dot{\boldsymbol{\varepsilon}}$, orthotropic perfect plasticity, b) $\mathbf{C} = \boldsymbol{\varepsilon}$, $\mathbf{A} = \boldsymbol{\sigma}$, $\dot{\mathbf{B}} = \dot{\boldsymbol{\varepsilon}}$, plasticity with hardening, c) $\mathbf{C} = \mathbf{H}$, $\mathbf{A} = \boldsymbol{\varepsilon}$, $\dot{\mathbf{B}} = \dot{\boldsymbol{\sigma}}$, orthotropic locking materials, d) $\mathbf{C} = \boldsymbol{\sigma}$, $\mathbf{A} = \boldsymbol{\varepsilon}$, $\dot{\mathbf{B}} = \dot{\boldsymbol{\sigma}}$, non-perfectly locking behaviour.

For

$$(\mathbf{A}, \dot{\mathbf{B}}) = (\boldsymbol{\sigma}, \dot{\boldsymbol{\varepsilon}}), \tag{3.28}$$

and $\mathbf{C} = \mathbf{H}$, perfect plasticity of at most orthotropic materials is recovered. Here $\boldsymbol{\sigma}$ denotes the stress tensor and $\dot{\boldsymbol{\varepsilon}}$ is the rate of plastic deformation. Assuming additionally (3.18), one obtains plasticity with isotropic hardening/softening. Incorporating (3.19), one describes an orthotropic hardening. For $\mathbf{C} = \boldsymbol{\varepsilon}$, the material behaviour is initially isotropic; $\boldsymbol{\varepsilon}$ denotes the strain tensor.

The second case (perfectly locking behaviour) is for $\mathbf{C} = \mathbf{H}$ and, cf. [18, 19]

$$(\mathbf{A}, \dot{\mathbf{B}}) = (\boldsymbol{\varepsilon}, \dot{\boldsymbol{\sigma}}). \tag{3.29}$$

For instance, if $\mathbf{C} = \boldsymbol{\sigma}$ then our approach describes non-perfectly locking behaviour; the orthotropy is then induced by the locking stress tensor $\boldsymbol{\sigma}$.

Summarizing, we conclude that (3.13) represents:

(a) The yield condition if $\mathbf{A} = \boldsymbol{\sigma}$,

(b) the locking condition if $\mathbf{A} = \boldsymbol{\varepsilon}$.

Similarly, (3.16) is:

(i) the flow rule if $(\mathbf{A}, \dot{\mathbf{B}}) = (\boldsymbol{\sigma}, \dot{\boldsymbol{\varepsilon}})$,

(ii) the locking rule if $(\mathbf{A}, \dot{\mathbf{B}}) = (\boldsymbol{\varepsilon}, \dot{\boldsymbol{\sigma}})$.

As we already know, in general both these rules are non-associated laws.

4. Representation of the tensor function (3.1)

To determine the general form of the tensor function (3.1) satisfying the isotropy condition (3.6) one can apply either the results due to SPENCER [32] on polynomial representation or those obtained by WANG [39] as well as BOEHLER and RACLIN [5], concerning nonpolynomial representations. For both representations the so-called canonical form of (3.1) is expressed by

$$\mathbf{A} = \widetilde{\alpha}_p \mathbf{G}_p\,, \tag{4.1}$$

where $p = 1, ..., 9$ for the polynomial representation, while $p = 1, ..., 8$ in the case of the nonpolynomial representation. The summation convention still applies, unless otherwise stated. Here $\widetilde{\alpha}_p$ are scalar functions of ϱ and of isotropic invariants of $\dot{\mathbf{B}}$ and $\mathbf{C}$:

$$\alpha_p = \widetilde{\alpha}_p(\varrho, I_r), \qquad r = 1, ..., 10. \tag{4.2}$$

The symmetric second-order tensors $\mathbf{G}_p$ are the so-called generators.

We have

$$\begin{aligned} \{I_r\} &= \{\operatorname{tr}\dot{\mathbf{B}}, \operatorname{tr}\dot{\mathbf{B}}^2, \operatorname{tr}\dot{\mathbf{B}}^3, \operatorname{tr}\dot{\mathbf{B}}\mathbf{C}, \operatorname{tr}\dot{\mathbf{B}}\mathbf{C}^2, \operatorname{tr}\dot{\mathbf{B}}^2\mathbf{C}, \operatorname{tr}\dot{\mathbf{B}}^2\mathbf{C}^2, \operatorname{tr}\mathbf{C}, \operatorname{tr}\mathbf{C}^2, \operatorname{tr}\mathbf{C}^3\} \\ &= \{J_s, \operatorname{tr}\mathbf{C}^i\}, \qquad s = 1, \ldots, 7, \quad i = 1, 2, 3, \end{aligned} \tag{4.3}$$

$$\{\mathbf{G}_p\} = \{\mathbf{I}, \dot{\mathbf{B}}, \dot{\mathbf{B}}^2, \mathbf{C}, \mathbf{C}^2, \mathbf{C}\dot{\mathbf{B}} + \dot{\mathbf{B}}\mathbf{C}, \mathbf{C}^2\dot{\mathbf{B}} + \dot{\mathbf{B}}\mathbf{C}^2, \mathbf{C}^2\dot{\mathbf{B}}^2 + \dot{\mathbf{B}}^2\mathbf{C}^2\}. \tag{4.4}$$

It can be easily shown that under the condition (3.20), the representation (4.1) simplifies since it contains only seven generators while $\widetilde{\alpha}_p$ satisfy additional relations, see below.

Since $\dot{\mathbf{B}}$ is a symmetric tensor, therefore one has

$$\begin{aligned} &\frac{\partial \operatorname{tr}\dot{\mathbf{B}}^i}{\partial \dot{\mathbf{B}}} = i\dot{\mathbf{B}}^{i-1}, \qquad i = 1, 2, 3, \quad \dot{\mathbf{B}}^0 = \mathbf{I}, \\ &\frac{\partial \operatorname{tr}\dot{\mathbf{B}}\mathbf{C}^\alpha}{\partial \dot{\mathbf{B}}} = \mathbf{C}^\alpha, \qquad \frac{\partial \operatorname{tr}\dot{\mathbf{B}}^2\mathbf{C}^\alpha}{\partial \dot{\mathbf{B}}} = \dot{\mathbf{B}}\mathbf{C}^\alpha + \mathbf{C}^\alpha\dot{\mathbf{B}}, \quad \alpha = 1, 2, \end{aligned} \tag{4.5}$$

and the representation (4.1) can be written in the following form:

$$\begin{aligned} \mathbf{A} = \gamma_1\mathbf{I} + 2\gamma_2\dot{\mathbf{B}} + 3\gamma_3\dot{\mathbf{B}}^2 + \gamma_4\mathbf{C} + \gamma_5\mathbf{C}^2 + \gamma_6(\dot{\mathbf{B}}\mathbf{C} + \mathbf{C}\dot{\mathbf{B}}) \\ + \gamma_7(\dot{\mathbf{B}}\mathbf{C}^2 + \mathbf{C}^2\dot{\mathbf{B}}) = \gamma_s\widetilde{\mathbf{G}}_s\,, \end{aligned} \tag{4.6}$$

provided that (3.20) is satisfied; $s = 1, ..., 7$. Here

$$\gamma_s = \frac{\partial \widetilde{g}}{\partial J_s}, \tag{4.7}$$

and $g = \widetilde{g}(\varrho, J_s, \mathrm{tr}\,\mathbf{C}^i)$, $i = 1, 2, 3$. The condition (3.20) implies

$$\frac{\partial \gamma_s}{\partial J_t} = \frac{\partial \gamma_t}{\partial J_s}, \qquad s, t = 1, ..., 7. \tag{4.8}$$

Note that (4.6) yields

$$\frac{\partial \mathbf{A}}{\partial \dot{\mathbf{B}}} = \frac{\partial \gamma_s}{\partial J_t} \widetilde{\mathbf{G}}_s \otimes \widetilde{\mathbf{G}}_t + \gamma_s \frac{\partial \widetilde{\mathbf{G}}_s}{\partial \dot{\mathbf{B}}} \tag{4.9}$$

and

$$\frac{\partial \widetilde{\mathbf{G}}_s}{\partial \dot{\mathbf{B}}} = \frac{\partial^2 J_s}{\partial \dot{\mathbf{B}} \otimes \partial \dot{\mathbf{B}}}. \tag{4.10}$$

Here the functions γ_s are interrelated by (4.8). Hence we conclude that (4.8) is equivalent to (3.20).

Comparing (4.1) with (4.6) and taking into account (4.7) and (4.8) one obtains additional conditions which have to be fulfilled by the scalar functions $\widetilde{\alpha}_p$. To satisfy (4.10) we conclude that the representation (4.6) has to be at least of class C^2 with respect to $\dot{\mathbf{B}}$.

The invariants (4.3) appearing in (4.2) constitute both polynomial and non-polynomial bases. A polynomial basis consisting of 10 invariants can be constructed from another set of independent invariants, provided, however, that they are polynomials in I_r. Obviously, we are discussing the general case of 3D tensors. In the remaining cases the relevant representations are simplified. The choice of a functional basis is also not unique. To satisfy the homogeneity condition (3.2) it is convenient to deal with the following functional basis consisting of invariants being functions of order one with respect to $\dot{\mathbf{B}}$ and including the invariants $\mathrm{tr}\,\mathbf{C}^i$ $(i = 1, 2, 3)$:

$$\{K_r\} = \left\{ \mathrm{tr}\dot{\mathbf{B}}, \sqrt{\mathrm{tr}\dot{\mathbf{B}}^2}, \sqrt[3]{\mathrm{tr}\dot{\mathbf{B}}^3}, \mathrm{tr}\dot{\mathbf{B}}\mathbf{C}, \mathrm{tr}\dot{\mathbf{B}}\mathbf{C}^2, \sqrt{\mathrm{tr}\dot{\mathbf{B}}^2\mathbf{C}}, \sqrt{\mathrm{tr}\dot{\mathbf{B}}^2\mathbf{C}^2}, \mathrm{tr}\mathbf{C}^i \right\} \tag{4.11}$$

$$r = 1, \ldots, 10, \qquad i = 1.2.3.$$

Note that $\sqrt{\mathrm{tr}\dot{\mathbf{B}}^2} = \|\dot{\mathbf{B}}\|$ is a norm of $\dot{\mathbf{B}}$.

In particular cases or in order to facilitate an experimental identification of material functions, one can choose the functional basis consisting of nine nonpolynomial invariants, say C_i and $\dot{B}_i$ $(i = 1,2,3)$ and, for instance, three Euler angles, which determine a mutual position of the eigenvectors of the tensors $\mathbf{C}$ and $\dot{\mathbf{B}}$; by C_i and $\dot{B}_i$ we have denoted the eigenvalues of $\mathbf{C}$ and $\dot{\mathbf{B}}$, respectively.

By using (2.11) and (2.16) one can construct a nine-element functional basis in the following way: six nonpolynomial invariants are just C_i and $\dot{B}_i$. To this end, M_i and $\mathbf{M}$ in (2.11) - (2.13) should be replaced first by C_i and $\mathbf{C}$ and next by $\dot{B}_i$ and $\dot{\mathbf{B}}$. To determine the remaining three invariants we apply (2.16) once again. For the tensor $\mathbf{C}$ we write

$$\text{(4.12)}\qquad \begin{aligned} \mathbf{i}_j \otimes \mathbf{i}_j &= \frac{1}{2C_j^2 - I_C C_j + III_C C_j^{-1}} \left[\mathbf{C}^2 - (I_C - C_j)\mathbf{C} + III_C C_j^{-1}\mathbf{I}\right] \\ &= \frac{1}{a_0^{(j)}} \left(\mathbf{C}^2 - a_1^{(j)}\mathbf{C} + a_2^{(j)}\mathbf{I}\right) \qquad \text{(no summation over } j) \end{aligned}$$

and similarly for the tensor $\dot{\mathbf{B}}$:

$$\text{(4.13)}\qquad \mathbf{j}_j \otimes \mathbf{j}_j = \frac{1}{b_0^{(j)}} \left(\dot{\mathbf{B}}^2 - b_1^{(j)}\dot{\mathbf{B}} + b_2^{(j)}\mathbf{I}\right),$$

where

$$\text{(4.14)}\qquad b_0^{(j)} = 2\dot{B}_j^2 - I_{\dot{B}}\dot{B}_j + III_{\dot{B}}\dot{B}_j^{-1}, \qquad b_1^{(j)} = I_{\dot{B}} - \dot{B}_j, \qquad b_2^{(j)} = III_{\dot{B}}\dot{B}_j^{-1}.$$

We recall that $\dot{\mathbf{B}}$ and $\mathbf{C}$ are symmetric, second-order tensors. The space T^s of symmetric second-order tensors is equipped with the scalar product defined by $\mathbf{T}\cdot\mathbf{Z} = \mathrm{tr}\,\mathbf{TZ}$, for each $\mathbf{T},\mathbf{Z} \in T^s$. Consequently, the natural norm of $\mathbf{T} \in T^s$ is given by

$$\text{(4.15)}\qquad \|\mathbf{T}\|^2 = \mathbf{T}\cdot\mathbf{T} = \mathrm{tr}\,\mathbf{T}^2.$$

The space $(T^s, \|\cdot\|)$ can be identified with the six-dimensional Euclidean space. Therefore one can calculate the scalar product of the eigenvectors (4.12) and (4.13)

$$\text{(4.16)}\qquad \begin{aligned} (\mathbf{i}_i \otimes \mathbf{i}_i)\cdot(\mathbf{j}_j \otimes \mathbf{j}_j) &= \mathrm{tr}_{(1,4)}\mathrm{tr}_{(2,3)}\mathbf{i}_i \otimes \mathbf{i}_i \otimes \mathbf{j}_j \otimes \mathbf{j}_j = (\mathbf{i}_i \cdot \mathbf{j}_j)^2 \\ &= \cos^2\alpha_{ij} \geq 0 \qquad \text{(no summation over } i \text{ and } j), \end{aligned}$$

where α_{ij} is the angle between the eigenvectors $\mathbf{i}_i$ and $\mathbf{j}_j$. We assume that the eigenvalues of C_i are different: $C_1 \neq C_2 \neq C_3 \neq C_1$ and similarly for $\dot{\mathbf{B}}$. The remaining cases are simpler since then a smaller number of invariants is involved.

We may write $[\cos\alpha_{ij}] = [Q_{ij}]$, obviously $\mathbf{Q} \in O(3)$. Further we find

$$\text{(4.17)}\qquad (\mathbf{j}_i \otimes \mathbf{j}_i)\cdot(\mathbf{i}_j \otimes \mathbf{i}_j) = \cos^2\beta_{ij} \qquad \text{(no summation over } i \text{ and } j),$$

and $[\cos\beta_{ij}]^T = [\cos\alpha_{ij}]$. Recalling that $Q_{ik}Q_{kj} = \delta_{ij}$, where δ_{ij} stands for Kronecker's delta, we deduce that only three of the angles α_{ij} are independent. From (4.12), (4.13) and (4.16) one obtains

$$\begin{aligned}(4.18)\qquad \cos^2\alpha_{ij} = \frac{1}{a_0^{(i)}b_0^{(j)}}\Big(&\operatorname{tr}\mathbf{C}^2\dot{\mathbf{B}}^2 - b_1^{(j)}\operatorname{tr}\mathbf{C}^2\dot{\mathbf{B}} + b_2^{(j)}\operatorname{tr}\mathbf{C}^2 - a_1^{(i)}\operatorname{tr}\mathbf{C}\dot{\mathbf{B}}^2\\ &+ a_1^{(i)}b_1^{(j)}\operatorname{tr}\mathbf{C}\dot{\mathbf{B}} - a_1^{(i)}b_2^{(j)}\operatorname{tr}\mathbf{C} + a_2^{(i)}\operatorname{tr}\dot{\mathbf{B}}^2 - a_2^{(i)}b_1^{(j)}\operatorname{tr}\dot{\mathbf{B}} + 3a_2^{(i)}b_2^{(j)}\Big)\\ &\text{(no summation over } i \text{ and } j).\end{aligned}$$

All in all, we conclude that it is possible to construct the functional basis consisting of nine invariants which in turn depend on ten polynomial invariants.

In the case of perfect plasticity or ideal locking, when $\mathbf{C} = \mathbf{H}$ is a fabric tensor, more convenient in applications seems to be a representation of the tensor function (3.1) different from (4.1). Suppose that $H_1 \neq H_2 \neq H_3 \neq H_1$ and assume that $\mathbf{i}_i$ $(i = 1, 2, 3)$ are the principal axes of orthotropy. We set

$$(4.19)\qquad \mathbf{M}_i = \mathbf{i}_i \otimes \mathbf{i}_i \qquad \text{(no summation over } i).$$

The representation (4.1) can then be written in the form

$$\begin{aligned}(4.20)\qquad \mathbf{A} = \beta_1\mathbf{M}_1 + \beta_2\mathbf{M}_2 + \beta_3\mathbf{M}_3 &+ \beta_4(\mathbf{M}_1\dot{\mathbf{B}} + \dot{\mathbf{B}}\mathbf{M}_1)\\ &+ \beta_5(\mathbf{M}_2\dot{\mathbf{B}} + \dot{\mathbf{B}}\mathbf{M}_2) + \beta_6(\mathbf{M}_3\dot{\mathbf{B}} + \dot{\mathbf{B}}\mathbf{M}_3) + \beta_7(\mathbf{M}_1\dot{\mathbf{B}}^2 + \dot{\mathbf{B}}^2\mathbf{M}_1)\\ &+ \beta_8(\mathbf{M}_2\dot{\mathbf{B}}^2 + \dot{\mathbf{B}}^2\mathbf{M}_2) + \beta_9(\mathbf{M}_3\dot{\mathbf{B}}^2 + \dot{\mathbf{B}}^2\mathbf{M}_3),\end{aligned}$$

where

$$\begin{aligned}\beta_i &= \alpha_1 + H_i\alpha_2 + H_i^2\alpha_6\,,\\ (4.21)\qquad \beta_{i+3} &= \alpha_4 + H_i\alpha_6 + H_i^2\alpha_7\,,\\ \beta_{i+6} &= \alpha_5 + H_i\alpha_8 + H_i^2\alpha_9\,, \qquad i = 1, 2, 3.\end{aligned}$$

Since

$$\begin{aligned}(4.22)\qquad \operatorname{tr}\dot{\mathbf{B}}^\alpha\mathbf{H}^\beta = H_1^\beta\operatorname{tr}\mathbf{M}_1\dot{\mathbf{B}}^\alpha + H_2^\beta\operatorname{tr}\mathbf{M}_2\dot{\mathbf{B}}^\alpha + H_3^\beta\operatorname{tr}\mathbf{M}_3\dot{\mathbf{B}}^\alpha,\\ \alpha = 1, 2, \qquad \beta = 0, 1, 2,\end{aligned}$$

therefore

$$(4.23)\qquad \beta_p = \tilde{\beta}_p(\varrho, \operatorname{tr}\mathbf{M}_i\dot{\mathbf{B}}^\alpha, \operatorname{tr}\dot{\mathbf{B}}^3, H_j).$$

The representation of the orthotropic symmetric tensor function has also been considered in the papers [2–5]. In those papers the same number of generators as in (4.20) describe such a representation. There is, however, a difference between our representation and the representation proposed in [2 – 5]. In the present paper the scalar functions associated with the generators depend on the invariants of the structural tensor $\mathbf{H}$. In this manner we can take into account material inhomogeneities, because $\mathbf{H}$ and ϱ depend on a position of the material point considered. Allowance for the eigenvalues of $\mathbf{H}$ in the functions (4.23) delivers a possibility of determination of the material orthotropy. This is not the case with the structural tensors used in [2 – 5]. For instance, as a "measure" of the material orthotropy one can take the quantities H_2/H_1 and H_3/H_1 provided that $H_1 > H_2 > H_3$.

Applying the approach proposed by BOEHLER [2, 3], it can be shown that an alternative nonpolynomial representation of (4.20) contains only seven generators:

$$\mathbf{A} = \delta_1\mathbf{M}_1 + \delta_2\mathbf{M}_2 + \delta_3\mathbf{M}_3 + \delta_4(\mathbf{M}_1\dot{\mathbf{B}} + \dot{\mathbf{B}}\mathbf{M}_1) + \delta_5(\mathbf{M}_2\dot{\mathbf{B}} + \dot{\mathbf{B}}\mathbf{M}_2) + \delta_6(\mathbf{M}_3\dot{\mathbf{B}} + \dot{\mathbf{B}}\mathbf{M}_3) + \delta_7\dot{\mathbf{B}}^2, \tag{4.24}$$

where

$$\delta_p = \tilde{\delta}(\varrho, \operatorname{tr}\mathbf{M}_i\dot{\mathbf{B}}^\alpha, \operatorname{tr}\dot{\mathbf{B}}^3, H_j), \qquad p = 1, \ldots, 7. \tag{4.25}$$

Another form includes two arbitrary tensors from the set $\{\mathbf{M}_i\}$ $(i = 1, 2, 3)$, say $\mathbf{M}_1$ and $\mathbf{M}_2$,

$$\mathbf{A} = \kappa_1\mathbf{I} + \kappa_2\mathbf{M}_1 + \kappa_3\mathbf{M}_2 + \kappa_4\dot{\mathbf{B}} + \kappa_5(\mathbf{M}_1\dot{\mathbf{B}} + \dot{\mathbf{B}}\mathbf{M}_1) + \kappa_6(\mathbf{M}_2\dot{\mathbf{B}} + \dot{\mathbf{B}}\mathbf{M}_2) + \kappa_7\dot{\mathbf{B}}^2. \tag{4.26}$$

Here

$$\kappa_p = \tilde{\kappa}(\varrho, \operatorname{tr}\dot{\mathbf{B}}^i, \operatorname{tr}\mathbf{M}_\alpha\dot{\mathbf{B}}^\beta, H_j), \qquad \alpha, \beta = 1, 2, \quad p = 1, \ldots, 7. \tag{4.27}$$

Obviously, the representations (4.24) and (4.26) are equivalent. Consequently, for $\mathbf{C} = \mathbf{H}$ the nonpolynomial representation of the tensor function (4.1) has the form (4.6), with γ_p being functions of J_s and H_i where $p, s = 1, ..., 7$ and $i = 1, 2, 3$.

In the case of transverse isotropy, for instance when $H_1 = H_2 = H$ and $\mathbf{i}_3$ is a privileged direction, one obtains the following additional relations between the invariants involved in (4.25):

$$\begin{aligned} \operatorname{tr}\mathbf{H}^i &= 2H^i + H_3^i, \qquad i = 1, 2, 3, \\ \operatorname{tr}\mathbf{H}^\alpha\dot{\mathbf{B}}^\beta &= H^\alpha \operatorname{tr}\dot{\mathbf{B}}^\beta + (H_3^\alpha - H^\alpha)\operatorname{tr}\mathbf{M}_3\dot{\mathbf{B}}^\beta, \qquad \alpha, \beta = 1, 2, \end{aligned} \tag{4.28}$$

as well as between the generators appearing in (4.24):

$$\begin{aligned} \mathbf{H}^\alpha &= H^\alpha\mathbf{I} + (H_3^\alpha - H^\alpha)\mathbf{M}_3, \\ \mathbf{H}^\alpha\dot{\mathbf{B}}^\beta + \dot{\mathbf{B}}^\beta\mathbf{H}^\alpha &= 2H^\alpha\dot{\mathbf{B}}^\beta + (H_3^\alpha - H^\alpha)(\dot{\mathbf{B}}^\beta\mathbf{M}_3 + \dot{\mathbf{B}}^\beta\mathbf{M}_3). \end{aligned} \tag{4.29}$$

Thus we arrive at the transversely isotropic representation of the tensor function (3.1), cf. also BOEHLER [2, 3],

$$\mathbf{A} = \varrho_1 \mathbf{I} + \varrho_2 \mathbf{M}_3 + \varrho_3 \dot{\mathbf{B}} + \varrho_4(\mathbf{M}_3\dot{\mathbf{B}} + \dot{\mathbf{B}}\mathbf{M}_3) + \varrho_5 \dot{\mathbf{B}}^2 + \varrho_6(\mathbf{M}_3\dot{\mathbf{B}}^2 + \dot{\mathbf{B}}^2\mathbf{M}_3), \tag{4.30}$$

where

$$\varrho_p = \tilde{\varrho}(\varrho, \operatorname{tr}\dot{\mathbf{B}}^i, \operatorname{tr}\mathbf{M}_3\dot{\mathbf{B}}^\alpha, H, H_3), \qquad \alpha = 1,2, \quad i = 1,2,3, \quad p = 1,\ldots,6. \tag{4.31}$$

The simplest is the case of isotropy: $H_1 = H_2 = H_3 = H$. Then we easily obtain

$$\mathbf{A} = \gamma_1 \mathbf{I} + \gamma_2 \dot{\mathbf{B}} + \gamma_3 \dot{\mathbf{B}}^2, \tag{4.32}$$

where

$$\gamma_i = \tilde{\gamma}_i(\varrho, \operatorname{tr}\dot{\mathbf{B}}^i, H). \tag{4.33}$$

REMARK 4. Assuming *a priori* that the tensor function (3.1) involves only symmetric two-dimensional tensors, (4.1) simplifies to

$$\mathbf{A} = \mathbf{F}(\varrho, \dot{\mathbf{B}}, \mathbf{C}) = \overline{\alpha}_1 \mathbf{I} + \overline{\alpha}_2 \mathbf{C} + \overline{\alpha}_3 \dot{\mathbf{B}}, \tag{4.34}$$

where $\mathbf{I}$ denotes the two-dimensional unit tensor; moreover

$$\overline{a}_i = \tilde{a}_i(\operatorname{tr}\dot{\mathbf{B}}, \operatorname{tr}\dot{\mathbf{B}}^2, \operatorname{tr}\dot{\mathbf{B}}\mathbf{C}, \operatorname{tr}\mathbf{C}, \operatorname{tr}\mathbf{C}^2). \tag{4.35}$$

The representation (4.34) with (4.35) is formally the same in the case of both polynomial and nonpolynomial representation.

For $\mathbf{C} = \mathbf{H}$ and $H_1 = H_2 = H$, (4.34) reduces to

$$\mathbf{A} = \overline{\beta}_1 \mathbf{I} + \overline{\beta}_2 \dot{\mathbf{B}}, \tag{4.36}$$

where

$$\overline{\beta}_\alpha = \tilde{\beta}_\alpha(\operatorname{tr}\dot{\mathbf{B}}, \operatorname{tr}\dot{\mathbf{B}}^2, H), \qquad \alpha = 1,2. \quad \triangleleft \tag{4.37}$$

5. Some specific cases

The aim of this section is to examine the tensor function (4.34) by imposing suitable homogeneity requirements. Next, particular cases of plastic flow laws and locking rules as well as yield and locking conditions will be investigated.

5.1. Two-dimensional case

Deleting the bar over functions in (4.34) we write

$$\mathbf{A} = \alpha_1\mathbf{I} + \alpha_2\mathbf{C} + \alpha_3\dot{\mathbf{B}} = \widetilde{\mathbf{F}}(\varrho, \dot{\mathbf{B}}, \mathbf{C}), \tag{5.1}$$

where

$$\alpha_i = \tilde{\alpha}_i(\varrho, \operatorname{tr}\dot{\mathbf{B}}, \|\dot{\mathbf{B}}\|, \operatorname{tr}\dot{\mathbf{B}}\mathbf{C}, \operatorname{tr}\mathbf{C}, \|\mathbf{C}\|), \quad i = 1,2,3. \tag{5.2}$$

Here, for the sake of convenience, a nonpolynomial basis has been chosen.

Since

$$\frac{\partial\widetilde{\mathbf{F}}}{\partial\dot{\mathbf{B}}} = \mathbf{I}\otimes\frac{\partial\tilde{\alpha}_1}{\partial\dot{\mathbf{B}}} + \mathbf{C}\otimes\frac{\partial\tilde{\alpha}_2}{\partial\dot{\mathbf{B}}} + \tilde{\alpha}_3\mathbf{1} + \dot{\mathbf{B}}\otimes\frac{\partial\tilde{\alpha}_3}{\partial\dot{\mathbf{B}}}, \tag{5.3}$$

$$\frac{\partial\tilde{\alpha}_i}{\partial\dot{\mathbf{B}}} = \frac{\partial\tilde{\alpha}_i}{\partial\operatorname{tr}\dot{\mathbf{B}}} + \frac{\partial\tilde{\alpha}_i}{\partial\|\dot{\mathbf{B}}\|}\frac{\dot{\mathbf{B}}}{\|\dot{\mathbf{B}}\|} + \frac{\partial\tilde{\alpha}_3}{\partial\operatorname{tr}\mathbf{C}\dot{\mathbf{B}}}\mathbf{C}, \tag{5.4}$$

therefore (5.1) is a potential law provided that

$$\frac{1}{\|\dot{\mathbf{B}}\|}\frac{\partial\tilde{\alpha}_1}{\partial\|\dot{\mathbf{B}}\|} = \frac{\partial\tilde{\alpha}_3}{\partial\operatorname{tr}\dot{\mathbf{B}}}, \qquad \frac{\partial\tilde{\alpha}_1}{\partial\operatorname{tr}\mathbf{C}\dot{\mathbf{B}}} = \frac{\partial\tilde{\alpha}_2}{\partial\operatorname{tr}\dot{\mathbf{B}}}, \qquad \frac{1}{\|\dot{\mathbf{B}}\|}\frac{\partial\tilde{\alpha}_2}{\partial\|\dot{\mathbf{B}}\|} = \frac{\partial\tilde{\alpha}_3}{\partial\operatorname{tr}\mathbf{C}\dot{\mathbf{B}}}. \tag{5.5}$$

In (5.3), $\mathbf{1}$ denotes the 2D unit tensor of the fourth order.

The homogeneity condition of degree zero now gives

$$\frac{\partial\widetilde{\mathbf{F}}}{\partial\dot{\mathbf{B}}}\cdot\dot{\mathbf{B}} = \left(\frac{\partial\tilde{\alpha}_1}{\partial\dot{\mathbf{B}}}\cdot\dot{\mathbf{B}}\right)\mathbf{I} + \left(\frac{\partial\tilde{\alpha}_2}{\partial\dot{\mathbf{B}}}\cdot\dot{\mathbf{B}}\right)\mathbf{C} + \left(\frac{\partial\tilde{\alpha}_3}{\partial\dot{\mathbf{B}}}\cdot\dot{\mathbf{B}} + \tilde{\alpha}_3\right)\dot{\mathbf{B}} = \mathbf{0}. \tag{5.6}$$

Equation (5.6) implies that $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are homogeneous functions of degree zero, while $\tilde{\alpha}_3$ is a homogeneous function of degree (-1) with respect to $\dot{\mathbf{B}}$.

Substituting (5.4) into (5.6) one obtains

$$\begin{aligned} \frac{\partial\tilde{\alpha}_\alpha}{\partial\dot{\mathbf{B}}}\cdot\dot{\mathbf{B}} &= \frac{\partial\tilde{\alpha}_\alpha}{\partial K_i}K_i = 0, \qquad \alpha = 1,2, \quad i = 1,2,3, \\ \frac{\partial\tilde{\alpha}_3}{\partial\dot{\mathbf{B}}}\cdot\dot{\mathbf{B}} + \tilde{\alpha}_3 &= \frac{\partial\tilde{\alpha}_3}{\partial K_i}K_i + \tilde{\alpha}_3 = 0, \end{aligned} \tag{5.7}$$

where

$$\{K_i\} = \left\{\operatorname{tr}\dot{\mathbf{B}}, \operatorname{tr}\dot{\mathbf{B}}\mathbf{C}, \|\dot{\mathbf{B}}\|\right\}. \tag{5.8}$$

Introducing new variables

$$x = \ln\|\dot{\mathbf{B}}\|, \qquad p_1 = \frac{\operatorname{tr}\dot{\mathbf{B}}}{\|\dot{\mathbf{B}}\|}, \qquad p_2 = \frac{\operatorname{tr}\mathbf{C}\dot{\mathbf{B}}}{\|\dot{\mathbf{B}}\|}, \tag{5.9}$$

from (5.7) one gets the equations

$$\frac{\partial \tilde{\alpha}_\alpha}{\partial x} = 0, \qquad \frac{\partial \tilde{\alpha}_3}{\partial x} + \alpha_3 = 0. \tag{5.10}$$

Their solutions are

$$\tilde{\alpha}_\alpha = a_\alpha(\varrho, p_1, p_2, \operatorname{tr}\mathbf{C}, \|\mathbf{C}\|), \qquad \tilde{\alpha}_3 = \frac{1}{\|\dot{\mathbf{B}}\|} a_3(\varrho, p_1, p_2, \operatorname{tr}\mathbf{C}, \|\mathbf{C}\|). \tag{5.11}$$

Taking into account (5.11) in (5.1) one obtains

$$\mathbf{A} = a_1\mathbf{I} + a_2\mathbf{C} + \frac{a_3}{\|\dot{\mathbf{B}}\|}\dot{\mathbf{B}}. \tag{5.12}$$

The last constitutive law is of a potential type provided that

$$\frac{\partial a_1}{\partial p_\alpha} p_\alpha = -\frac{\partial a_3}{\partial p_1}, \qquad \frac{\partial a_2}{\partial p_\alpha} p_\alpha = -\frac{\partial a_3}{\partial p_2}, \qquad \frac{\partial a_1}{\partial p_2} = \frac{\partial a_2}{\partial p_1}, \quad \alpha = 1, 2. \tag{5.13}$$

These relations are obtained by substitution of (5.11) into (5.5).

From (5.12) one gets

$$\begin{aligned} \operatorname{tr}\mathbf{A} &= 2a_1 + a_2 \operatorname{tr}\mathbf{C} + a_3 p_1, \\ \operatorname{tr}\mathbf{AC} &= a_1 \operatorname{tr}\mathbf{C} + a_2\|\mathbf{C}\|^2 + a_3 p_2, \\ \|\mathbf{A}\|^2 &= 2a_1^2 + a_3^2 + a_2\left(2a_1 \operatorname{tr}\mathbf{C} + a_2\|\mathbf{C}\|^2\right) + 2a_3(a_1 p_1 + a_2 p_2). \end{aligned} \tag{5.14}$$

In the case when $\mathbf{A} = \boldsymbol{\sigma}$, the set (5.14) is called a parametric yield condition; similarly if $\mathbf{A} = \boldsymbol{\varepsilon}$, a parametric locking condition is obtained.

If the parameters p_α can be eliminated from (5.14), then the invariants of $\mathbf{A}$ are interrelated by a scalar relation:

$$f(\varrho, \operatorname{tr}\mathbf{A}, \|\mathbf{A}\|, \operatorname{tr}\mathbf{AC}, \operatorname{tr}\mathbf{C}, \|\mathbf{C}\|) = 0. \tag{5.15}$$

From (5.12) one can easily derive the semi-inverse relation

$$\frac{\dot{\mathbf{B}}}{\|\dot{\mathbf{B}}\|} = \frac{1}{a_3}(\mathbf{A} - a_1\mathbf{I} - a_2\mathbf{C}), \tag{5.16}$$

which represents either a flow rule ($\mathbf{A} = \boldsymbol{\sigma}$), or a locking law ($\mathbf{A} = \boldsymbol{\varepsilon}$).

To derive specific forms of the constitutive relationships, one has to postulate concrete forms of the functions a_i or b_i ($i = 1, 2, 3$). For instance, simple polynomial forms were used by the second author in the case of perfectly plastic isotropic materials [34].

As we already know, the spherical tensor $\mathbf{C} = \mathbf{H}$ characterises an isotropic material. The functions appearing in (4.36) are then given by

$$\beta_1 = \tilde{\beta}_1 \left(\frac{\operatorname{tr}\dot{\mathbf{B}}}{\|\dot{\mathbf{B}}\|}, H \right), \qquad \beta_2 = \frac{1}{\|\dot{\mathbf{B}}\|} \tilde{\beta}_2 \left(\frac{\operatorname{tr}\dot{\mathbf{B}}}{\|\dot{\mathbf{B}}\|}, H \right). \tag{5.17}$$

We pass now to providing simple examples.

EXAMPLE 1

Consider the two-dimensional case when the perfectly locking constraint affects only the volumetric part of the strain tensor of an isotropic material, cf. [11, 33]. We set

$$\mathbf{A} = \frac{1}{2}(\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{I}, \qquad \dot{\mathbf{B}} = \frac{1}{2}(\operatorname{tr}\dot{\boldsymbol{\sigma}})\mathbf{I}.$$

Now we have

$$\|\dot{\mathbf{B}}\| = \frac{1}{\sqrt{2}} |\operatorname{tr}\dot{\boldsymbol{\sigma}}|. \tag{5.18}$$

Taking into account (5.17), the constitutive relationship (4.36) reduces to

$$(\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{I} = \sqrt{2} c \frac{\operatorname{tr}\dot{\boldsymbol{\sigma}}}{|\operatorname{tr}\dot{\boldsymbol{\sigma}}|} \mathbf{I}, \tag{5.19}$$

where c is a material coefficient. Hence the locking condition is given by

$$|\operatorname{tr}\boldsymbol{\varepsilon}| = \sqrt{2} c. \tag{5.20}$$

EXAMPLE 2 (Perfect plasticity, isotropic material)

For the case of plane stresses, we set $\mathbf{A} = \boldsymbol{\sigma}$, $\dot{\mathbf{B}} = \dot{\boldsymbol{\varepsilon}}$ while the functions appearing in (5.17) are assumed in the form:

$$\tilde{\beta}_1 = \frac{\sqrt{2} k p}{\sqrt{1 + p^2}}, \qquad \tilde{\beta}_2 = \frac{\sqrt{2} k}{\sqrt{1 + p^2}}, \qquad p = \frac{\operatorname{tr}\dot{\boldsymbol{\varepsilon}}}{\|\dot{\boldsymbol{\varepsilon}}\|}, \quad k = \text{const}. \tag{5.21}$$

Then (4.26) gives

$$\boldsymbol{\sigma} = \frac{\sqrt{2} k}{\sqrt{1 + p^2}} \left(p\mathbf{I} + \frac{\dot{\boldsymbol{\varepsilon}}}{\|\dot{\boldsymbol{\varepsilon}}\|} \right), \tag{5.22}$$

and the yield condition has the following form

$$\frac{\|\mathbf{s}\|^2}{2k^2} + \frac{\operatorname{tr}^2 \boldsymbol{\sigma}}{12 k^2} = 1, \tag{5.23}$$

where $\mathbf{s} = \boldsymbol{\sigma} - \frac{1}{2}(\mathrm{tr}\boldsymbol{\sigma})\mathbf{I}$; k is the yield limit in shear. Obviously, (5.23) represents the Huber–Mises yield condition. The flow rule (5.22) is associated with this condition. The plastic dissipation density is given by

$$D(\dot{\boldsymbol{\varepsilon}}) = \mathrm{tr}\boldsymbol{\sigma}\dot{\boldsymbol{\varepsilon}} = \sqrt{2}k\|\dot{\boldsymbol{\varepsilon}}\|\sqrt{1+p^2}. \tag{5.24}$$

EXAMPLE 3

We shall now consider a perfectly plastic, incompressible and orthotropic material in the case of plane deformations, $\mathbf{C} = \mathbf{H}$. In this case one obviously has $\mathrm{tr}\dot{\boldsymbol{\varepsilon}} = 0$ and (5.12) gives

$$\mathrm{tr}\boldsymbol{\sigma} = 2c_1\,, \qquad \mathbf{s} = c_2\mathbf{H}_d + \frac{c_3}{\|\dot{\mathbf{d}}\|}\dot{\mathbf{d}}, \tag{5.25}$$

where

$$\begin{gathered} \mathbf{H}_d = \mathbf{H} - \frac{1}{2}(\mathrm{tr}\mathbf{H})\mathbf{I}, \qquad \dot{\mathbf{d}} = \dot{\boldsymbol{\varepsilon}} - \frac{1}{2}(\mathrm{tr}\dot{\boldsymbol{\varepsilon}})\mathbf{I} = \dot{\boldsymbol{\varepsilon}}, \\ c_\alpha = \tilde{c}_\alpha\left(\varrho, \frac{\mathrm{tr}\dot{\mathbf{d}}\mathbf{H}_d}{\|\dot{\mathbf{d}}\|}, \mathrm{tr}\mathbf{H}, \|\mathbf{H}_d\|\right). \end{gathered} \tag{5.26}$$

Note that for $c_1 = 0$ the yield condition and flow rule do not depend upon $\mathrm{tr}\boldsymbol{\sigma}$. The parametric yield condition resulting from (5.25) has then the form

$$\mathrm{tr}\mathbf{s}\mathbf{H}_d = c_2\|\mathbf{H}_d\|^2 + c_3 q, \qquad \|\mathbf{s}\|^2 = c_2^2\|\mathbf{H}_d\|^2 + 2c_2c_3q + c_3^2\,, \tag{5.27}$$

where

$$q = \frac{\mathrm{tr}\dot{\mathbf{d}}\mathbf{H}_d}{\|\dot{\mathbf{d}}\|}. \tag{5.28}$$

For $q = 0$ the yield condition is given by

$$(1 - \cos^2\alpha)\|\mathbf{s}\|^2 = c_3^2, \tag{5.29}$$

where

$$\cos\alpha = \frac{\mathrm{tr}\mathbf{s}\mathbf{H}_d}{\|\mathbf{s}\|\|\mathbf{H}_d\|}, \qquad \text{if} \quad \mathbf{s} \neq \mathbf{0}. \tag{5.30}$$

Obviously, for an isotropic material one has the well known flow rule

$$\mathbf{s} = \frac{\sqrt{2}k}{\|\dot{\mathbf{d}}\|}\dot{\mathbf{d}}, \qquad k = \mathrm{const}, \tag{5.31}$$

and the yield condition

$$\|\mathbf{s}\|^2 = 2k^2. \tag{5.32}$$

EXAMPLE 4

Consider now a plastic incompressible material, still in the case of plane deformation. Let now

$$\mathbf{A}_d = \mathbf{s}, \quad \dot{\mathbf{B}}_d = \dot{\mathbf{d}}, \quad \mathbf{C}_d = \mathbf{d}, \quad \mathbf{d} = \boldsymbol{\varepsilon} - \frac{1}{2}(\mathrm{tr}\boldsymbol{\varepsilon})\mathbf{I}, \quad \mathrm{tr}\dot{\boldsymbol{\varepsilon}} = 0.$$

Equation $(5.25)_2$ is written in the form

$$\overset{*}{\mathbf{s}} = \mathbf{s} - c_2\mathbf{d} = \frac{c_3}{\|\dot{\mathbf{d}}\|}\dot{\mathbf{d}}. \tag{5.33}$$

Here we assume that

$$c_\alpha = \tilde{c}_\alpha(\|\mathbf{d}\|), \qquad \alpha = 2, 3. \tag{5.34}$$

The yield condition following from (5.33) is given by

$$\mathrm{tr}\left(\overset{*}{\mathbf{s}}\right)^2 = c_3^2. \tag{5.35}$$

It takes into account the simple kinematic hardening and the isotropic hardening. Under the influence of plastic deformations, initially isotropic material becomes an orthotropic one. We note that specific cases studied in [30, 31] fall within our general framework.

So far only simple cases of constitutive relationships have been discussed, provided that deformations are plane. Here we shall not study more complex and essentially new constitutive relationships, which naturally result from our rather general approach. Note that there still remains the problem of identification of material functions, particularly for locking materials.

6. Three-dimensional case

Proceeding similarly to the two-dimensional case, one obtains the following constitutive relationships satisfying (3.2):

$$\begin{aligned} \mathbf{A} = {} & a_1\mathbf{I} + a_2\mathbf{C} + a_3\mathbf{C}^2 + \frac{a_4}{\|\dot{\mathbf{B}}\|}\dot{\mathbf{B}} + \frac{a_5}{\|\dot{\mathbf{B}}\|^2}\dot{\mathbf{B}}^2 + \frac{a_6}{\|\dot{\mathbf{B}}\|}\left(\mathbf{C}\dot{\mathbf{B}} + \dot{\mathbf{B}}\mathbf{C}\right) \\ & + \frac{a_7}{\|\dot{\mathbf{B}}\|}\left(\mathbf{C}^2\dot{\mathbf{B}} + \dot{\mathbf{B}}\mathbf{C}^2\right) + \frac{a_8}{\|\dot{\mathbf{B}}\|^2}\left(\mathbf{C}\dot{\mathbf{B}}^2 + \dot{\mathbf{B}}^2\mathbf{C}\right) + \frac{a_9}{\|\dot{\mathbf{B}}\|^2}\left(\mathbf{C}^2\dot{\mathbf{B}}^2 + \dot{\mathbf{B}}^2\mathbf{C}^2\right), \end{aligned} \tag{6.1}$$

where

$$a_p = \widetilde{a}_p(\varrho, p_m, \operatorname{tr}\mathbf{C}^i), \qquad p = 1,\ldots,9, \quad m = 1,\ldots,6, \quad i = 1,2,3. \tag{6.2}$$

Here

$$\begin{gathered} p_m = \frac{r_m}{\|\dot{\mathbf{B}}\|}, \qquad r_1 = \operatorname{tr}\dot{\mathbf{B}}, \qquad r_2 = \operatorname{tr}\dot{\mathbf{B}}\mathbf{C}, \qquad r_3 = \operatorname{tr}\dot{\mathbf{B}}\mathbf{C}^2, \\ r_4 = \sqrt{\operatorname{tr}\dot{\mathbf{B}}^2\mathbf{C}}, \qquad r_5 = \sqrt{\operatorname{tr}\dot{\mathbf{B}}^2\mathbf{C}^2}, \qquad r_6 = \sqrt[3]{\operatorname{tr}\dot{\mathbf{B}}^3}. \end{gathered} \tag{6.3}$$

Note that the functions $\widetilde{\alpha}_p$ cannot be polynomials constructed from the elements of the integrity basis (4.3). Moreover, $\widetilde{\alpha}_1$, $\widetilde{\alpha}_4$ and $\widetilde{\alpha}_5$ are homogeneous functions of degree zero; $\widetilde{\alpha}_2$, $\widetilde{\alpha}_6$, $\widetilde{\alpha}_7$ are homogeneous of degree (-1) while $\widetilde{\alpha}_3$, $\widetilde{\alpha}_8$, $\widetilde{\alpha}_9$ are homogeneous of degree (-2). Obviously, the homogeneity holds with respect to $\dot{\mathbf{B}}$.

Applying the generalized Cayley-Hamilton theorem due to RIVLIN [26], from (6.1) one can construct 7 invariants K_s $(s = 1,\ldots,7)$ as follows,

$$\{K_s\} = \{\operatorname{tr}\mathbf{A}^i, \operatorname{tr}\mathbf{A}^\alpha\mathbf{C}^\beta\}, \qquad i = 1,2,3, \quad \alpha,\beta = 1,2. \tag{6.4}$$

Taking into account (6.2) we write

$$K_s = g_s(\varrho, p_m, \operatorname{tr}\mathbf{C}^i). \tag{6.5}$$

For $\mathbf{A} = \boldsymbol{\sigma}$ the last relation represents the yield condition in a parametric form while for $\mathbf{A} = \boldsymbol{\varepsilon}$ a parametric form of the locking condition is obtained.

Suppose that the parameters p_m $(m = 1,\ldots,6)$ can be eliminated from (6.5). Then

$$f(\varrho, K_s, \operatorname{tr}\mathbf{C}^i) = 0, \tag{6.6}$$

is the yield condition when $\mathbf{A} = \boldsymbol{\sigma}$ or locking condition for $\mathbf{A} = \boldsymbol{\varepsilon}$.

To derive the semi-inverse form of (6.1) one has to find the following generators, cf. BOEHLER [2]:

$$\mathbf{A}^i, \ \mathbf{A}^\alpha\mathbf{C}^\beta + \mathbf{C}^\beta\mathbf{A}^\alpha, \qquad i = 1,2,3, \quad \alpha,\beta = 1,2. \tag{6.7}$$

After some algebraic manipulations we finally obtain

$$\begin{aligned} \frac{\dot{\mathbf{B}}}{\sqrt{\operatorname{tr}\dot{\mathbf{B}}^2}} = \tilde{\lambda}\Big[& b_1\mathbf{I} + b_2\mathbf{C} + b_3\mathbf{C}^2 + b_4\mathbf{A} + b_5\mathbf{A}^2 + b_6(\mathbf{CA} + \mathbf{AC}) \\ & + b_7(\mathbf{C}^2 A + \mathbf{AC}^2) + b_8(\mathbf{CA}^2 + \mathbf{A}^2\mathbf{C}) + b_9(\mathbf{C}^2\mathbf{A}^2 + \mathbf{A}^2\mathbf{C}^2)\Big], \qquad \tilde{\lambda} \geq 0, \end{aligned} \tag{6.8}$$

where $\tilde{\lambda}$ and b_p $(p = 1,\ldots,9)$ are functions of ϱ, p_m $(m = 1,\ldots,6)$ and $\operatorname{tr}\mathbf{C}^i$.

The general constitutive relationships just derived provide a rational basis for more specific equations, which model orthotropic, perfectly plastic or perfectly locking materials for $\mathbf{C} = \mathbf{H}$. For instance, associated laws follow by assuming (6.6) as a yield or locking condition, which is equivalent to the condition, see (6.11) below,

$$\frac{\partial \tilde{b}_r}{\partial K_s} = \frac{\partial \tilde{b}_s}{\partial K_r}, \qquad s, r = 1, \ldots, 7. \tag{6.9}$$

If the function f is sufficiently regular, then (6.8) takes the form

$$\dot{\mathbf{B}} = \lambda\Big[\tilde{b}_1 \mathbf{I} + 2\tilde{b}_2 \mathbf{A} + 3\tilde{b}_3 \operatorname{tr}\mathbf{A}^2 + \tilde{b}_4 \operatorname{tr}\mathbf{H} + \tilde{b}_5 \operatorname{tr}\mathbf{H}^2 + \tilde{b}_6(\mathbf{HA} + \mathbf{AH}) + \tilde{b}_7(\mathbf{H}^2\mathbf{A} + \mathbf{AH}^2)\Big], \qquad \lambda \geq 0, \tag{6.10}$$

where

$$\tilde{b}_s = \frac{\partial f}{\partial K_s}, \qquad s = 1, \ldots, 7; \tag{6.11}$$

and $\{K_s\} = \{\operatorname{tr}\mathbf{A}, \operatorname{tr}\mathbf{A}^2, \operatorname{tr}\mathbf{A}^3, \operatorname{tr}\mathbf{HA}, \operatorname{tr}\mathbf{H}^2\mathbf{A}, \operatorname{tr}\mathbf{HA}^2, \operatorname{tr}\mathbf{H}^2\mathbf{A}^2\}$. Note that the plastic or locking multiplier λ can be calculated from (3.3) as follows:

$$\lambda = \frac{D}{D'}, \tag{6.12}$$

where

$$D' = \tilde{b}_1 \operatorname{tr}\mathbf{A} + 2\tilde{b}_2 \operatorname{tr}\mathbf{A}^2 + 3\tilde{b}_3 \operatorname{tr}\mathbf{A}^3 + \tilde{b}_4 \operatorname{tr}\mathbf{HA} + \tilde{b}_5 \operatorname{tr}\mathbf{H}^2\mathbf{A} + 2\tilde{b}_6 \operatorname{tr}\mathbf{HA}^2 + 2\tilde{b}_7 \operatorname{tr}\mathbf{H}^2\mathbf{A}^2.$$

The last form of λ is important for numerical computations.

REMARK 5. Suppose that (6.8) has been derived from a potential $g(\varrho, K_s, \operatorname{tr}\mathbf{C}^i)$ and $\mathbf{C} = \mathbf{H}$. One readily verifies that in such case b_8 and b_9 vanish while the remaining material functions satisfy (6.9) and (6.11), with f being replaced by g.

REMARK 6. Following BOEHLER [4] and BOEHLER and SAWCZUK [6, 7], a simpler version of (3.1) is obtained when

$$\mathbf{A}^* = \mathbf{F}^*(\varrho, \dot{\mathbf{B}}), \tag{6.13}$$

where

$$\mathbf{A}^* = \mathbb{C}(\mathbf{H}) \cdot \mathbf{A}. \tag{6.14}$$

Here $\mathbb{C}$ is a fourth order tensor function of $\mathbf{H}$, satisfying $\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{klij}$. The representation of such tensor function was considered by the second author in [35]. The tensor function appearing on the r.h.s. of (6.13) and satisfying the homogeneity condition

$$\frac{\partial \mathbf{F}^*}{\partial \dot{\mathbf{B}}} \cdot \dot{\mathbf{B}} = \mathbf{0} \qquad \text{if} \qquad \frac{\partial \mathbf{F}^*}{\partial \dot{\mathbf{B}}} \neq \tilde{\mathbf{0}}, \tag{6.15}$$

was discussed in detail in our previous papers [18, 19]. To use the results presented in those two papers, $\mathbf{A}$ has to be replaced with $\mathbf{A}^*$.

EXAMPLE 5. Of practical interest is the following specific case of (6.6) for $\mathbf{C} = \mathbf{H}$, independent of $\operatorname{tr}\mathbf{A}^3$:

$$\begin{aligned} f(\varrho, K_s \operatorname{tr}\mathbf{H}^i) = {} & c_1 \operatorname{tr}\mathbf{A} + c_2 \operatorname{tr}\mathbf{AH} + c_3 \operatorname{tr}\mathbf{AH}^2 + d_1 \operatorname{tr}^2\mathbf{A} + d_2 \operatorname{tr}\mathbf{A}^2 \\ & + d_3 \operatorname{tr}^2\mathbf{AH} + d_4 \operatorname{tr}\mathbf{A} \operatorname{tr}\mathbf{AH} + d_5 \operatorname{tr}^2\mathbf{AH}^2 + d_6 \operatorname{tr}\mathbf{A} \operatorname{tr}\mathbf{AH}^2 \\ & + d_7 \operatorname{tr}\mathbf{AH} \operatorname{tr}\mathbf{AH}^2 + d_8 \operatorname{tr}\mathbf{A}^2\mathbf{H} + d_9 \operatorname{tr}\mathbf{A}^2\mathbf{H}^2 - 1, \end{aligned} \tag{6.16}$$

where c_i $(i = 1, 2, 3)$ and d_p $(p = 1, \ldots, 9)$ are scalar functions of ϱ and $\operatorname{tr}\mathbf{H}^i$. Note that for a given material with prescribed ϱ, one has $c_i = \text{const}$, $d_p = \text{const}$; obviously c_i and d_p may also depend on the position of a material point.

The condition (6.16) represents an orthotropic yield condition for $\mathbf{A} = \boldsymbol{\sigma}$; similarly, if $\mathbf{A} = \boldsymbol{\varepsilon}$ then it describes an orthotropic locking condition. The yield condition of the form (6.16) is an invariant form of the condition proposed by TSAI and WU [37]. For more details including the description of experimental tests for the determination of the constants c_i and d_p we refer the reader to the paper by COWIN [9].

The anisotropic yield conditions due to HOFFMAN [16] and Mises-Hill [15] are particular cases of (6.16), cf. also [35].

The functions $\tilde{b}_r$ appearing in the associated law (6.10) are now given by

$$\begin{aligned} \tilde{b}_1 &= \frac{\partial f}{\partial \operatorname{tr}\mathbf{A}} = c_1 + 2d_1 \operatorname{tr}\mathbf{A} + d_4 \operatorname{tr}\mathbf{AH} + d_6 \operatorname{tr}\mathbf{AH}^2, \\ \tilde{b}_2 &= \frac{\partial f}{\partial \operatorname{tr}\mathbf{A}^2} = d_2\,, \qquad \tilde{b}_3 = \frac{\partial f}{\partial \operatorname{tr}\mathbf{A}^3} = 0, \\ \tilde{b}_4 &= \frac{\partial f}{\partial \operatorname{tr}\mathbf{AH}} = c_2 + 2d_3 \operatorname{tr}\mathbf{AH} + d_4 \operatorname{tr}\mathbf{A} + d_7 \operatorname{tr}\mathbf{AH}^2, \\ \tilde{b}_5 &= \frac{\partial f}{\partial \operatorname{tr}\mathbf{AH}^2} = c_3 + 2d_5 \operatorname{tr}\mathbf{AH}^2 + d_6 \operatorname{tr}\mathbf{A} + d_7 \operatorname{tr}\mathbf{AH}, \\ \tilde{b}_6 &= \frac{\partial f}{\partial \operatorname{tr}\mathbf{A}^2\mathbf{H}} = d_8\,, \qquad \tilde{b}_7 = \frac{\partial f}{\partial \operatorname{tr}\mathbf{A}^2\mathbf{H}^2} = d_9\,, \end{aligned} \tag{6.17}$$

where f is obviously specified by (6.16). It can easily be verified that (6.9) is satisfied.

Let us set $d_p = 0$ in (6.16). Then the resulting locking condition, i.e. for $\mathbf{A} = \boldsymbol{\varepsilon}$, represents a generalization of the isotropic condition (5.25) to orthotropic materials.

7. Final remarks

Real materials may exhibit elastic, plastic and locking behaviour. In the case of small deformations the tensor of elastic deformations can easily be included into our scheme. In a separate paper we shall be concerned with modelling the elastic-plastic-locking behaviour. It seems that locking behaviour does not affect all components of the strain tensor, i.e. the locking condition should probably be imposed on particular modes of deformations. Unfortunately, in the relevant literature one cannot find reliable experimental data. Our considerations clearly reveal the role of nonpolynomial representations for modelling plastic and locking behaviour of materials. For instance, such representations involve quotients of polynomials, square roots, etc.

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WARSAW UNIVERSITY OF TECHNOLOGY
CIVIL ENGINEERING FACULTY, INSTITUTE OF STRUCTURAL MECHANICS

and

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH

e-mail: jtelega@ippt.gov.pl

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