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On the description of the consolidation phenomenon by means of a two-component continuum

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THE PURPOSE of the present paper is the consistent formulation of the initial-boundary value problem for the consolidation phenomenon within the frame of a new two-component continuum model. The new class of models of two-component continua characterized by the balance equation for porosity is presented. The initial-boundary value problem with regard to the physical features of the consolidation is formulated. Some additional constitutive relations for the boundary quantities are proposed. Bearing in mind these constitutive relations, an example of a one-dimensional structure is calculated. The results of the numerical simulation are the basis for the parameter study of some constants of the model.

1. Introduction

THE HISTORY of development of the mathematical description of the consolidation phenomenon goes back to the twenties of this century. At this time *von Terzaghi* has derived the consolidation equation under strongly simplifying assumptions and presented it in his work [1]. The most important assumption was that the influence of the inertia forces has been neglected in his investigations. The consequences of this assumption are that the disturbance propagates in the domain with infinite speed what does not agree with the reality. This equation is the basis of consolidation calculation within the scope of soil mechanics up to the present time.

The development of the mixture theories and, especially, of the porous media theories based on the principles of the continuum mechanics has allowed to describe this phenomenon on the macroscopical level in mathematically exact and physically more accurate way.

The most of the existing macroscopic models of this sort are based on the model of multicomponent continuum with the so-called incompressible components that have been introduced by BOWEN [2]. From the point of view of continuum mechanics these models differ from each other only in the choice of constitutive relations. However, the model of BOWEN and its different modifications cause serious mathematical problems, particularly in their numerical treatment because the number of the governing equations and the number of the unknown fields are not equal.

One of the consequences of this fact is that the consistent formulation of the boundary value problem for most models of this type is not possible. The existing numerical calculations could be obtained only with additional conditions on the boundary. These conditions are justified neither from the mathematical nor from the physical point of view. HUTTER, JÖHNK and SVENDSEN show in their work [3] an example of such a model yielding results only in one of the limit cases namely only then if solely one of the constituting components exists! We shall not discuss the problems connected with the construction of the class of Bowen-like models, nevertheless it can be easily proved that most of these models are mathematically not consistent.

Moreover, there exist also models proposed in the works [4, 5, 6 and 7] based on the concept of the so-called equilibrated forces. The system of the governing equations in such models is closed with an additional equation which is motivated by the information won from the microscopic level of observation. However, the identification of the microscopic quantities is not obvious and it causes some problems.

A detailed overview of the existing porous media models and the discussion of some new tendencies in the theories of porous media can be found in the paper by DE BOER [8].

The class of models which is the basis for the formulation of the initial-boundary value problem and for the numerical calculation in this work is the new one. It has been developed by WILMAŃSKI and presented first in the paper [9].

The system of governing equations of this model includes the balance equation for porosity. This equation allows the macroscopical description of the properties of the semimicroscopical level of observation. Due to the fact that the porosity is a scalar variable, only one of the properties of the semimicroscopic domain, namely the volume contribution of the pores, can be reflected on the macroscopic level in this way. The physical motivation for this equation and its derivation can be taken from the work [10] of WILMAŃSKI.

The aim of this work is the formulation of the initial-boundary value problem for the consolidation phenomenon and the numerical parameter study of some model constants appearing in the class of models presented in the paper [9]. In the second section, a simplified model constituting the basis for our calculation is described. Then, we go over to the main part of this paper. We formulate the initial-boundary value problem for the simplified model in a consistent way. The fourth section is devoted to the description of the methods enabling the calculation of a simple one-dimensional example. Finally, the work is closed with the discussion of the numerical results and completed with the concluding remarks.

2. Basic concepts

The thermodynamical behaviour of a two-component continuum constituting of materially homogeneous components, a solid and a fluid one, can be described

generally in a similar way as the behaviour of a mixture of two immiscible fluids, by using the spatial description for the fluid and the material description for the solid through the following set of unknown fields

$$\{\varrho_t^F, \mathbf{v}_F\} \times \{\varrho^S, \boldsymbol{\chi}_S, n\} \,. \tag{2.1}$$

The only difference between a two-component porous medium and a mixture of two immiscible fluids is the occurence of the variable n standing for the porosity which apears in the set (2.1). The quantities ϱ_t^F and ϱ^S describe the macroscopical partial mass densities of the fluid (current) and of the solid (initial), respectively. The motion of the solid component is described by the function $\boldsymbol{\chi}_S$ and the kinematics of the fluid is defined by the *Eulerian* description by the velocity field $\mathbf{v}_F$.

In order to avoid the difficulties in the formulation of the boundary value problem we choose after WILMAŃSKI [11] the *Lagrangean* uniform description for both components. In such a case we have for a homogeneous solid without mass exchange between components

$$\varrho^S = \text{const} \,. \tag{2.2}$$

Then, the kinematic behaviour of the fluid is determined by the so-called *Lagrangean* fluid velocity $\mathbf{X}'_F$, which has been derived in [11].

The set (2.1) of the unknown fields takes now the form

$$f := \{\varrho^F, \mathbf{X}'_F, \boldsymbol{\chi}_S, n\} \,, \tag{2.3}$$

where ϱ^F is the mass density of the fluid referred to the reference configuration of the solid.

Due to the *Lagrangean* uniform description for both components, the functions of the set f have the same domain, i.e. they are functions of the material points X of the skeleton and of the time t.

The material points X of the skeleton belong to the domain $\mathcal{B}(X \in \mathcal{B})$ of the three-dimensional differentiable manifold. In this work we identify the domain $\mathcal{B}$ with a chosen configuration of the skeleton with the positions $\mathbf{X}$ of the material points X at the instant of time $t \equiv t_0$. Then, the current position of a material point X of the solid is defined in the following way

$$\mathbf{x} = \boldsymbol{\chi}_S(\mathbf{X}, t), \qquad \mathbf{x} \in \mathcal{B}_t \,, \qquad \mathcal{B}_t := \boldsymbol{\chi}_S(\mathcal{B}, t) \subset \mathbf{R}^3, \quad t \in \mathcal{T} \subset \mathbf{R}^1, \tag{2.4}$$

where $\mathcal{B}_t$ is the actual configuration.

If the set f of mappings is the solution of an appriopriate initial-boundary value problem for the set of field equations then isothermal processes taking place in the above continuum can be defined in the following way

$$\bigwedge_{\mathbf{x}\in\mathcal{B},\, t\in\mathcal{T}} : (\mathbf{X}, t) \mapsto (\varrho^F, \mathbf{X}'_F, \mathbf{x}, n) \in \mathcal{V}^8, \tag{2.5}$$

where $\mathcal{V}^8$ is the eight-dimensional vector space of the values of the fields.

The fields (2.3) should be determined through the system of partial differential equations following from the balance equations. In the case of the linear model, considered in this work, we do not have to distinguish between the material (*Lagrangean*) and *Eulerian* description of motion [11]. We can also use the displacement vector $\mathbf{u}_S$ instead of the function of motion $\boldsymbol{\chi}_S$. Consequently, the set of unknown fields for such a case can be chosen in the following manner

$$F := \left\{\varrho_t^F, n^\triangle, \mathbf{v}_F, \mathbf{u}_S\right\}, \qquad n^\triangle := n - n_0\,, \tag{2.6}$$

where n_0 is the constant equilibrium value of the porosity.

The corresponding field equations follow from the balance laws. For the elastic skeleton and the ideal fluid, they have the form

$$\begin{aligned}
&\frac{\partial \varrho_t^F}{\partial t} + \operatorname{div}(\varrho_t^F \mathbf{v}_F) = 0,\\
&\frac{\partial n^\triangle}{\partial t} + \gamma_0 \operatorname{div} \mathbf{v}_F = -\frac{n^\triangle}{\tau}\,,\\
&\varrho^S \frac{\partial^2 \mathbf{u}_S}{\partial t^2} - \operatorname{div} \mathbf{T}_S - \bar{\pi}\mathbf{w} - \varrho^S \mathbf{b}_S = \mathbf{0},\\
&\varrho_t^F \frac{\partial \mathbf{v}_F}{\partial t} + \varrho_t^F \operatorname{grad} \mathbf{v}_F \mathbf{v}_F - \operatorname{div} \mathbf{T}_F + \bar{\pi}\mathbf{w} - \varrho_t^F \mathbf{b}_F = \mathbf{0}, \quad \mathbf{w} := \mathbf{v}_F - \frac{\partial \mathbf{u}_S}{\partial t}\,,
\end{aligned} \tag{2.7}$$

where the partial stress tensors $\mathbf{T}_S$ and $\mathbf{T}_F$ satisfy the following constitutive relations

$$\begin{aligned}
&\mathbf{T}_S = \mathbf{T}_S^{\mathrm{cl}} + p^{\mathrm{int}}\mathbf{I}, \qquad \mathbf{T}_S^{cl} = \lambda^S (\mathbf{E}_S \cdot \mathbf{I})\mathbf{I} + 2\mu^S \mathbf{E}_S\,,\\
&\mathbf{T}_F = -p_{c1}^F \mathbf{I} - p^{\mathrm{int}}\mathbf{I}, \quad p_{c1}^F := p_0^F + \frac{1}{\kappa^F} \ln\left(\frac{\varrho_t^F}{\varrho_{t0}^F}\right), \qquad p^{\mathrm{int}} := \gamma_0 \frac{\varrho_t^F}{\tau \mathcal{N}} n^\triangle\,.
\end{aligned} \tag{2.8}$$

The above relations contain the following material constants depending on the equilibrium porosity n_0

$$\mathcal{C} := \{\lambda^S, \mu^S, \kappa^F, \bar{\pi}, \mathcal{N}, \tau, \gamma_0\}\,. \tag{2.9}$$

As shown in the earlier papers on the subject (e.g. [9, 12]), they can be found for many materials by means of dynamical experiments.

Otherwise, the tensor $\mathbf{E}_S$ in (2.8) denotes the *Green–St. Venant* deformation tensor for small deformations

$$\mathbf{E}_S \approx \frac{1}{2}\left[\operatorname{grad} \mathbf{u}_S + (\operatorname{grad} \mathbf{u}_S)^T\right] \tag{2.10}$$

and ϱ_{t0}^F denotes the reference value of the fluid mass density corresponding to the pressure p_0^F.

3. The consistent formulation of the initial-boundary value problem for consolidation

The consolidation phenomenon as a special kind of physical processes is already well known. During such processes, which take place in multicomponent continua, one or more components flow out of the domain of the multicomponent continuum due to the action of the external load. Consequently, the concept of free surface must be accounted for by the calculation of the consolidation phenomena. In contrast to the one-component materials, in which the free surface problems appear only then if wave propagation, phase changes, plastic deformation and some other problems of the change of material structure are being treated, the free surface in the multicomponent continua appears already if the free boundary is permeable.

If we assume that the boundary surface is material with respect to the solid boundary $\partial\mathcal{B}$, i.e. we identify the boundary of the treated continuum with the boundary of the skeleton then it is singular as well as non-material for the outflowing component. This fact is then reflected in the relation

$$\mathbf{c} \equiv \mathbf{v}_S|_{\partial\mathcal{B}}\,, \tag{3.1}$$

where $\mathbf{c}$ is the velocity of the free boundary.

Further, we shall assume that the boundary surface is ideal, i.e. it does not possess any intrinsic structure. We denote such a boundary surface as S.

In order to describe this surface we have to formulate the balance equations on the ideal surface S. We do not need to derive these conditions because their derivation is standard and it has been found by WILMAŃSKI in his work [11]. If we write the governing equations (2.7) in the integral form and extend them to hold in the limit on the singular surfaces then we obtain the following local dynamic compatibility conditions:

- for the solid component

$$[\mathbf{T}_S]\,\mathbf{n} = \mathbf{0}, \tag{3.2}$$

- for the fluid component

$$[\varrho_t^F(\mathbf{v}_F - \mathbf{v}_S)]\cdot\mathbf{n} = 0, \tag{3.3}$$

$$[\mathbf{T}_F]\,\mathbf{n} = [\varrho_t^F(\mathbf{v}_F - \mathbf{v}_S)\cdot\mathbf{n}\,\mathbf{v}_F]\,, \tag{3.4}$$

- for the porosity

$$[N_0(\mathbf{v}_F - \mathbf{v}_S)]\cdot\mathbf{n} = 0, \tag{3.5}$$

where N_0 is a constant and the square brackets denote the difference of the limit values between the positive and the negative sides of the surface S, i.e.

$$[...] := (...)^+ - (...)^- \tag{3.6}$$

and the internal side of the boundary has been chosen as the negative one. It is convenient to introduce the following definition

$$m^F := \varrho_t^F (\mathbf{v}_F - \mathbf{v}_S) \cdot \mathbf{n}. \tag{3.7}$$

The relation (3.7) represents namely the mass flux of the fluid component through the boundary surface S per time unit and area unit. The physical meaning of the flow continuity through the non-material surface for the fluid is that the fluid component does not stick to this surface.

In contrast to the relation (3.4), the relation (3.2) is identical with the classical *Poisson* condition. The relation (3.4) shows that the contact force in the fluid is not continuous. We will see in the sequel that this fact has a great influence on the formulation of the boundary value problem because such an inhomogeneous dynamic compatibility condition indicates the existence of the free surface. The compatibility condition (3.5) for the porosity does not influence the formulation of the boundary value problem in the simplified case considered in this work. Therefore we skip it in our consideration. Finally it should be mentioned that the mass conservation law for the skeleton does not appear as a dynamic compatibility condition because it is identically fulfilled.

We can now pass over to the formulation of the boundary value problem. Since we have to formulate the boundary quantity neither for the mass conservation law $(2.7)_1$ nor for the equation for porosity $(2.7)_2$, the mathematical structure of the governing equations (2.7) requires only the formulation of two vector quantities on the boundary. We can see that in the case of the chosen simplified model, the balance equation for porosity transforms to the evolution equation $(2.7)_2$.

For this reason, we shall treat in the sequel only both the balance equations of momentum. First, we integrate the relations $(2.7)_3$ and $(2.7)_4$ over the domain $\mathcal{B}$ and obtain, after using the compatibility conditions (3.2), (3.4) and the definition (3.7),

$$\begin{aligned}
&\int_{\mathcal{B}} \left(\varrho^S \frac{\partial^2 \mathbf{u}_S}{\partial t^2} - \bar{\pi}\mathbf{w} - \varrho^S \mathbf{b}_S \right) dv = \int_{\partial\mathcal{B}} \mathbf{t}_S^+ \, da \,, \\
&\int_{\mathcal{B}} \left(\varrho_t^F \frac{\partial \mathbf{v}_F}{\partial t} + \varrho_t^F \operatorname{grad} \mathbf{v}_F \mathbf{v}_F + \bar{\pi}\mathbf{w} - \varrho_t^F \mathbf{b}_F \right) dv \\
&\qquad = \int_{\partial\mathcal{B}} \left[\mathbf{t}_F^+ - m^{F+} \left(\mathbf{v}_F^+ - \mathbf{v}_F^- \right) \cdot \mathbf{n}\, \mathbf{n} \right] da \,.
\end{aligned} \tag{3.8}$$

In this transformation we have employed the following relation

$$\mathbf{v}_F^+ - \mathbf{v}_F^- = \left(\mathbf{v}_F^+ \cdot \mathbf{n} - \mathbf{v}_F^- \cdot \mathbf{n}\right)\mathbf{n} \tag{3.9}$$

resulting from the fact that the partial fluid stress tensor contains only the spherical part.

The analysis of the relations (3.8) leads to the conclusion that due to the outflow of the fluid component, two vector and two scalar boundary quantities for two-component continua described by similar equations as the relations (2.7) must be specified on the boundary. These are in our case

$$\mathcal{R} := \left\{\mathbf{t}_S^+, \mathbf{t}_F^+, m^{F^+}, \mathbf{v}_F^+ \cdot \mathbf{n}\right\}. \tag{3.10}$$

It should also be mentioned that the treatment of one-component continua does not require the formulation of an additional scalar quantity until we have to solve a problem with the non-material surface such as, for instance, the wave propagation problems. This fact has been illustrated in the paper [13] in the discussion of an example of propagation of surface waves.

Next, we formulate the set of boundary quantities (3.10) in accordance with the physics of the consolidation phenomenon. Before we do so, the class of boundaries which can appear in the consolidation problem must be defined.

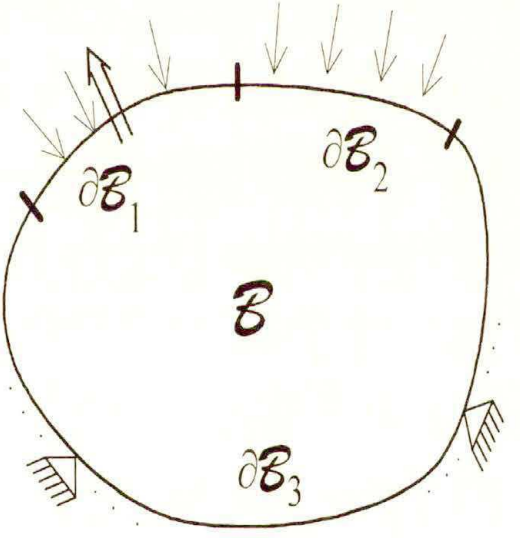

FIG. 1. The class of boundaries by consolidation – a possibility.

In Fig. 1 we show schematically these possible classes of boundary conditions on the boundary $\partial\mathcal{B}$ $(= \partial\mathcal{B}_1 + \partial\mathcal{B}_2 + \partial\mathcal{B}_3)$ of the body $\mathcal{B}$. Namely

- $\partial\mathcal{B}_1$ – loaded and permeable free boundary,

- $\partial\mathcal{B}_2$ – loaded and impermeable boundary,
- $\partial\mathcal{B}_3$ – impermeable fixed boundary.

By the definition of the boundary quantities the physical features of consolidation must be taken into consideration. As we have already mentioned, the consolidation process will be characterized through the outflow of one or more fluid components outside from the domain of the multicomponent continuum due to the action of the external load. One of the consequences of the outflow process is the existence of the free surface, whose description, as we have shown already, requires an additional scalar quantity prescribed on the boundary. Moreover, the acting external load cannot be *a priori* divided into separate parts acting on the fluid and on the solid, respectively, because it does not happen in the reality.

According to this, we add up the partial loads $\mathbf{t}_S^+$ and $\mathbf{t}_F^+$ in the following constitutive way

$$\mathbf{t}^{\text{ext}} = \mathbf{t}_S^+ + \mathbf{t}_F^+ , \tag{3.11}$$

where the index "ext" indicates the whole external acting load. If we make the above assumption we cannot use the set (3.10) in the formulation of the boundary value problem because we are missing one vector quantity. However, this additional vector quantity can be defined on the boundary in accordance with the physical features of the consolidation phenomenon.

The outflow of the fluid component through the boundary of the domain of the multicomponent continuum means that the velocity difference $\mathbf{w}$ on the boundary is not equal to zero. So we choose this vector quantity as the missing second vector boundary relation. It can be also written in the form

$$\mathbf{w}^+ = w^+\mathbf{n} + \mathbf{w}_\perp^+ , \tag{3.12}$$

where $\mathbf{w}_\perp$ has the meaning of the velocity difference, perpedicular to the unit outward normal vector $\mathbf{n}$. For all components of the vector $\mathbf{w}^+$ we must propose constitutive relations. Before we pass over to the formulation of the boundary quantities, we want to replace the scalar quantity $(\mathbf{v}_F^+ \cdot \mathbf{n})$ by $\varrho_t^{F^+}$ by means of the definition (3.7) in the following way

$$\left(\mathbf{v}_F^+ - \mathbf{v}_F^-\right) \cdot \mathbf{n} = m^{F^+} \left(\frac{1}{\varrho_t^{F+}} - \frac{1}{\varrho_t^{F-}}\right). \tag{3.13}$$

With (3.13) the right-hand side of the relation $(3.8)_2$ transforms to

$$\int_{\mathcal{B}} (...)\, dv = \int_{\partial\mathcal{B}} \left[\mathbf{t}_F^+ - \left(m^{F^+}\right)^2 \left(\frac{1}{\varrho_t^{F+}} - \frac{1}{\varrho_t^{F-}}\right)\mathbf{n}\right] da. \tag{3.14}$$

It has been shown in the paper [13] that the second term of the surface integral can be neglected in consolidation problems because the value of the

acting force normal to the boundary is much greater than the value of the second term, the so-called flow force. However, this force must be taken into account in cases characterized through the rapid outflow processes such as combustion. This problem is nonlinear, even if the operator is linear because the nonlinearity results from the nonlinear boundary condition.

The above considerations show that the set (3.10) must be replaced by the following one

$$\mathcal{R}' := \left\{ \mathbf{t}^{\text{ext}}, \mathbf{w}^+, m^{F^+}, \varrho_t^{F^+} \right\}. \tag{3.15}$$

For the last three quantities the constitutive realtions must be formulated. We begin with the vector quantity $\mathbf{w}^+$. We assume after WILMAŃSKI ([11]) that the outflowing fluid cannot slip along the boundary. Therefore, we obtain for both components which are perpedicular to the normal vector

$$\mathbf{w}_\perp^+ \equiv \mathbf{0}. \tag{3.16}$$

In the case of an ideal fluid, this result can be obtained indepedently as a mathematical consequence of the material properties.

The component in the direction of the normal unit vector can be directly formulated from the relation (3.7) using the definition $(2.7)_5$

$$w^+ = \frac{m^{F^+}}{\varrho_t^{F^+}}. \tag{3.17}$$

We see, that the normal component of one of the two vector boundary quantities is determined by the remaining scalar boundary quantities. One can also observe that the vector boundary quantity (3.12) is an inhomogeneous one. This is an additional characteristic property of the multicomponent continua. For the mass flux of the fluid component we assume the following constitutive relation

$$m^{F^+} = \beta \left(p^{F^-}\big|_{\partial\mathcal{B}} - n^-|_{\partial\mathcal{B}}\, p^A \right). \tag{3.18}$$

A similar relation has been proposed for the consolidation phenomenon by RUNESSON in his Ph.D. Thesis [14]. However, relations of this form are also well known in other fields of physics (e.g. heat transfer through thin walls). Some indications of the difficulties connected with the formulation of the boundary value problem can be also found in the papers [20 – 23].

We see, that the mass flux of the fluid is determined by the partial pressure of the fluid and by one more external quantity, namely the atmospheric pressure p^A, weighted, according to *Dalton*'s law, with the porosity. The coefficient β describes the physical properties of the boundary and has, due to this interpretation, the meaning of surface permeability. We shall show in the parameter study in the Sec. 5, how great is the influence of the value of this coefficient on the value of the

mass transport through the boundary. For the purpose of this work we assume that the properties of the boundary are independent of the time and piecewise homogeneous. We obtain then

$$\frac{\partial \beta}{\partial t} \equiv 0, \qquad \frac{\partial \beta}{\partial \mathbf{x}} \equiv \mathbf{0} \;\Rightarrow\; \beta = \text{const}. \tag{3.19}$$

We shall assume for the second scalar boundary quantity that the fluid mass density on the exterior of the boundary is constant, i.e.

$$\varrho_t^{F^+} = \text{const}. \tag{3.20}$$

We see that the relations (3.11) ÷ (3.12), (3.16) ÷ (3.18) and (3.20) determine the quantities included in the set $\mathcal{R}'$. Summarizing the above analysis we obtain the following relations for the whole boundary (see Fig. 1):

- loaded and permeable free boundary

$$\bigwedge_{\mathbf{x}\in\partial\mathcal{B}_1} \begin{cases} \mathbf{t}^{\text{ext}} \neq \mathbf{0}, \\ \mathbf{w}^+ = \dfrac{m^{F^+}}{\varrho_t^{F^+}}\mathbf{n}, \\ m^{F^+} = \beta\left(p^{F^-} - n^- p^A\right), \\ \varrho_t^{F^+} = \text{const}, \end{cases} \tag{3.21}$$

- loaded and impermeable boundary

$$\bigwedge_{\mathbf{x}\in\partial\mathcal{B}_2} \begin{cases} \mathbf{t}^{\text{ext}} \neq \mathbf{0}, \\ \mathbf{w}^+ = \dfrac{m^{F^+}}{\varrho_t^{F^+}}\mathbf{n}, \\ m^{F^+} = 0, \\ \varrho_t^{F^+} = \text{const}, \end{cases} \tag{3.22}$$

- impermeable fixed boundary

$$\bigwedge_{\mathbf{x}\in\partial\mathcal{B}_3} \begin{cases} \mathbf{v}_F^+ = \mathbf{0}, \\ \mathbf{w}^+ = \mathbf{0}, \end{cases} \tag{3.23}$$

where the boundary quantities can take the forms defined above.

Let us mention that the conditions (3.21) and (3.22) are the so-called mixed or *Robbin*'s boundary conditions and the conditions (3.23) are the well known essential or *Dirichlet*'s boundary conditions. In the case of a fixed boundary we can accept another possible definition of the boundary quantities since we need only two from the three existing kinematic quantities.

Due to the choice of the second vector quantity, namely the velocity difference $\mathbf{w}$, the system of the governing equations (2.7) must be transformed in a suitable way because the velocity difference $\mathbf{w}$ does not belong to the set (2.6) of the unknown fields. There are two ways of transforming the governing equations. The first one is the extension of the set of fields and of the equations through the relation $(2.7)_5$. The second one is the variable transformation

$$\left\{\varrho_t^F, n^{\triangle}, \mathbf{v}_F, \mathbf{u}_S\right\} \to \left\{\varrho_t^F, n^{\triangle}, \mathbf{v}_F, \mathbf{w}\right\}. \tag{3.24}$$

We have decided to go the first way because it is technically easier to handle. Let us notice that the boundary value problem for the additional equation does not need to be formulated since the relation $(2.7)_5$ belongs to the class of the evolution equations. The set of the unknown fields (2.6) takes now the form

$$F' := \left\{\varrho_t^F, n^{\triangle}, \mathbf{v}_F, \mathbf{u}_S, \mathbf{w}\right\}. \tag{3.25}$$

Finally, we formulate the initial value problem for the above set of unknown fields. We choose

$$\bigwedge_{\mathbf{x}\in\mathcal{B}} \begin{cases} \varrho_t^F(\mathbf{x},t)|_{t=0} = \varrho_0^F(\mathbf{x}), \\ n^{\triangle}(\mathbf{x},t)\big|_{t=0} = 0, \\ \mathbf{u}_S(\mathbf{x},t)|_{t=0} = \mathbf{0}, \\ \mathbf{v}_F(\mathbf{x},t)|_{t=0} = \mathbf{0}, \\ \mathbf{w}(\mathbf{x},t)|_{t=0} = \mathbf{0}, \end{cases} \tag{3.26}$$

where the function $\varrho_0^F(\mathbf{x})$ is the one-dimensional static solution of the balance law of momentum for fluid. If we introduce the coordinates shown in the Fig. 2 a, we obtain then the following distribution of the initial fluid mass density

$$\varrho_0^F \equiv \varrho^F(z, t=0) \cong \varrho_{t0}^F\left[1 + \kappa^F \varrho_{t0}^F g(h-z)\right] \tag{3.27}$$

with g as the value of the gravity acceleration.

With the last assertion we have closed the formulation of the initial-boundary value problem and now we pass over to the treatment of the numerical example.

4. Numerical simulation

In the present section we proceed to construct the weak formulation with respect to the already mentioned physical features of the consolidation.

The numerical simulation will be carried by the finite element method. We shall treat a simple one-dimensional example shown below in the Fig. 2 a. In this case the set (3.25) of unknown fields takes the following form

$$F^1 := \left\{\varrho_t^F, n^{\triangle}, u^S, v^F, w\right\}. \tag{4.1}$$

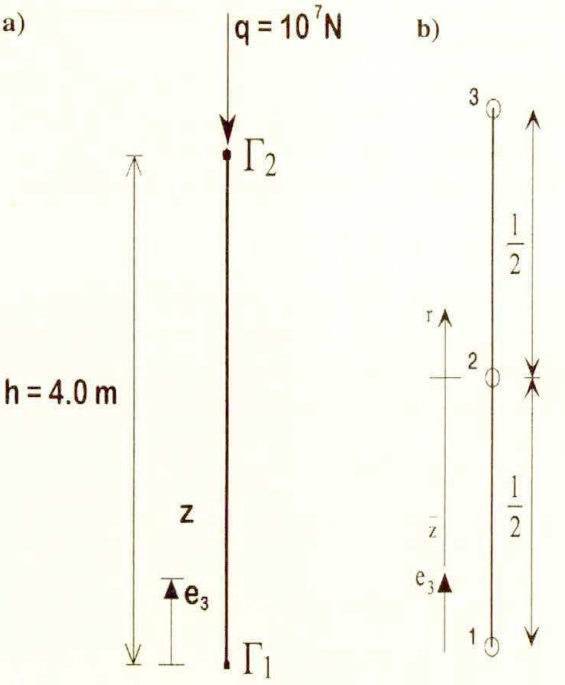

FIG. 2. a. The one-dimensional domain. b. Three-nodal continuum element.

For the purpose of the space discretization we use the finite element method and proceed in the following way. We identify a finite number G of points in the whole one-dimensional domain D, shown in the Fig. 2 a, and these shall be called nodal points. The whole domain will be divided in the standard way into E subdomains (finite elements) connected at nodes on their boundaries. The functions which form the set F^1 will be approximated locally over each finite element by continuous functions, the so-called trial functions, which are uniquely defined in terms of the values of the functions (or also their derivatives) at the nodal points belonging to each element. Furthermore, we make use of isoparametric elements by which the approximation of the element coordinates and of the functions appears by applying the same interpolation functions.

In accordance with the chosen one-dimensional domain, we use for the discretization three-nodal elements (see Fig. 2 b) with the well known quadratic linearly independent trial functions

$$\begin{aligned} \varphi_1 &= \frac{1}{2}(1-r) - \frac{1}{2}(1-r^2), \\ \varphi_2 &= 1 - r^2, \\ \varphi_3 &= \frac{1}{2}(1+r) - \frac{1}{2}(1-r^2), \end{aligned} \tag{4.2}$$

where the relation $r \equiv (2/1)\bar{z} - 1$ to calculate the natural coordinates holds true. We construct then the approximating functions $\check{\xi}$ for all unknown fields in the following manner

$$\check{\xi} = \sum_{j=1}^{3} \check{\xi}_j \varphi_j \,, \qquad \check{\xi} \equiv \left\{ \check{\varrho}_t^F, \check{n}^{\triangle}, \check{u}^S, \check{v}^F, \check{w} \right\}. \tag{4.3}$$

The most important property of the above relations is the fact that they fulfil each boundary condition for the chosen finite element because the trial function takes the value one on the node it is defined for, and vanishes identically on the other nodes of the element.

It is clear that the approximating functions do not fulfil exactly the set of the governing equations. If we insert the relations (4.3) into the governing equations they shall not be satisfied. There remains the error ε which is also called the residuum. In order to minimize the residuum ε, the undetermined parameters $\check{\xi}_j$ must be properly chosen. Let us mention that the above form (4.3) of the approximating functions is the conventional one in which the parameters $\check{\xi}_j$ depend on time and the trial functions (4.2) are functions of the space variable.

For the optimization of the approximating functions (4.3) the *Galerkin* method has been chosen. In the same way as by the application of other weighted residual methods, the arising errors will be projected one after another on the test functions, which are in the case of the *Galerkin* method the same as the trial functions φ_j. We obtain then for each equation an orthogonality condition of the form

$$\bigwedge_{i=\{1,2,3\}} (\varepsilon, \varphi_i) \equiv \int_{\mathcal{B}} \varepsilon \varphi_i \, dv = 0. \tag{4.4}$$

Using this procedure, the errors ε_j will be minimized in the averaging sense because the limited number of the trial functions spans only a finite-dimensional subspace of the space of exact solutions.

The use of the *Galerkin* method does not allow to define the essential boundary conditions. Therefore, we define in the sequel the coefficients which have the sense of the arbitrary increments of the prescribed kinematic boundary quantities. These arbitrary increments vanish identically for the prescribed value of the corresponding quantity and are undetermined if the quantity is unknown. Due to the relation (3.23), such arbitrary increments must be defined in our one-dimensional case for v^F and for w.

If we now apply the orthogonality conditions (4.4) to each of the governing equations (2.7) using the definition of the scalar product, we obtain in the

one-dimensional case the following equations:

$$
\bigwedge_{i:\ i=1,2,3}
\begin{cases}
\displaystyle\int_{-1}^{1}\left\{\frac{\partial\check{\varrho}_t^F}{\partial t}+\frac{\partial\check{\varrho}_t^F}{\partial z}\check{v}^F+\check{\varrho}_t^F\frac{\partial\check{v}^F}{\partial z}\right\}\varphi_i\,dr=0,\\
\displaystyle\int_{-1}^{1}\left\{\frac{\partial\check{n}^{\triangle}}{\partial t}+\gamma_0\frac{\partial\check{v}^F}{\partial z}+\frac{\check{n}^{\triangle}}{\tau}\right\}\varphi_i\,dr=0,\\
\displaystyle\int_{-1}^{1}\left\{\varrho^S\frac{\partial\check{v}^F}{\partial t}-\varrho^S\frac{\partial\check{w}}{\partial t}-\frac{\partial\check{\sigma}^S}{\partial z}-\bar{\pi}\check{w}+\varrho^S g\right\}\varphi_i\,dr=0,\\
\displaystyle\int_{-1}^{1}\left\{\check{\varrho}_t^F\frac{\partial\check{v}^F}{\partial t}+\check{\varrho}_t^F\frac{\partial\check{v}^F}{\partial z}\check{v}^F-\frac{\partial\check{\sigma}^F}{\partial z}+\bar{\pi}\check{w}\right\}\check{v}_i^{F^\delta}\varphi_i=0,\\
\displaystyle\int_{-1}^{1}\left\{\frac{\partial\check{u}^S}{\partial t}-\check{v}^F+\check{w}\right\}\check{w}_i^\delta\varphi_i\,dr=0,
\end{cases}
\tag{4.5}
$$

where $\check{v}_i^{F^\delta}$ and $\check{w}_i^\delta$ are the arbitrary increments described above, and the following definitions have been used

$$
\begin{aligned}
\check{\sigma}^S &:= (\lambda^S+2\mu^S)\frac{\partial\check{u}^S}{\partial z}+\frac{\gamma_0}{\mathcal{N}\tau}\check{\varrho}_t^F\check{n}^{\triangle},\\
\check{\sigma}^F &:= -\left(p_0^F+\frac{1}{\kappa^F}\ln\left(\frac{\check{\varrho}_t^F}{\varrho_{t0}^F}\right)+\frac{\gamma_0}{\mathcal{N}\tau}\check{\varrho}_t^F\check{n}^{\triangle}\right).
\end{aligned}
\tag{4.6}
$$

Let us mention that, due to prescribed distribution of the initial partial fluid mass density (see (3.27)), the influence of the gravity force on the fluid component has already been accounted for. In such a case the gravity force does not appear in the relation $(4.5)_4$.

If we prescribe the natural boundary condition on the boundary Γ_1 (see Fig. 2 a) and neglect (as mentioned in the Sec. 3) the nonlinear boundary contributions, then, bearing in mind the structure of the unspecified boundary quantities, the dynamic compatibility conditions (3.2)÷(3.4) and the definition (3.7), we obtain from the equation $(4.5)_4$

$$
\int_{-1}^{1}\left\{\left[\check{\varrho}_t^F\frac{\partial\check{v}^F}{\partial t}+\check{\varrho}_t^F\frac{\partial\check{v}^F}{\partial z}\check{v}^F+\bar{\pi}\check{w}\right]\check{v}^{F^\delta}+\check{\sigma}^F\frac{\partial\check{v}^{F^\delta}}{\partial z}+\frac{\partial(\check{\sigma}^S\check{v}^{F^\delta})}{\partial z}\right\}dr
= t^{\mathrm{ext}}\check{v}^{F^\delta}\Big|_{\Gamma=\Gamma_1+\Gamma_2},
\tag{4.7}
$$

where the relations $t^{\mathrm{ext}}\big|_{\Gamma_1}=-q$ for the top element and $\check{v}_1^{F^\delta}\big|_{\Gamma_2}\equiv 0$ for the

bottom element hold. To simplify the notation the following definition

$$\check{v}^{F^\delta} := \sum_{i=1}^{3} \check{v}_i^{F^\delta} \varphi_i \tag{4.8}$$

has been introduced.

Replacing Eq. $(4.5)_4$ by the relation (4.7) and integrating the system of Eqs. (4.5) over the treated one-dimensional domain we obtain a nonlinear system of algebraic equations which can be written in the following matrix form

$$\mathsf{D}\,\dot{\mathsf{u}} + \mathsf{K}\,\mathsf{u} = \mathsf{R}\,, \tag{4.9}$$

where D is the so-called damping matrix, K the so-called stiffness matrix, u and $\dot{\mathsf{u}}$ denote the process vector and its time derivative, respectively, and R denotes the load vector. We skip here the presentation of their explicit form. The solution of the system of nonlinear equations can be obtained by means of the *Newton – Raphson* method and the time integration by means of the *Newmark* method. Since the *Newmark* method is an implicit one, the relation (4.9) will be solved for the time $t + \triangle t$. The matrix equation (4.9) takes then the form

$$^{t+\triangle t}\mathsf{D}\,^{t+\triangle t}\dot{\mathsf{u}} + {}^{t+\triangle t}\mathsf{K}\,^{t+\triangle t}\mathsf{u} = {}^{t+\triangle t}\mathsf{R}\,. \tag{4.10}$$

The linearization of the above relation by means of the *Newton – Raphson* method leads to

$$^{t+\triangle t}_{(\iota-1)}\mathsf{D}\,^{t+\triangle t}_{(\iota)}\triangle\dot{\mathsf{u}} + {}^{t+\triangle t}_{(\iota-1)}\mathsf{K}\,^{t+\triangle t}_{(\iota)}\triangle\mathsf{u} = {}^{t+\triangle t}\mathsf{R} - {}^{t+\triangle t}_{(\iota-1)}\mathsf{F}\,, \tag{4.11}$$

where the index ι denotes the number of the iteration step, $^{t+\triangle t}_{(\iota-1)}\mathsf{F}$ is the internal force vector corresponding to the stresses, and $\triangle\mathsf{u}$ and $\triangle\dot{\mathsf{u}}$ denote the increments of the process vector and of its time derivative, respectively.

For the approximation of the increment of the time derivative of the process vector, the following linearized *Newmark* ansatz will be used

$$^{t+\triangle t}\triangle\dot{\mathsf{u}} = \frac{\delta}{\alpha\triangle t}\,{}^{t+\triangle t}\triangle\mathsf{u}\,. \tag{4.12}$$

Substitution of the above relation into Eq. (4.11) yields

$$^{t+\triangle t}_{(\iota-1)}\hat{\mathsf{K}}\,^{t+\triangle t}_{(\iota)}\triangle\mathsf{u} = {}^{t+\triangle t}\mathsf{R} - {}^{t+\triangle t}_{(\iota-1)}\mathsf{F}\,, \tag{4.13}$$

with the following definition of the so-called effective stiffness matrix

$$^{t+\triangle t}_{(\iota-1)}\hat{\mathsf{K}} := \frac{\delta}{\alpha\triangle t}\,{}^{t+\triangle t}_{(\iota-1)}\mathsf{D} + {}^{t+\triangle t}_{(\iota-1)}\mathsf{K}\,. \tag{4.14}$$

Furthermore, we choose for the constants α and δ the values 0.25 and 0.50, respectively. In this case the *Newmark* method is identical with the constant average acceleration method.

With the solution of Eq. (4.13) the process vector at the time $t + \triangle t$ and for the iteration step ι can be determined from the relation

$$(4.15) \qquad {}^{t+\triangle t}_{(\iota)}\mathsf{u} = {}^{t+\triangle}_{(\iota-1)}\mathsf{u} + {}^{t+\triangle t}_{(\iota)} \triangle \mathsf{u}\,.$$

If a chosen breaking off criterion is fulfilled, the value ${}^{t+\triangle t}_{(\iota)}\mathsf{u}$ is simultaneously the initial value of the process vector for the next time step.

5. Parameter study and concluding remarks

The purpose of this section is to study the constants of the chosen simplified model. We present here the qualitative comparison of the results obtained in this work with the classical results of the consolidation theory as well as we study the influence of variation the material constants on the quantitative results. The values of the following constants: λ^S, μ^S, $\mathcal{N}$ and τ have been taken from the work [15] of WILMAŃSKI. He has determined of values for these constants by means of the wave analysis using the experimental data quoted in the book [16] of BOURBIE, COUSSY and ZINSZNER. For the Massillon sandstone with empty pores and the porosity $n_0 = 0.23$ he obtained the following values:

$$(5.1) \qquad \begin{aligned} \lambda^S &= 10.766 \times 10^7\,\mathrm{Pa}\,, \\ \mu^S &= 6.144 \times 10^7\,\mathrm{Pa}\,, \\ \mathcal{N} &= 1.986 \times 10^{-2}\,\mathrm{sm}^{-2}\,, \\ \tau &= 3.699 \times 10^{-6}\,\mathrm{s}\,. \end{aligned}$$

The remaining quantities of the set (2.9) of the model constants are κ^F, $\bar{\pi}$, γ_0 and the additional constant β.

Let us treat a two-component material constituted by the solid component whose properties are described by the constants (5.1) and which is fully saturated with water. Simultaneously, we assume the effective compressibility of water

$$(5.2) \qquad \kappa^F = 0.452 \times 10^{-9}\,\mathrm{Pa}^{-1}\,.$$

The permeability coefficient $\bar{\pi}$ has been investigating of CHAMSAZ in his dissertation [17] and it takes the value $2.602 \times 10^{-9}\,\mathrm{Pa\,m^{-2}s}$ for the Massillon sandstone. For this order of magnitude of $\bar{\pi}$ CHAMSAZ obtains the value $w \simeq 0.14\,\mathrm{mh}^{-1}$ for the velocity difference which is a realistic one in the consolidation processes. Using the above value $\bar{\pi}$, the order of magnitude of the surface permeability β has been investigated in the paper [13]. It has been shown that

we can assume its value to be $\approx 1.0 \times 10^{-8}\,\mathrm{sm}^{-1}$. Both coefficients $\bar{\pi}$ and β will be varied in the following parameter study.

Finally let us assume for the last constant

$$\gamma_0 \equiv n_0 = 0.23\,. \tag{5.3}$$

This assumption can be motivated by the analysis of the limits of the present model. We shall skip these considerations in this paper. We assume the remaining constants to be

$$\begin{aligned} \varrho^S &= 1000.0\,\mathrm{kgm}^{-3}, \\ \varrho^F_{t0} &= 230.0\,\mathrm{kgm}^{-3}, \\ g &= 10.0\,\mathrm{ms}^{-2}, \\ p^A &= 100\,000.0\,\mathrm{N} \end{aligned} \tag{5.4}$$

and define the integration constant as follows

$$p^F_0 := n_0 p^A = 23\,000.0\ \mathrm{N}\,. \tag{5.5}$$

The above values of constants reflect the order of magnitude of the real constants which should be obtained by the proper averaging procedure. Such procedures are being presently investigated([1]). We shall not discuss this very important problem here.

As we have already mentioned, the character of the numerical simulation will be characterized by the parameter study. The parameters which have been chosen to be variable are:

- the number of elements: 10 or 30,
- the value of the permeability coefficient $\bar{\pi} = 2.602{\times}10^8 \div 2.602{\times}10^{10}\ \mathrm{Pa\,m^{-2}s}$,
- the value of the surface permeability $\beta = 10^{-6} \div 10^{-11}\ \mathrm{kg^{-1}m^2s}$.

For the numerical simulation we have developed the finite element program LFEP which has been written using the macro-language of the program system MAPLE V2. Although the capacity of this program is limited, it is large enough for our purposes.

We begin with the variation of the permeability coefficient $\bar{\pi}$. Figure 3 shows how the permeability coefficient influences the growth and the relaxation of the partial fluid pressure. Due to the well known fact that the whole load will be carried at the beginning of the consolidation process only through the fluid component, the partial pressure p^F increases quickly and then in the second step relaxes from the fluid component to the solid one. This fact shall be illustrated later. The speed of this process depends also on the value of the source of momentum $\bar{\mathbf{p}}$ which is proportional to the difference velocity $\mathbf{w}$ through the coefficient $\bar{\pi}$.

([1]) One of such procedures can be found for example by SHAFIRO and KACHANOV [18].

The Fig. 3 shows the relaxation behaviour of the partial fluid pressure for three different values of $\bar{\pi}$ for a chosen depth and for a chosen value of β. It is easy to see the characteristic excess of the partial pressure of the fluid component which has also been observed in experiments, and which is called the *Mandel – Cryer* effect. The different maximum values can be attributed to the inertia effects which have been neglected in the classical consolidation theory.

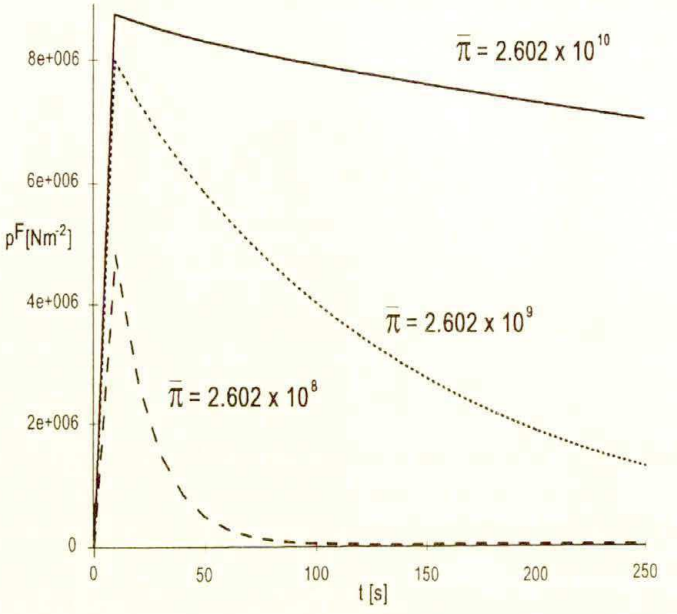

FIG. 3. Partial fluid pressure p^F vs. time: 10 elements, $\beta = 10^{-6}$, $z = 1.80\,\mathrm{m}$.

In the next figure it is shown how the value of the permeability coefficient influences the values of the partial fluid pressure as a function of the depth of the chosen structure.

Let us remind that the variable z is measured from the bottom of the structure, i.e., for instance, the point $z = 4.0\,\mathrm{m}$ in Fig. 4 corresponds to the loaded boundary.

We see that for the chosen time step $t = 100\,\mathrm{s}$ the highest value of the pressure p^F appears at the bottom of the structure for the smallest coefficient $\bar{\pi}$, i.e. then, if the permeability is high. In such a case the friction force acting as the source of momentum between the components is the lowest one, and the external load will be quickly distributed through the whole structure down to its bottom.

Figure 5 illustrates the relaxation behaviour of the partial fluid pressure p^F as a function of depth and chosen time steps for the fixed values of $\bar{\pi} = 2.602 \times 10^9$ and $\beta = 10^{-6}$. As we can see, the partial fluid pressure decreases for increasing time. This relaxation behaviour has been also observed in the Fig. 3. We return to this property in the sequel, where it shall be shown how the acting external load relaxes from the fluid to the solid component.

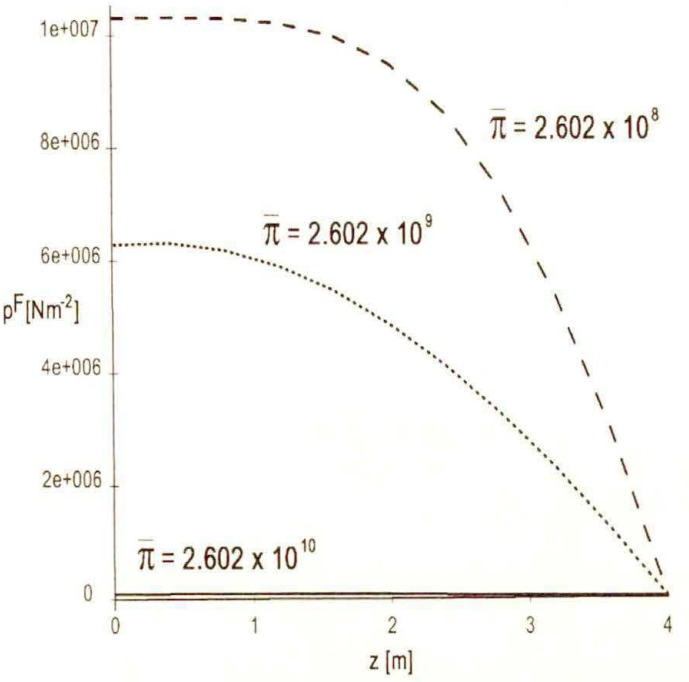

FIG. 4. Partial fluid pressure p^F vs. depth; 10 elements, $\beta = 10^{-6}$, $t = 100\,\mathrm{s}$.

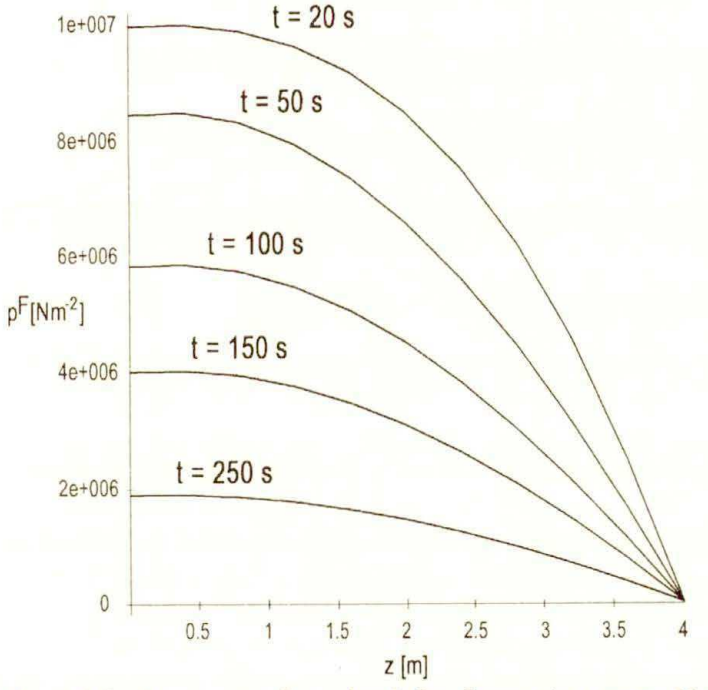

FIG. 5. Partial fluid pressure p^F vs. depth for chosen time steps; 10 elements, $\bar{\pi} = 2.602 \times 10^9$, $\beta = 10^{-6}$.

In the Fig. 6 we can observe a similar behaviour as that in the Fig. 3, however, for another parameter of the function, namely the depth. The surface permeability has been chosen to be $\beta = 10^{-6}$. We see that for the bottom of the structure ($z = 0.00$ m) the partial fluid pressure reaches its maximum. The same value of the pressure can be taken from the figures for t equal 20 s. Due to the velocity of the propagating waves, the exact course of the function $p^F(t)$ can be reached only for very small time steps. This unfortunately exceeds the capacity of our program.

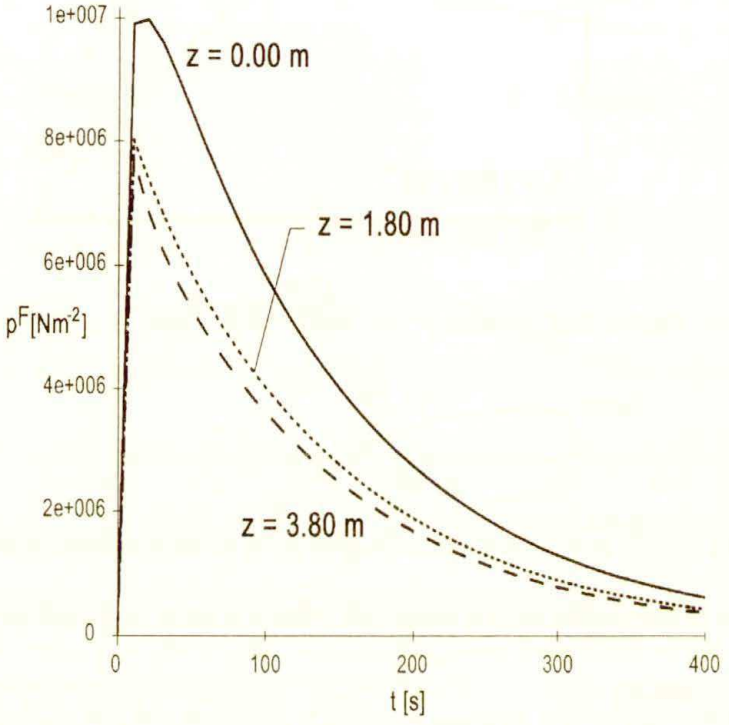

FIG. 6. Partial fluid pressure p^F vs. time for chosen depths; 10 elements, $\bar{\pi} = 2.602 \times 10^9$, $\beta = 10^{-6}$.

Figure 7 illustrates the relaxation behaviour discussed above for different values of the permeability $\bar{\pi}$. The value of the fluid pressure increases and reaches ist maximum after a short time. We have explained this behaviour by the analysis of Fig. 3. It is clear that at the beginning of the consolidation process nearly the whole external load shall be carried through the fluid component. In contrast to the classical simple theory of *von Terzaghi*, a part of the load is also carried through the solid component. As we have already mentioned, the difference in the results between these models is caused by the fact that *von Terzaghi* has neglected the acceleration terms, i.e. the influence of the inertial forces in his model. Nevertheless, the courses of the figure coincide very well with the results

obtained in experiments, where the fluid pressure relaxes to its initial value and finally, at the end of the consolidation process the solid component carries the whole external load.

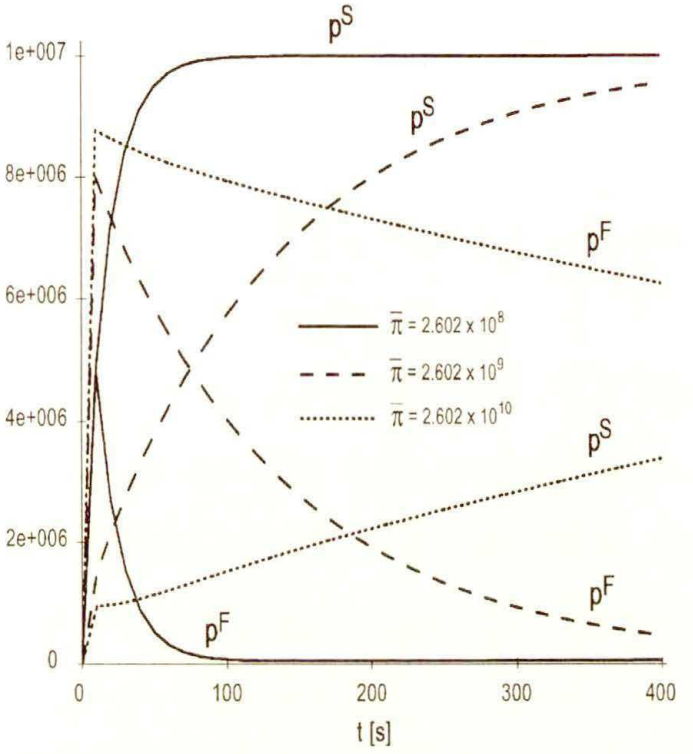

FIG. 7. Time relaxation between the fluid and the solid pressure for $z = 1.80\,\mathrm{m}$; 10 elements, $\bar{\pi} = 2.602 \times 10^9$, $\beta = 10^{-6}$.

In the next figure we present the courses of the functions of the hydraulic gradient $i \equiv \partial p^F / \partial z$ for the chosen points of the structure having in this case the depth of $h = 12\,\mathrm{m}$. The graphs of the Fig. 8 correspond qualitatively to the curves appearing in the literature [17, 19] in the range of short times. Neither the quantitative comparison nor the asymptotic behaviour for large time could be carried through, due to the limited efficiency of the present numerical code. It should be also borne in mind that the classical curves for the hydraulic gradient have been obtained for the semi-infinite medium and not for the layer of the finite thickness which is the subject of this work. It means that we can expect considerable deviations in the boundary regions. These deviations are also due to the novel formulation of the inhomogeneous outflow condition. For this reason we present in Fig. 8 the results for the middle region of the layer. This point shall be discussed again in the forthcoming paper on the two-dimensional consolidation problem.

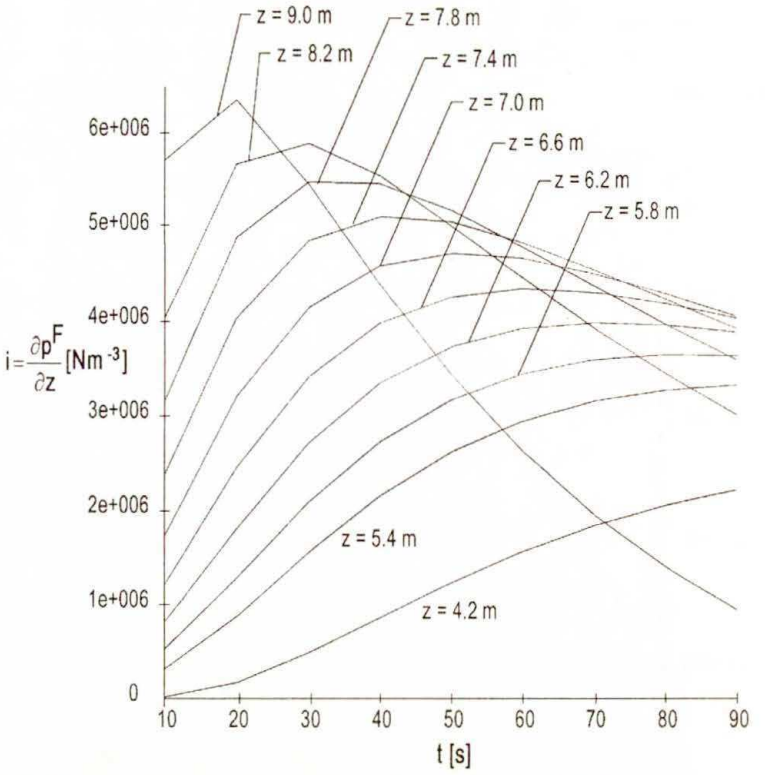

FIG. 8. Hydraulic gradient vs. time for chosen depths; 30 elements, $\bar{\pi} = 2.602 \times 10^9$, $\beta = 10^{-6}$.

We pass over to the last two figures. The former shows the courses of the time-dependent outflow of the fluid component through the free boundary as the function of the surface permeability β. It is clear that for low values of this coefficient, the mass transport of the fluid through the boundary is very low. This fact confirms the physical meaning of the coefficient β as the surface permeability. It has been anticipated earlier in this work. We can also see that for the range $10^{-6} \div 10^{-8}$ of the values of the surface permeability the courses of the function $m^F(t)$ are approximately the same. It seems that these values determine a kind of the asymptote for the magnitude of the surface permeability, of the treated simplified model. Moreover, we see in this figure, due to the relation (5.18), the same effect of the excess of the $m^F(t)$-function which we have observed in the Fig. 3 for the partial fluid pressure.

The latter figure shows the time-depedence of the solid displacement for two chosen points of the structure presented in the Fig. 2 a. Both courses of these functions reproduce the assumption of the linear behaviour of the solid component. In other words, in the case of $z = 3.80$ m the maximum strain amounts approximately to four per cent, what should be expected in the linear case.

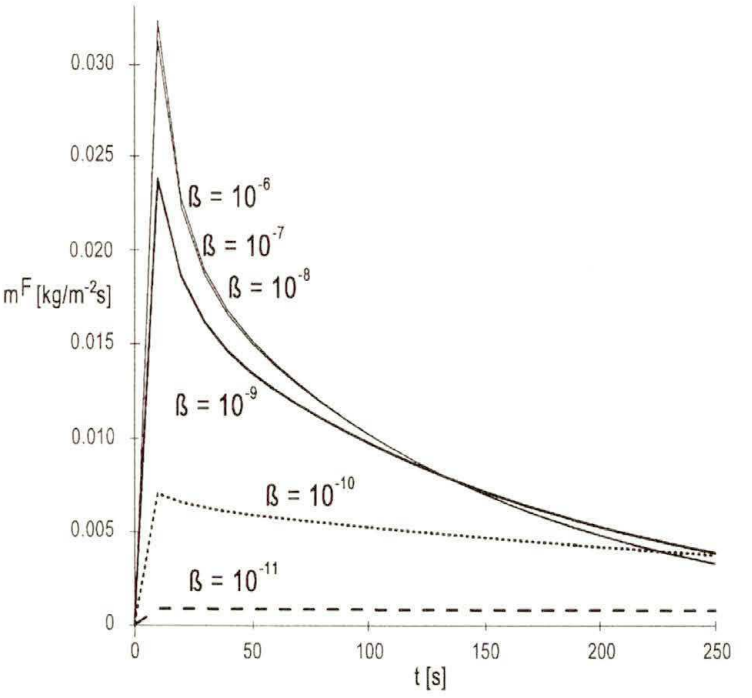

FIG. 9. Mass transport m^F of the fluid through the boundary vs. time; 10 elements, $\bar{\pi} = 2.602 \times 10^9$.

The numerical results presented in this section show that the new class of models developed by WILMAŃSKI, in particular the chosen simplified model described in Sec. 2, agree not only with the well known results obtained by application of the classical model of *von Terzaghi* but also with the phenomena observed in experiments. By means of this model one can simulate the phenomena taking place in the two-component continua. However, we have only focused on the qualitative comparison of the results. The adaptation of the constants of this model as well as their calculation from the point of view of their effective values was not the topic of this paper. Nevertheless the presented results make it clear that it is worthwhile to develop an efficient finite element program to calculate two-dimensional structures with nonlinearities. This is the subject of the current research.

Another important issue of this work was the consistent formulation of the initial-boundary value problem for the consolidation phenomenon. The most important part of this topic was the analysis of the boundary value problem on the free boundary and the formulation of the inhomogeneous boundary condition on it, as well as the formulation of additional boundary quantities for the free surface. We shall present an overview of the existing and possible types of boundaries in two-component continua in a forthcoming paper.

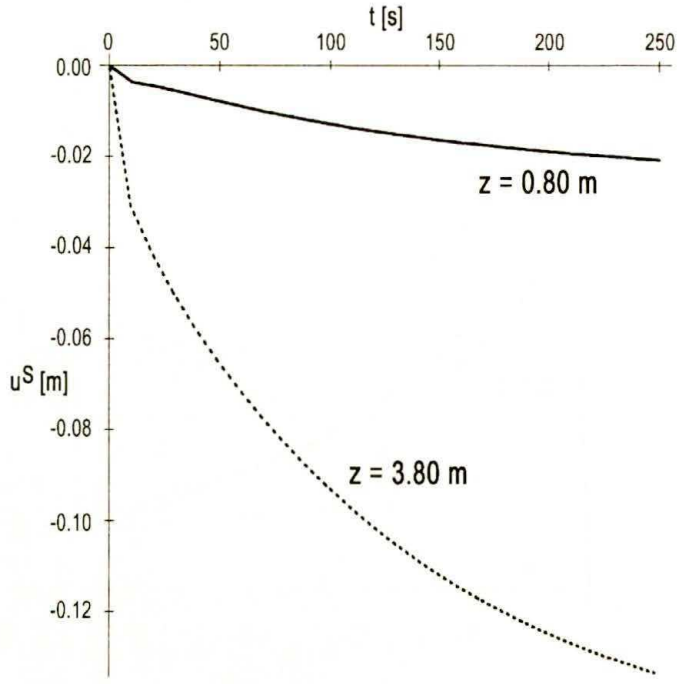

FIG. 10. Displacement u^s vs. time for two chosen points; 10 elements, $\beta = 10^{-6}$, $\bar{\pi} = 2.602 \times 10^9$.

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References

1. K. V. TERZAGHI, *Die Berechnung der Durchlässigkeit des Tones aus dem Verlauf der hydrodynamischen Spannungserscheinungen*, Akademie der Wissenschaften in Wien, Sitzungsberichte der math.-naturw. Kl., Abt. IIa, **132**, 3/4, 125–138, 1923.
2. R.M. BOWEN, *Incompressible porous media models by use of the theory of mixtures*, Int. J. Engng. Sci., **18**, 1129–1148, 1980.
3. K. HUTTER, K. JÖHNK and B. SVENDSEN, *On interfacial transition conditions in two-phase gravity flow*, ZAMP, **45**, 1994.
4. M.A. GOODMAN and S.C. COWIN, *A continuum theory for granular materials*, Arch. Rat. Mech. Anal., **44**, 249–266, 1972.

5. S.L. PASSMAN, *Mixtures of granular materials*, Int. J. Engng. Sci., **15**, 117–129, 1977.
6. S.L. PASSMAN, I.W. NUNZIATO and E.K. WALSH, *A theory of multiphase mixtures*, [in:] Rational Thermodynamics, C. TRUESDELL [Ed.], Second Edition, 286–325, Springer-Verlag, N.Y. 1984.
7. M.R. BAER and I.W. NUNZIATO, *A two-phase mixture theory for the deflagration-to-detonation transition (DDT) in reactive granular materials*, Int. J. Multiphase Flow, **12**, 861–889, 1986.
8. R. DE BOER, *Highlights in the historical development of the porous media theory*, Appl. Mech. Rev., **49**, 4, 201–262, 1996.
9. K. WILMAŃSKI, *On weak discontinuity waves in porous materials*, [in:] Interactions of Mechanics and Mathematics, J.F. RODRIGUEZ [Ed.], STAMM 94, Longman Scientific & Technical, Essex 1995.
10. K. WILMAŃSKI, *The thermodynamical model of the compressible porous material with the balance equation for porosity*, J. Non-Equilibrium Thermodyn., [to appear 1996].
11. K. WILMAŃSKI, *Lagrangean model of two-phase porous material*, J. Non-Equilibrium Thermodyn., **20**, 1, 50–78, 1995.
12. K. WILMAŃSKI, *Porous media at finite strains. The new model with the balance equation for porosity*, Arch. Mech., **48**, 4, 591–628, 1996.
13. W. KEMPA, *Das Grenzwertproblem bei Konsolidation – eine konsistente Konstruktion der schwachen Formulierung*, MECH-Report 95/13, Universität-GH Essen, Essen 1995.
14. K. RUNESSON, *On non-linear consolidation of soft clay*, Ph.D. Thesis, Dept. of Struct. Mech., Chalmers Univ. of Technology, Göteborg 1978.
15. K. WILMAŃSKI, *Acceleration waves in two-component porous media. Part II. Supplementary remarks on the model and the evolution of amplitudes*, MECH-Bericht 95/2, Universität-GH Essen, Essen 1995.
16. T. BOURBIE, O. COUSSY and B. ZINSZNER, *Acoustics of porous media*, Editions Technip, Paris 1987.
17. B. CHAMSAZ, *Einige quasistatische Probleme ebener Verzerrungen bei kompressiblen porösen Körpern*, Diplomarbeit 95/3, Institut für Mechanik, Fachbereich Bauwesen, Universität-GH Essen, Essen 1995.
18. B. SHAFIRO and M. KACHANOV, *Materials with fluid-filled pores of various shapes: Effective elastic properties and fluid pressure polarization*, Int. J. Solids & Struct., [in press].
19. J.K. MITCHELL, *Fundamentals of soil behavior*, John Wiley & Sons, Inc., University of California, Berkeley 1976.
20. M. CIESZKO and J. KUBIK, *On the compatibility conditions in the fluid-fluid saturated porous solid contact problems*, Arch. Mech., **45**, 1, 77–91, 1993.
21. T. LEVY and E. SANCHEZ-PALENCIA, *On boundary conditions for fluid flow in porous media*, Int. J. Engng. Sci., **13**, 923–940, 1975.
22. K. VAFAI and C.L. TIEN, *Boundary and inertia effects on flow and heat transfer in porous media*, Int. J. Heat Maff Transfer, **24**, 195–203, 1981.
23. K.R. RAJAGOPAL, A.S. WINEMAN and M. GANDHI, *On boundary conditions for a certain class of problems in mixture theory*, Int. J. Engng. Sci., **24**, 1453–1463, 1986.

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