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Transverse Stokes flow through a square array of cylinders

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THE WORK PRESENTS results of calculations of the transverse Stokes flow through a square array of cylinders. The new functional basis has been derived and the solution is sought in the form of series expansions in this basis, the terms of which are given explicitly as functions of the volume fraction φ. The presented method enabled us to evaluate the expression for the drag force of high accuracy using symbolic computations.

1. Introduction

SLOW FLOW of a viscous fluid through an array of cylinders is observed in many technical applications such as heat exchangers, fibre filters and bundles of man-made fibres in spinning processes. In this paper we shall study the flow through a square array of cylinders in a direction transverse to the cylinder axes. This problem was first studied in 1959 by HAPPEL who calculated the Stokes flow, taking into account the periodic structure of the array with the aid of a so-called free surface model [1–2]. As a result, he got an expression for the drag force F' exerted by the fluid on a unit length of a cylinder. The drag force was a function of the volume fraction φ of cylinders for a given mean velocity U of the fluid. The volume fraction φ is defined as

$$\varphi = \frac{\pi a'^2}{A}, \tag{1.1}$$

where a' is the cylinder radius and A is the cross-sectional area of the array per single cylinder. In the case of a square area it takes the form

$$A = l^2, \tag{1.2}$$

l being the distance between the cylinder axes.

The expression for the drag force F' may be presented in the following general form which involves approximations of various order

$$\frac{F'}{\mu U} = \frac{1}{K(\varphi)}, \tag{1.3}$$

where

$$K(\varphi) = \frac{1}{8\pi}\left[\ln(1/\varphi) + T^{(n)}(\varphi)\right], \tag{1.4}$$

$$T^{(n)}(\varphi) = \sum_{i=0}^{n} T_i\,\varphi^i, \tag{1.5}$$

and μ is the dynamic viscosity of the fluid.

The approximation of HAPPEL [1–2] was rather rough and the results obtained were of a reasonable accuracy merely for very dilute arrays ($\varphi \ll 1$). At the same time, an alternative approach to the investigation of a viscous fluid flow through periodic arrays of particles was proposed by HASIMOTO [3]. Using Fourier series expansions, he obtained spatially periodic fundamental solutions of the Stokes flow for arrays of spheres as well as of cylinders. Lattice sums which appeared in this method were calculated using rapidly converging Evald's technique. He introduced then two functions S_1 and S_2 with the aid of which it was possible to construct the infinite system of algebraic equations, in which one of unknown quantities was the drag force. After truncation the system was solved and the expression for $K(\varphi)$ (1.4) has been obtained with the accuracy of $O(\varphi)$. The method of Hasimoto was then developed by SANGANI and ACRIVOS [4] who obtained the expression for $K(\varphi)$ to $O(\varphi^3)$. It is also worth to mention the paper of DRUMMOND and TAHIR [5]. They calculated $K(\varphi)$ using the method of singularities, matching a solution outside a cylinder to a sum of solutions inside each cylinder in an infinite array. The obtained expression for $K(\varphi)$ was determined with the accuracy up to $O(\varphi^4)$.

SANGANI and ACRIVOS [6] made also numerical calculations of a Stokes flow past a periodic array of cylinders and evaluated the drag force F' in a wide range $0.05 < \varphi < 0.75$. The results [6] may then be treated as reference data for analytical approximations.

The expressions derived for the drag force may be used to calculate filtration flow through the array of cylindrical rods, treated as a porous medium [7]. It can be shown that the force F' is related to the mean pressure gradient exerted on the fluid in the array of cylinders [1–2]

$$\frac{F'}{A} = -\frac{dp'}{dx}\,. \tag{1.6}$$

Inserting (1.6) to (1.3) we obtain the relation

$$U = -\frac{A}{\mu} K(\varphi) \frac{dp'}{dx}\,, \tag{1.7}$$

which has the form of a linear Darcy equation, where $K(\varphi)$ (1.4) plays the role of a permeability coefficient [7].

These results, together with the results for the parallel case [1–2], were applied by SZANIAWSKI and ZACHARA [8, 9] for calculation of filtration flow through a bundle of man-made fibers in a formation processes. It allowed them to obtain velocity and pressure distribution inside the bundle of fibers.

In the present paper, the approach of Hasimoto as well as of Sangani and Acrivos has been modified. A new functional basis has been derived. It allowed to derive explicit expressions for matrix components of the infinite system of equations and its solution could be obtained using symbolic computations [10].

The system was truncated and as an example, the coefficient $K(\varphi)$ was calculated with the accuracy of $O(\varphi^5)$. This procedure can be easily extended to solutions of higher accuracy.

2. Governing equations

We consider the slow flow of a viscous fluid through a square array of cylinders, each of them of radius a'. They are infinitely long, so the problem may be treated as two-dimensional. With respect to the periodicity of the array, we shall limit ourselves to a unit cell which is repeated throughout the system (Fig. 1). The dimension of a unit cell is l. To describe the problem, we shall use both the Cartesian (x_1', x_2') and cylindrical (r', θ) coordinate systems. Position of the cylinder axes in the plane $x_1' x_2'$ are given by the vector

$$\mathbf{n}' = l(n_1\mathbf{a}_{(1)} + n_2\mathbf{a}_{(2)}), \qquad n_1,\ n_2 = 0, \pm 1, \pm 2, \dots, \tag{2.1}$$

where $\mathbf{a}_{(1)}$ and $\mathbf{a}_{(2)}$ are basic unit vectors in x_1' and x_2' direction, respectively. We assume that the fluid flows in x_1' direction with the mean velocity U. According to the assumption that the Reynolds number is very small, the flow may be described by the Stokes equations which are given below in a non-dimensional form

$$\frac{\partial^2 u_i}{\partial x_k \partial x_k} = \frac{\partial p}{\partial x_i}, \tag{2.2}$$

$$\frac{\partial u_k}{\partial x_k} = 0, \tag{2.3}$$

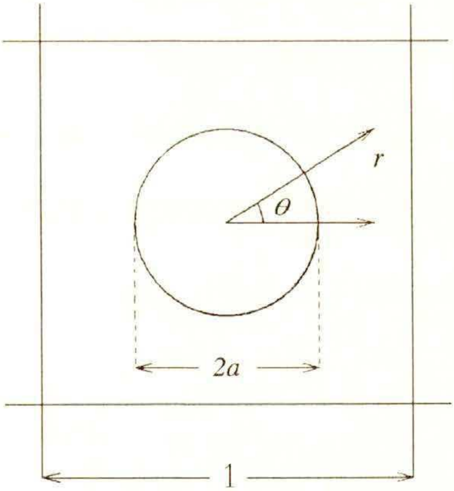

FIG. 1. Unit cell of a square array of cylinders.

where u_i is the velocity component of the fluid and p_i is the pressure. The co-ordinates have been non-dimesionalized with l and velocity components with U. The non-dimensional pressure p is determined as

$$p = \frac{p' \cdot l}{U \cdot \mu}.$$

The velocity field must satisfy the periodicity conditions and the no-slip boundary condition at the cylinder surface

$$\tag{2.4} \begin{aligned} \mathbf{u}(\mathbf{r}+\mathbf{n}) &= \mathbf{u}(\mathbf{r}), \\ \mathbf{u}(a,\theta) &= 0. \end{aligned}$$

Following the approach of HASIMOTO [3] and SANGANI and ACRIVOS [4], we first consider the Stokes flow through the periodic system where cylindrical particles have been replaced with singular multipole force distributions located at their axes. In this case, the Stokes equations may be presented in the following form:

$$\tag{2.5} \begin{aligned} \frac{\partial^2 v_i}{\partial x_k \partial x_k} &= \frac{\partial q}{\partial x_i} + F_i \sum_{\{n\}} \delta(\mathbf{r}-\mathbf{n}), \\ \frac{\partial v_k}{\partial x_k} &= 0, \end{aligned}$$

where $\delta(\mathbf{r}-\mathbf{n})$ is Dirac's delta function, while v_i and q are velocity component and pressure, respectively. The components F_i are $F_1 = F$, $F_2 = 0$.

The non-dimensional drag force F acting on a unit length of the cylinder is determined by

$$F = \frac{F'}{U\mu}.$$

The symbol $\{n\}$ at the sign $\sum$ in $(2.5)_1$ denotes summation in the directions x_1 and x_2 to infinity,

$$\sum_{\{n\}}^{\infty} = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty}.$$

HASIMOTO [3] found the periodic fundamental solution of Eq. (2.5) in the form

$$\tag{2.6} v_i = U_0 \delta_{i1} - \frac{F}{4\pi}\left[S_1 \delta_{i1} - \frac{\partial^2 S_2}{\partial x_1 \partial x_i}\right],$$

$$\tag{2.7} \frac{\partial q}{\partial x_i} = F\left[-\delta_{i1} + \frac{1}{4\pi}\frac{\partial^2 S_1}{\partial x_1 \partial x_i}\right].$$

The functions S_1 and S_2 which are periodic throughout the lattice are as follows [11]:

$$S_1 = \frac{1}{\pi}\sum_{\{n\}}{}' \frac{\exp[-2\pi i(\mathbf{n}\cdot\mathbf{r})]}{|\mathbf{n}|^2}, \tag{2.8}$$

$$S_2 = -\frac{1}{4\pi^3}\sum_{\{n\}}{}' \frac{\exp[-2\pi i(\mathbf{n}\cdot\mathbf{r})]}{|\mathbf{n}|^4}, \tag{2.9}$$

where i is an imaginary unit and the prime ($'$) over the summation symbol indicates that the term $|\mathbf{n}| = 0$ is excluded.

They are solutions of the equations

$$\nabla^2 S_1 = -4\pi\left[\sum_{\{n\}}\delta(\mathbf{r}-\mathbf{n}) - 1\right], \tag{2.10}$$

$$\nabla^2 S_2 = S_1, \tag{2.11}$$

what may be proved by the finite Fourier transforms.

Now we choose the unit cell of the array, indicated by the point $\mathbf{n} = 0$ at its centre. The functions S_1 and S_2 (2.8)–(2.9) are here calculated using the Evald summation [3] and expanded in planar harmonics near $r = 0$. The corresponding expressions are as follows

$$S_1 = -2\ln r - C_0 + \pi r^2 + 2\sum_{m=4}^{\infty} A_m r^m \cos m\theta, \tag{2.12}$$

$$S_2 = \frac{1}{2}r^2(1-\ln r) - C_0\frac{r^2}{4} + \frac{\pi r^4}{16} + \sum_{m=4}^{\infty}\Big[A_m/(2(m+1))r^2 + B_m\Big] r^m \cos m\theta, \tag{2.13}$$

where

$$r = (x_1^2 + x_2^2)^{1/2},$$
$$\theta = \tan^{-1}(x_2/x_1).$$

Both the functions (2.12)–(2.13) fulfil Eqs. (2.10), (2.11) in the unit cell $\mathbf{n} = 0$ where the Dirac's delta $\delta(0)$ corresponds to the properties of $\ln r$ and its derivatives at $r = 0$. The function S_1 is simply related to the Wigner potential and the first non-vanishing coefficients A_m are evaluated in [12]. With respect to the symmetry of the square array, the coefficients A_m and B_m are different from zero only for m which are multiples of 4. The method of evaluation and numerical values of A_m, B_m and C_0 are given in the Appendix 1.

The importance of these results exceeds the frames of the fluid dynamics since S_1 is equivalent to the electrostatic potential in a periodic system of charge particles embedded in a neutralizing uniform background [12]. It is worth to note that this background corresponds to the mean pressure gradient which is able to balance the drag of the cylinders in the flow. The approach of Hasimoto was also successfully applied to the calculation of the effective conductivity of composite materials of a regular structure [13–16].

The fundamental solution (2.6) is a starting point to construct a general solution of Eqs. (2.2)–(2.3) where velocity components u_1 and u_2 defined for $r \geq a$ satisfy the no-slip boundary condition on the cylinder surface $r = a$ with the required accuracy. To this end, following [3] and [4], we add to the solution (2.6) the even derivatives of v_i and S_1 multiplied by unknown coefficients. This operation satisfies the symmetry conditions of the periodic flow through the array and leads to the following expressions for the velocity components u_1 and u_2:

$$u_1(r,\theta) = U_0 - \frac{1}{4\pi}\left[\mathbf{G}\left(S_1 - \frac{\partial^2 S_2}{\partial x_1^2}\right) + \mathbf{H}\frac{\partial^2 S_1}{\partial x_1^2}\right], \tag{2.14}$$

$$u_2(r,\theta) = \frac{1}{4\pi}\left[\mathbf{G}\frac{\partial^2 S_2}{\partial x_1 \partial x_2} - \mathbf{H}\frac{\partial^2 S_1}{\partial x_1 \partial x_2}\right], \tag{2.15}$$

where $\mathbf{G}$ and $\mathbf{H}$ are differential operators

$$\begin{aligned}\mathbf{G} &= \sum_{n=0}^{\infty} P_n \frac{\partial^{2n}}{\partial x_1^{2n}}, \\ \mathbf{H} &= \sum_{n=0}^{\infty} Q_n \frac{\partial^{2n}}{\partial x_1^{2n}},\end{aligned} \tag{2.16}$$

and

$$U_0 = 1 + Q_0 . \tag{2.17}$$

We perform differentiation of $S_1(r,\theta)$ and $S_2(r,\theta)$, Eqs. (2.12)–(2.13), with respect to x_1 and x_2 in (2.14)–(2.16) using operators

$$\begin{aligned}\frac{\partial}{\partial x_1} &= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}, \\ \frac{\partial}{\partial x_2} &= \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}.\end{aligned} \tag{2.18}$$

To calculate the coefficients P_n and Q_n we make use of the no-slip boundary condition on the surface of the cylinder $(2.4)_2$. Thus we have

$$u_1(a,\theta) = 0, \qquad u_2(a,\theta) = 0. \tag{2.19}$$

Hence Eqs. (2.14)–(2.19) lead to the system of algebraic equations for the coefficients P_n and Q_n. If we compare (2.6), (2.14) and $(2.16)_1$ we can see that the force F exerted by the fluid on the cylinder is equal to the coefficient P_0,

$$F = P_0\,, \tag{2.20}$$

and from (1.3) we have

$$K(\varphi) = 1/P_0\,. \tag{2.21}$$

3. The basis functions

It is useful to define and derive the basis functions which may help to carry out calculations of the coefficients P_n and Q_n in an efficient and tractable way. These functions, which appear in (2.14) and (2.15) are

$$U^1 = S_1 - \frac{\partial^2 S_2}{\partial x_1^2}\,, \qquad U^2 = \frac{\partial^2 S_1}{\partial x_1^2}\,, \tag{3.1}$$

$$V^1 = \frac{\partial^2 S_2}{\partial x_1 \partial x_2}\,, \qquad V^2 = \frac{\partial^2 S_1}{\partial x_1 \partial x_2}\,, \tag{3.2}$$

where S_1 and S_2 are determined by (2.12) and (2.13). Performing differentiation of S_1 and S_2 with the use of operators (2.18), we get expressions for the function (3.1) and (3.2) given below

$$\begin{aligned} U^1(r,\theta) = \frac{1}{2}\left[\ln 1/r^2 - C_0\right] + \frac{\pi r^2}{2} + \frac{1}{4}(2-\pi r^2)\cos 2\theta \\ + \sum_{m=0}^{\infty} A_m E_m(r,\theta) - \sum_{m=0}^{\infty}(m+2) D_{m0}(r) E_m(r,\theta), \end{aligned} \tag{3.3}$$

$$U^2(r,\theta) = 2\pi + \frac{2\cos 2\theta}{r^2} + 2\sum_{m=0}^{\infty} A_{m+2}\frac{(m+2)!}{m!}E_m(r,\theta), \tag{3.4}$$

$$V^1(r,\theta) = -\frac{1}{4}(2-\pi r^2)\sin 2\theta - \sum_{m=1}^{\infty}(m+2) D_{m0}(r) F_m(r,\theta), \tag{3.5}$$

$$V^2(r,\theta) = \frac{2\sin 2\theta}{r^2} - 2\sum_{m=1}^{\infty} A_{m+2}\frac{(m+2)!}{m!}F_m(r,\theta). \tag{3.6}$$

The auxiliary functions which appear in (3.3) – (3.6) are defined as follows:

$$D_{mn}(r) = \frac{1}{2}A_{m+2n+2}r^2 + (m+1)B_{m+2n+2}\,, \tag{3.7}$$

$$E_m(r,\theta) = r^m \cos m\theta\,, \tag{3.8}$$

$$F_m(r,\theta) = r^m \sin m\theta\,. \tag{3.9}$$

The even derivatives of the basis functions which appear in the operators (2.16) are denoted by the following symbols

$$\tag{3.10} \begin{aligned} U_n^1 &= \frac{\partial^{2n} U^1}{\partial x_1^{2n}}, & U_n^2 &= \frac{\partial^{2n} U^2}{\partial x_1^{2n}}, \\ V_n^1 &= \frac{\partial^{2n} V^1}{\partial x_1^{2n}}, & V_n^2 &= \frac{\partial^{2n} V^2}{\partial x_1^{2n}}, \end{aligned}$$

where $n = 1,\ 2,\ 3, \dots$.

After differentiation of the basis functions (3.3) – (3.6), we obtain the expressions for their derivatives (3.10) which are

$$\tag{3.11} \begin{aligned} U_n^1 = \frac{\pi}{2}\delta_{n1} + \frac{1}{2r^{2n}}\left[(2n)!\cos 2(n+1)\theta - 2(n-1)(2n-1)!\cos 2n\theta\right] \\ - \sum_{m=0}^{\infty} \frac{(m+2n+2)!}{(m+1)!} D_{mn}(r) E_m(r,\theta) \\ - (n-1)\sum_{m=0}^{\infty} A_{m+2n} \frac{(m+2n)!}{m!} E_m(r,\theta), \qquad \text{for} \quad n \geq 1, \end{aligned}$$

while

$$U_0^1 = U^1,$$

$$\tag{3.12} \begin{aligned} U_n^2 = 2\pi\delta_{n0} + \frac{2(2n+1)!}{r^{2(n+1)}}\cos 2(n+1)\theta \\ + 2\sum_{m=0}^{\infty} A_{m+2n+2}\frac{(m+2n+2)!}{m!} E_m(r,\theta) \qquad \text{for} \quad n \geq 0, \end{aligned}$$

$$\tag{3.13} \begin{aligned} V_n^1 = \frac{(2n)!}{2r^{2n}}\left[\sin 2n\theta - \sin 2(n+1)\theta\right] - n\sum_{m=1}^{\infty} A_{m+2n}\frac{(m+2n)!}{m!} F_m(r,\theta) \\ - \sum_{m=1}^{\infty} \frac{(m+2n+2)!}{(m+1)!} D_{mn}(r) F_m(r,\theta) \qquad \text{for} \quad n \geq 1, \end{aligned}$$

while $V_0^1 = V^1$,

$$\tag{3.14} \begin{aligned} V_n^2 = \frac{2(2n+1)!}{r^{2n+2}}\sin 2(n+1)\theta \\ - 2\sum_{m=1}^{\infty} A_{m+2n+2}\frac{(m+2n+2)!}{m!} F_m(r,\theta) \qquad \text{for} \quad n \geq 0. \end{aligned}$$

4. Calculation of the drag force

The drag force F (2.20) can be calculated from the system of Eqs. (2.19) where the velocity components u_1 and u_2 are determined by (2.14) and (2.15). The differential operators $\mathbf{G}$ and $\mathbf{H}$ (2.16) act on the functions (3.3) – (3.6). The system of equations (2.19) may thus be written

$$\mathbf{G}U^1 + \mathbf{H}U^2 = 4\pi U_0\,, \tag{4.1}$$

$$\mathbf{G}V^1 - \mathbf{H}V^2 = 0. \tag{4.2}$$

Using (2.16) – (2.18) we present Eqs. (4.1) – (4.2) in the form

$$\sum_{i=0}^{\infty}\left[P_i U_i^1 + Q_i\left(U^2 i - 4\pi\delta_{i0}\right)\right] = 4\pi\,, \tag{4.3}$$

$$\sum_{i=0}^{\infty}\left[P_i V_i^1 - Q_i V_i^2\right] = 0. \tag{4.4}$$

The basis functions (3.3) – (3.6) and their derivatives (3.11) – (3.14) may be presented in a form of $\cos 2i\theta$ and $\sin 2i\theta$ expansions whose coefficients are elements of a matrix Z_{lm}. Thus we have the following expansions of U_n^k and V_n^k,

$$\begin{aligned} U_n^k(a,\theta) &= \sum_{i=0}^{\infty}(4\pi\delta_{i0}\delta_{n0}\delta_{k2} + Z_{2i+1,2n+k})\cos 2i\theta\,,\\ V_n^k(a,\theta) &= -(-1)^k\sum_{i=1}^{\infty} Z_{2i,2n+k}\sin 2i\theta\,, \end{aligned} \tag{4.5}$$

where $k = 1$ and 2, $n = 0,\ 1,\ 2,\ 3\ldots$.

Now we insert the basis function expansions (4.5) to Eqs. (4.3) – (4.4) and collect terms of Eq. (4.3) containing $\cos 2i\theta$ and terms of Eq. (4.4) containing $\sin 2i\theta$. Then, after some rearrangement we present Eqs. (4.3) – (4.4) in the following form:

$$\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}(Z_{2i+1,j}X_j)\cos 2i\theta = 4\pi\,, \tag{4.6}$$

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(Z_{2i,j}X_j)\sin 2i\theta = 0. \tag{4.7}$$

Here X_j are unknown quantities related to P_i and Q_i

$$X_{2n+1} = P_n\,, \qquad X_{2n+2} = Q_n\,, \tag{4.8}$$

while the coefficients Z_{ij} are known elements of the matrix evaluated from the basis functions (3.3) – (3.6) and their derivatives (3.11) – (3.14). The details are given in the Appendix 2. Thus we can transform Eqs. (4.6), (4.7) to the infinite system of algebraic equations where X_j are unknown components of the vector $\mathbf{X}$ and Z_{ij} are known elements of the matrix $\hat{Z}$,

$$\hat{Z}\mathbf{X} = 4\pi\mathbf{I}\,, \tag{4.9}$$

where $\mathbf{I}$ is a column vector whose first component is 1 and all other components are equal to zero.

We can see from (4.8) and (2.20) that the drag force F is equal to X_1. For calculation of X_1 it is useful to separate from the matrix Z_{ij} the logarithmic term included in $U^1(a,\theta)$ (3.3), which we denote by K_0

$$K_0 = \ln 1/a^2 - C_0\,. \tag{4.10}$$

Thus we can present the elements of the matrix $\hat{Z}$ in the form

$$Z_{ij} = \frac{1}{2}K_0\delta_{1i}\delta_{1j} + W_{ij}\,, \tag{4.11}$$

while the elements of the matrix $\hat{W}$ are given in the Appendix 2.

The unknown X_1 can be written formally as

$$X_1 = \frac{4\pi|\hat{Z}^S|}{|\hat{Z}|}\,, \tag{4.12}$$

where the superscript S denotes a submatrix of the original matrix, corresponding to its first element (1,1).

It follows from (4.11) that

$$|\hat{Z}| = \frac{1}{2}K_0|\hat{Z}^S| + |\hat{W}|, \tag{4.13}$$

and

$$\hat{Z}^S = \hat{W}^S. \tag{4.14}$$

Inserting (4.13) – (4.14) to (4.12) we get

$$X_1 = \frac{8\pi}{K_0 + \beta}\,, \tag{4.15}$$

where

$$\beta = 2/Y_1\,, \tag{4.16}$$

while Y_1 is the first component of the vector $\mathbf{Y}$ which is the solution of the equation

$$\hat{W}\mathbf{Y} = \mathbf{I}\,. \tag{4.17}$$

According to (2.21), (4.8) and (4.15) we have

$$K(\varphi) = \frac{1}{X_1} = \frac{1}{8\pi}(K_0 + \beta). \tag{4.18}$$

5. Results

The infinite system of equations (4.17) was truncated to five equations and solved with matrix elements W_{ij} taken from the Appendix 2. The solution Y_1, which is the subject of our interest, has been obtained with the accuracy of $O(a^{10})$. Inserting Y_1 to (4.16) we get the following expression

$$\beta = \frac{2}{Y_1} = C_1 a^2 + C_2 a^4 + C_3 a^6 + C_4 a^8 + C_5 a^{10}, \tag{5.1}$$

where the coefficients C_n have the following numerical values:

$$\begin{aligned}
C_1 &= 2\pi\,, \\
C_2 &= -\left(\frac{\pi^2}{2} + 1152\, B_4^2\right), \\
C_3 &= -1536\, A_4\, B_4\,, \\
C_4 &= 576\pi A_4 B_4 - 520 A_4^2\,, \\
C_5 &= 384\pi A_4^2\,.
\end{aligned} \tag{5.2}$$

Numerical values of C_0, A_4 and B_4 are given in the Appendix 1.

The expressions $(5.2)_{1-3}$ are completely equivalent to those used by SANGANI and ACRIVOS [4] who calculated β to $O(a^6)$. The expression $(5.2)_4$ cannot be directly compared with the corresponding one of DRUMMOND and TAHIR [5] since they used a different calculation method than ours, and these expressions are of a quite different form. We shall come back to this question later and compare the numerical values of the coefficients with the literature data.

It is however more convenient to express the force F as a function of the volume fraction φ (1.1) which is related to the non-dimensional radius a as

$$\varphi = \pi a^2. \tag{5.3}$$

Inserting (5.3) into (5.1) and then combining (5.1), (4.10) and (4.18), we obtain the expression for $K(\varphi)$ presented in Sec. 1 (1.4), where $T^{(n)}(\varphi)$ is a power function expansion in φ (1.5). The drag force is related to $K(\varphi)$ according to (1.3)

$$F = \frac{1}{K(\varphi)}\,. \tag{5.4}$$

The initial coefficient of the expansion $T^{(n)}$ is

$$T_0 = \ln\pi - C_0 = -1.47633597. \tag{5.5}$$

The other coefficients are related to the coefficients C_i of (5.1) using expression (5.3)

$$T_i = \frac{C_i}{\pi^i}\,. \tag{5.6}$$

Table 1.

i	T_i
0	−1.47633597
1	2.00000000
2	−1.77428264
3	4.07770444
4	−4.84227403
5	2.44662267

The numerical values of all these coefficients $T_0 - T_5$ are collected in Table 1.

We can now compare the numerical values of the coefficients from Table 1 with the values obtained by the previous authors [4] and [5]. The coefficients of SANGANI and ACRIVOS [4], calculated up to four decimal places in frames of the approximation to $O(\varphi^3)$, i.e. for $i = 3$, are equivalent to the corresponding values from Table 1. The coefficients of DRUMMOND and TAHIR [5] ($i = 4$) are equal to those from Table 1 up to nine decimal places, although they were evaluated by different procedures, as it was previously indicated. It seems to confirm the conclusion that both the procedures are equivalent, and calculations in [5] and in the present paper were carried out correctly. The coefficient T_5 is a new value

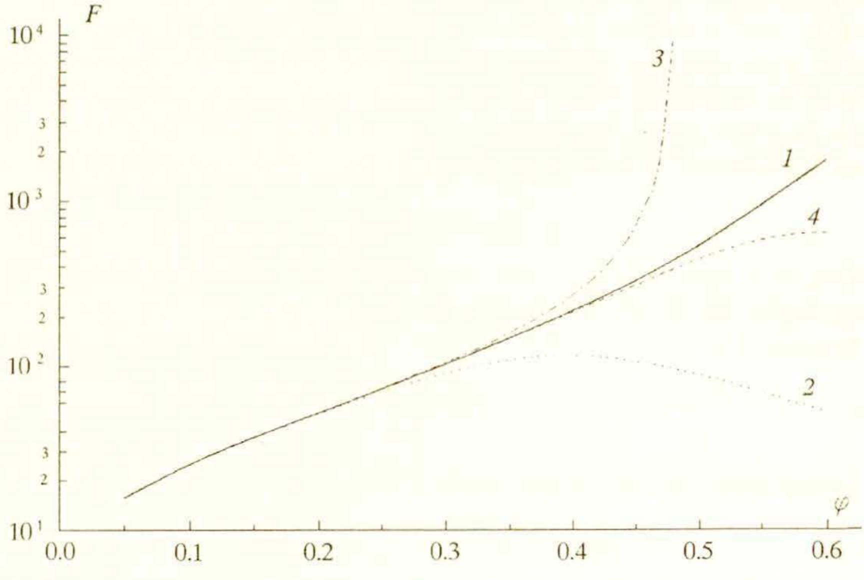

FIG. 2. The non-dimensional drag force F vs. the volume fraction φ. Comparison of the numerical reference data of SANGANI and ACRIVOS [6], (line 1) with various approximations. Line 2 – Sangani and Acrivos to $O(\varphi^3)$ [4], line 3 – Drummond and Tahir to $O(\varphi^4)$ [5], line 4 – the present results to $O(\varphi^5)$.

obtained in the frames of the present approximation corresponding to $O(\varphi^5)$. The results of [4] and [5] were collected and presented in the monograph of P. ADLER [7]. The literature data concerning approximations of higher order than that of DRUMMOND and TAHIR, of $O(\varphi^4)$, are not known to the author.

We calculated the drag force $F(\varphi)$ from (5.4) for approximations of various order using coefficients from Table 1. The results are presented in Fig. 2. The results of SANGANI and ACRIVOS [6] are here included as the reference data. They were obtained by numerical integration of the Stokes equations in a range of φ from 0.05 up to 0.75. It is almost the full range of φ since the maximum value of φ, which corresponds the case of touching cylinders, is $\varphi_{\max} = \pi/4 = 0.785398...$. We can see how the accuracy of calculations increases with the order of approximation. The expression (5.4) with the series expansion $T^{(3)}(\varphi)$ estimates the drag force within the error of about 2% at $\varphi = 0.2$. This error is kept with $T^{(4)}(\varphi)$ at $\varphi = 0.3$ and with $T^{(5)}(\varphi)$ at $\varphi = 0.4$. For $\varphi > 0.4$ all these expressions diverge and a new formula of better accuracy is needed.

6. Conclusions

A new functional basis derived in this paper enabled us to obtain expressions for matrix elements W_{ij} (see Appendix 2). The matrix is involved in Eq. (4.17) which is subjected to truncation of a chosen order, and its solution enters the formula (4.15) for a drag force. The explicit form of the expressions W_{ij} makes the calculations very tractable and allows to derive the solution using symbolic computations of *Mathematica* [10]. This procedure was here applied to the system of five equations and the results obtained to $O(\varphi^5)$ were of higher accuracy than the results of the previous authors. Extension of these calculations for larger systems of equations is straightforward.

Appendix 1

We present below numerical values of the first non-vanishing coefficients A_m and B_m as well as the constant C_0, which appear in (2.12) and (2.13). The coefficients were evaluated from the expressions derived in [4] and adopted here for the square array.

$$\text{(A1.1)} \qquad A_m = \frac{\alpha(2\pi)^m}{2^m(m)!}\left[\alpha^{-(m+1)}\sum_{\{n\}} E_m(|n|,\theta)\Phi_{m-1}\left(\frac{\pi}{\alpha}|n|^2\right)\right.$$
$$\left. + \sum_{\{n\}} E_m\left(|n|,\theta\right)\Phi_0(\pi|n|^2\alpha)\right],$$

$$B_m = \frac{-\alpha^2(2\pi)^m}{\pi 2^{m+1} m!}\left[\alpha^{-(m+1)}\sum_{\{n\}} E_m(|n|,\theta)\,\Phi_{m-2}\left(\frac{\pi|n|^2}{\alpha}\right) + \sum_{\{n\}} E_m(|n|,\theta)\,\Phi_0(\pi|n|^2\alpha)\right], \tag{A1.2}$$

$$C_0 = \gamma + \ln(\pi/\alpha) + \alpha - \sum_{\{n\}}\Phi_{-1}(\pi|n|^2/\alpha) - \alpha\sum_{\{n\}}\Phi_0(\pi|n|^2\alpha). \tag{A1.3}$$

We used here the following notation:

$$\Phi_\nu(x) = \int_1^\infty e^{-xt}t^\nu\,dt$$

is the incomplete gamma function. In particular we have

$$\Phi_{-1}(x) = -Ei(-x) = -\gamma - \ln x + \sum_{n=1}^\infty \frac{x^n}{n\,n!}(-1)^{n+1},$$

where $\gamma = 0.5772156649...$ is the Euler constant. The functions $\Phi_\nu(x)$ of higher rank can be obtained from the recurrence formula

$$x\,\Phi_\nu = e^{-x} + \nu\Phi_{\nu-1}.$$

Arguments of the functions can be calculated by taking $|n| = (n_1^2 + n_2^2)^{1/2}$, a modulus of the vector $\mathbf{n} = \mathbf{n}'/l$, (2.1),

$$\theta = \begin{cases} \operatorname{Arctan}(n_2/n_1), & n_1 \neq 0, \\ \pi - \operatorname{sgn}(n_2)\pi/2, & n_1 = 0. \end{cases}$$

The parameter α which appears in (A1.1) – (A1.3) is a moderate constant involved in Evald's summation method, and is of a very small influence on the final result of calculations.

The constants evaluated from (A1.1) – (A1.3) are as follows:

$$\begin{aligned} A_4 &= 0.7878030005, & A_8 &= 0.5319716294,\\ A_{12} &= 0.3282374177, & A_{16} &= 0.2509809396,\\ B_4 &= -1.044856181\ 10^{-1}, & B_8 &= -4.031710210\ 10^{-2},\\ B_{12} &= -1.469973805\ 10^{-2}, & B_{16} &= -8.399042320\ 10^{-3},\\ C_0 &= 2.6210658523. \end{aligned}$$

The coefficients A_m are related to the Rayleigh sums S_m

$$A_m = \frac{S_m}{m},$$

which were evaluated up to five decimal places and presented in [16]. Numerical values of the coefficients B_m, as far as it is known to the present author, were not presented in literature except the coefficient B_4 [4], the value of which given there is however not correct.

Appendix 2

In this section the expressions are presented which enable us to calculate elements W_{ij} of the matrix $\hat{W}$. They have been derived from Eqs. (4.11), (4.5), and (3.11)–(3.14). The expressions W_{ij} are different for odd and even subscripts j corresponding to the matrix columns. To simplify the form of expression for odd j, we exclude here elements of the first column ($j = 1$).

Matrix elements W_{ij} for $j = 1$:

$$\begin{aligned} W_{i1} = {} & \frac{\pi a^2}{2}\delta_{i1} + \frac{1}{4}(2 - \pi a^2)(\delta_{i3} - \delta_{i2}) + A_{i-1}a^{i-1} \\ & - (i+1)\left[\frac{1}{2}A_{i+1}a^2 + iB_{i+1}\right]a^{i-1} \\ & - (i+2)\left[\frac{1}{2}A_{i+2}a^2 + (i+1)B_{i+2}\right]a^{i}. \end{aligned} \tag{A2.1}$$

Matrix elements W_{ij} for other odd subscripts $j = 3,\ 5,\ 7, \dots$:

$$\begin{aligned} W_{ij} = {} & \frac{\pi}{2}\delta_{i1}\delta_{j3} + \frac{(j-1)!}{2a^{j-1}}(\delta_{i,j+2} - \delta_{i,j+1} + \delta_{i,j-1}) \\ & - \frac{(j-3)(j-2)!}{2a^{j-1}}\delta_{ij} - \frac{(j-3)(i+j-2)!}{2(i-1)!}A_{i+j-2}a^{i-1} \\ & - \frac{(j-1)(i+j-1)!}{2i!}A_{i+j-1}a^{i} - \frac{(i+j)!}{i!}\left[\frac{1}{2}A_{i+j}a^2 + iB_{i+j}\right]a^{i-1} \\ & - \frac{(i+j+1)!}{(i+1)!}\left[\frac{1}{2}A_{i+j+1}a^2 + (i+1)B_{i+j+1}\right]a^{i}. \end{aligned} \tag{A2.2}$$

Matrix elements W_{ij} for even subscripts $j = 2,\ 4,\ 6, \dots$:

$$\begin{aligned} W_{ij} = {} & -2\pi\delta_{i1}\delta_{j2} + \frac{2(j-1)!}{a^j}(\delta_{i,j+1} - \delta_{ij}) \\ & + \frac{2(i+j-1)!}{(i-1)!}A_{i+j-1}a^{i-1} + \frac{2(i+j)!}{i!}A_{i+j}a^{i}. \end{aligned} \tag{A2.3}$$

References

1. J. HAPPEL, *Viscous flow relative to arrays of cylinders*, AIChE J., **5**, 174–177, 1959.
2. J. HAPPEL and H. BRENNER, *Low Reynolds number hydrodynamics*, Nordhoff International Publ., Leyden 1973.

3. H. HASIMOTO, *On the periodic fundamental solutions of the Stokes equations and their application to viscous flow past a cubic array of spheres*, J. Fluid Mech., **5**, 317–328, 1959.
4. A.S. SANGANI and A. ACRIVOS, *Slow flow through a periodic array of spheres*, Int. J. Multiphase Flow, **8**, 4, 343–360, 1982.
5. J.E. DRUMMOND and M.I. TAHIR, *Laminar viscous flow through regular arrays of parallel solid cylinders*, Int. J. Multiphase Flow, **10**, 5, 515–540, 1984.
6. A.S. SANGANI and A. ACRIVOS, *Slow flow past periodic arrays of cylinders with application to heat transfer*, Int. J. Multiphase Flow, **8**, 193–206, 1982.
7. P.M. ADLER, *Porous media*, Butterworth - Heinemann, 1992.
8. A. SZANIAWSKI and A. ZACHARA, *Penetration of spinning bath into the filament bundle* [in Polish], Polimery, **19**, 143, 1974.
9. A. ZACHARA, *How to estimate the flow characteristics in multifilament spinning*, ATI Fiber World, Part I, May, 13–17, and Part II, July, 52–60, 1987.
10. S. WOLFRAM, *Mathematica. A system for doing mathematics by computer*, Addison-Wesley Publ. Co., 1991.
11. B. CICHOCKI and B.U. FELDERHOF, *Periodic fundamental solution of the linear Navier–Stokes equations*, Physica, **A 159**, 19–27, 1989.
12. B. CICHOCKI and B.U. FELDERHOF, *Electrostatic interactions in two-dimensional Coulomb systems with periodic boundary conditions*, Physica, **A 158**, 706–722, 1989.
13. M. ZUZOVSKY and H. BRENNER, *Effective conductivities of composite materials composed of cubic arrangements of spherical particles embedded in an isotropic matrix*, ZAMP, **28**, 6, 979–992, 1977.
14. A.S. SANGANI and A. ACRIVOS, *The effective conductivity of a periodic array of spheres*, Proc. R. Soc. London, **A 386**, 263–275, 1983.
15. S. MAY, S. TOKARZEWSKI, A. ZACHARA and B. CICHOCKI, *Continued fraction representation for the effective conductivity coefficient of a regular two-component composite*, Int. J. Heat Mass Transfer, **17**, 2165–2173, 1994.
16. W.T. PERRINS, D.R. MCKENZIE and R.C. MCPHEDRAN, *Transport properties of regular arrays of cylinders*, Proc. R. Soc. London, **A 369**, 207–225, 1979.

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