Arch. Mech., **49**, 4, pp. 677–695, Warszawa 1997

Remarks on Il'iushin's postulate

CH. TSAKMAKIS (KARLSRUHE)

IN ITS ORIGINAL VERSION (strong form), the postulate of Il'iushin states that the integral of the stress power of an elastic-plastic material must be non-negative for any closed strain path. As a consequence, it has been shown that the so-called simple endochronic theory of plasticity violates this postulate of "material stability". The characteristic feature of this theory is that yield surfaces and related loading conditions are not involved in the governing constitutive equations. In the present paper it is shown, with reference to a well-established class of plasticity laws, that the strong form of Il'iushin's postulate may be violated as well, if the constitutive theory is constructed on the basis of a yield surface and related loading conditions. The question arises if such strain-stress relations preserve some weaker stability conditions in the sense of Il'iushin. It turns out that they satisfy a weaker form of Il'iushin's postulate, in which the integral of the stress power is required to be non-negative only for special, so-called small cycles of deformation, as defined by Lucchesi and Silhavy. From a physical point of view on a phenomenological level, it seems that there is no experimental evidence to exclude from a general theory of plasticity such material behaviour which complies with the weak form of Il'iushin's postulate. Moreover, if the validity of the weak form of Il'iushin's postulate is assumed, then it is shown that the simple endochronic theory of plasticity is no longer in conflict with this version of the postulate.

1. Introduction

WHEN FORMULATING the framework for constitutive relations, it is convenient to introduce some constitutive inequalities limiting the types of the mechanical behaviour to be modeled. Such inequalities represent in some sense "material stability" conditions (see i.e. DRUCKER [8, 9], PALMER, MAIER and DRUCKER [24], HILL [13, 14], WANG and TRUESDELL [30, Ch. III.7–III.9], MARTIN [21, Sec. 2.4, 2.5], OGDEN [22, Sec. 6.2.8], LUBLINER [19, Sec. 3.2], HAVNER [12, Sec. 3.6] and the references cited therein). In the context of a pure mechanical theory concerning elastic-plastic material properties, a material stability condition frequently assumed is the postulate of Il'iushin. In its original form (strong form), this postulate is referred to small deformations and states that the integral of the stress power of a material element must be non-negative in any closed strain path (see IL'IUSHIN [16]). As a consequence (see SANDLER [26] as well as RIVLIN [25]), the linear isotropic version of the so-called simple endochronic theory of plasticity (see VALANIS [28]) violates this postulate. This fact was a reason for the simple endochronic theory of plasticity to be discredited.

The purpose of the present paper is to motivate the introduction of a weaker formulation of the postulate of Il'iushin and then to discuss the simple endochronic theory with respect to this weaker formulation of the postulate. Specifically, in Sec. 2 we will consider a class of plasticity laws which is defined in terms of a yield function and the related loading conditions. These plasticity laws exhibit kinematic hardening only, the well-known hardening rule of ARMSTRONG and FREDERICK [1] included as a particular case. Also, the linear isotropic version of the simple endochronic theory of plasticity is contained as a limiting case. Section 3 deals with an elementary proof of the fact that well-established plasticity laws, included in the class defined in Sec. 2, with kinematic hardening of the Armstrong–Frederick type, may violate the postulate of Il'iushin, if the material parameters do not have appropriate values. There arises then the question if these plasticity laws satisfy some kind of material stability in a weaker form.

Indeed, it is proved in Sec. 4, that the material behaviour predicted by these plasticity laws is in agreement with a weaker form of Il'iushin's postulate, in which the integral of the stress power is required to be non-negative only for special, so-called small strain cycles, as defined by LUCCHESI and SILHAVY [20]. Finally, it is shown in Sec. 5 that if the weak form of Il'iushin's postulate is assumed to apply in a general theory of plasticity, then the simple endochronic theory is no longer in conflict with this postulate.

In the following, we use bold-face and calligraphic letters for second-order and fourth-order tensors, respectively. In particular, $\mathbf{1}$ represents the identity second-order tensor and $\mathbf{A}^T$ denotes the transpose of $\mathbf{A}$. We write $\operatorname{tr}\mathbf{A}$ for the trace of $\mathbf{A}$, $\mathbf{A}^D = \mathbf{A} - \frac{1}{3}(\operatorname{tr}\mathbf{A})\mathbf{1}$ for the deviator of $\mathbf{A}$, as well as $\mathbf{A}\cdot\mathbf{B} = \operatorname{tr}(\mathbf{A}\mathbf{B}^T)$ for the inner product of $\mathbf{A}$ and $\mathbf{B}$. We denote by $\mathcal{E}$ the fourth-order tensor, which has components

$$\frac{1}{2}\left(\delta_{ij}\delta_{mn} + \delta_{in}\delta_{mj}\right)\mathbf{e}_i \otimes \mathbf{e}_m \otimes \mathbf{e}_j \otimes \mathbf{e}_n$$

relative to the orthonormal basis $\{\mathbf{e}_i\}$, $i = 1, 2, 3$, where the symbol $\otimes$ denotes the tensor product, and δ_{ij} is the Kronecker delta. Further, if $\mathcal{K}$ and $\mathbf{A}$ are fourth-order and second-order tensors, respectively, represented by $\mathcal{K} = \mathcal{K}_{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$ and $\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, then $\mathcal{K}[\mathbf{A}] := \mathcal{K}_{ijmn}A_{mn}\mathbf{e}_i \otimes \mathbf{e}_j$. For a real a, $|a|$ is the absolute value of a, while $(\ )^{\cdot}$ denotes the material time derivative of $(\ )$, the variable time being represented by t. If nothing is said to the contrary, the material parameters used will take values on the real interval $[0, \infty)$. The deformations considered are small isothermal deformations. Since the following is not affected by a space dependence, an explicit reference to space will be dropped. All components of tensor variables are related to a Cartesian coordinate system.

2. Plasticity laws with kinematic hardening

Consider the class of plasticity models with kinematic hardening summarized in Table 1.

Table 1. A class of plasticity models with kinematic hardening.

(2.1) $\mathbf{E} = \mathbf{E}_\mathrm{e} + \mathbf{E}_\mathrm{p}$,

(2.2) $\mathbf{T} = 2\mu\mathbf{E}_\mathrm{e} + \lambda(\operatorname{tr}\mathbf{E}_\mathrm{e})\mathbf{1}$,

(2.3) $\mu,\ \lambda$ – Elasticity constants ($\mu > 0,\ 2\mu + 3\lambda > 0$),

(2.4) F – yield function,

(2.5) $\mathsf{F}(t) = \bar{\mathsf{F}}(\mathbf{T}(t), \boldsymbol{\xi}(t)) := \sqrt{\frac{3}{2}(\mathbf{T}-\boldsymbol{\xi})^D\cdot(\mathbf{T}-\boldsymbol{\xi})^D + \alpha(\operatorname{tr}[\mathbf{T}-\boldsymbol{\xi}])^2} - k$,

(2.6) $k = \mathrm{const}, \quad 0 \leq \alpha \leq \frac{1}{2}$,

(2.7) yield condition $\iff$ $\mathsf{F} = 0$,

(2.8) $$\dot{\mathbf{E}}_\mathrm{p} := \begin{cases} \Lambda\dfrac{\partial\bar{\mathsf{F}}}{\partial\mathbf{T}} & \text{in plastic loading,} \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

(2.9) plastic loading $\iff$ $\mathsf{F} = 0 \ \ \&\ \ (\dot{\mathsf{F}})_{s=\mathrm{const}} > 0$,

(2.10) $\dot{s} = \sqrt{\dot{\mathbf{E}}_\mathrm{p}\cdot\mathcal{P}[\dot{\mathbf{E}}_\mathrm{p}]}$,

(2.11) $\mathcal{P} = p_1\mathcal{E} + p_2\mathbf{1}\otimes\mathbf{1}$,

(2.12) $p_1,\ p_2$ – const $(p_1 \geq 0, \quad p_1 + 3p_2 \geq 0)$,

(2.13) $\dot{z} = g\dot{s}$,

(2.14) g – constitutive function,

(2.15) $\boldsymbol{\xi} := \sum_{j=1}^{N}\mathbf{Z}_j + \sum_{r=1}^{n}\frac{1}{3}M_r\mathbf{1}$,

(2.16) $\dfrac{d\mathbf{Z}_j}{dz} = c_j\dfrac{d\mathbf{E}_\mathrm{p}}{dz} - b_j\mathbf{Z}_j$,

(2.17) $c_j,\ b_j > 0 \ \text{ if } \ 1 \leq j < N, \quad c_N > 0, \quad b_N \geq 0$,

(2.18) $\dfrac{dM_r}{dz} = c_r^*\dfrac{d(\operatorname{tr}\mathbf{E}_\mathrm{p})}{dz} - b_r^*M_r$,

(2.19) $c_r^*,\ b_r^* > 0 \ \text{ if } \ 1 \leq r < n, \quad c_n^* > 0, \quad b_n^* \geq 0$.

In these formulas, $\mathbf{E}_\mathrm{e}$ and $\mathbf{E}_\mathrm{p}$ denote the elastic and plastic parts in the additive decomposition of the linearized Green strain tensor $\mathbf{E}$, $\mathbf{T}$ is the Cauchy stress tensor and $\boldsymbol{\xi}$ represents the variable describing kinematic hardening (back stress). The scalar factor Λ is to be determined from the so-called consistency equation $\dot{\mathsf{F}} = 0$. The characteristic features of these plasticity laws are the yield function (2.5), (2.6) and the evolution equations (2.10) – (2.19) for the back stress. The yield function (2.5) is related to the general definition for yield functions, given by SHRIVASTAVA, MROZ and DUBEY [27]. A yield function of the form (2.5), (2.6) as well as the hardening rule (2.10) – (2.19) with $p_2 = 0$, $p_1 = 1$ and g as positive continuous monotonic and bounded function of s were proposed by KORZEN [17, Ch.5] as a generalization of the hardening rule introduced by CHABOCHE, DANG VAN and CORDIER [5]. However, it should be mentioned that relations of the form (2.10) – (2.19) indicate the typical structure of constitutive equations according to the simple endochronic theory of plasticity and were introduced by VALANIS [28] at first. Note in passing that g can be chosen to be a constitutive function of additional variables, for which then appropriate evolution equations are needed. Such examples are given e.g. in HAUPT, KAMLAH and TSAKMAKIS [11]. Aslo, it is perhaps of interest to remark that, for $\boldsymbol{\xi} = \mathbf{0}$, (2.5) reduces to a yield function proposed earlier by Burzynski (see Życzkowski [31, Sec. 11], where the original work of Burzynski is cited).

There are two particular cases in the class of plasticity laws (2.1) – (2.19), which will now be discussed briefly.

2.1. Plasticity laws with kinematic hardening of the Armstrong–Frederick type

On setting $\alpha = 0$, Eq. (2.5) reduces to a v. Mises yield function with kinematic hardening. Hence, by (2.8), $\mathbf{E}_\mathrm{p}$ becomes deviatoric. Further, we set $p_1 = 2/3$, $p_2 = 0$ and $g \equiv 1$. Then,

$$\dot{s} = \dot{z} = \sqrt{\frac{2}{3}\,\dot{\mathbf{E}}_\mathrm{p}\cdot\dot{\mathbf{E}}_\mathrm{p}}\,, \tag{2.20}$$

i.e., $\dot{s}$ reduces to the well-known plastic arc length formula. In addition, we assume homogeneous initial conditions, such that $M_r \equiv 0$ for every r. Finally, if we set $N = 1$ and $c_1 = c$, $b_1 = b$, Eqs. (2.15) – (2.19) yield the well-known hardening rule of ARMSTRONG and FREDERICK [1] (see also CHABOCHE [6]):

$$\dot{\boldsymbol{\xi}} = c\dot{\mathbf{E}}_\mathrm{p} - b\dot{s}\boldsymbol{\xi}\,. \tag{2.21}$$

This hardening rule together with the relations (2.1) – (2.9) and $\alpha = 0$ form a well established plasticity model, which will be discussed in Sec. 3 for one-dimensional loading histories. The one-dimensional relations needed for this discussion read as follows. Using the definitions $(\mathbf{T})_{11} = \sigma$, $(\mathbf{E})_{11} = \varepsilon$, $(\mathbf{E}_\mathrm{e})_{11} = \varepsilon_\mathrm{e}$, $(\mathbf{E}_\mathrm{p})_{11} = \varepsilon_\mathrm{p}$, $\dot{s} = |\dot{\varepsilon}_\mathrm{p}|$, $\frac{3}{2}(\boldsymbol{\xi})_{11} = \xi$, $E = \dfrac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$ (E – elasticity modulus), we get:

$$\varepsilon = \varepsilon_\mathrm{e} + \varepsilon_\mathrm{p}\,, \tag{2.22}$$

$$\sigma = E\varepsilon_\mathrm{e}\,, \tag{2.23}$$

$$\left.\begin{aligned} &|\sigma - \xi| = k\,, && (2.24)\\ &\dot{\sigma} = \dot{\xi}\,, && (2.25)\\ &\dot{\xi} = \frac{3}{2}c\dot{\varepsilon}_\mathrm{p} - b|\dot{\varepsilon}_\mathrm{p}|\xi\,, && (2.26) \end{aligned}\right\}\quad \text{for plastic loading.}$$

Thus,

$$\frac{d\mathcal{E}_{\mathrm{p}}}{d\mathcal{E}} = \frac{E}{E + \frac{3c}{2} \pm b\xi}, \tag{2.27}$$

where + denotes compressive loading and − tensile loading.

2.2. The case $k = 0$

Let $k = 0$ in Eq. (2.5). Then, from (2.5) – (2.7),

$$\mathbf{T} \equiv \boldsymbol{\xi}, \tag{2.28}$$

i.e., a pure elastic range is no longer present. Roughly speaking, the yield surface shrinks to a point in the stress space. However, according to (2.1), (2.2), the decomposition of the total strain into elastic and plastic parts still remains valid. Note that $k = 0$ always implies plastic loading. Further, (2.8) is meaningless now, for a unique normal on the yield surface no longer exists. But such an equation is no longer needed. In other words, the plastic strain $\mathbf{E}_{\mathrm{p}}$ is now postulated to depend upon the previous loading history by means of an implicitly defined functional. This functional is represented through the constitutive relations for the kinematic hardening. To be more specific, if we are given a loading history, then the decomposition for the strain (2.1), the elasticity law (2.2), the identity (2.28) and the kinematic hardening rule form a system of four equations for the determination of the four unknown $\boldsymbol{\xi}$, $\mathbf{E}_{\mathrm{e}}$, $\mathbf{E}_{\mathrm{p}}$ and $\mathbf{T}$ or $\mathbf{E}$.

Next, we note that if μ, $\lambda \to \infty$, then $\mathbf{E}_{\mathrm{e}} \to \mathbf{0}$ in order for the stress to remain finite. This means that the assumption μ, $\lambda \to \infty$ implies

$$\mathbf{E}_{\mathrm{p}} \to \mathbf{E}. \tag{2.29}$$

Now, let us integrate Eqs. (2.15) – (2.19) for homogeneous initial conditions and assume g to be a function of the arc length s. Then, by (2.28), (2.29),

$$\mathbf{T} = \int_0^z \Psi(z-\bar{z}) \left(\frac{d}{d\bar{z}}\mathbf{E}(\bar{z})\right) d\bar{z} + \mathbf{1} \int_0^z \chi(z-\bar{z}) \left(\frac{d}{d\bar{z}} \operatorname{tr}\mathbf{E}(\bar{z})\right) d\bar{z}, \tag{2.30}$$

$$\Psi(z) = \sum_{j=1}^{N} c_j e^{-b_j z}, \tag{2.31}$$

$$\chi(z) = \sum_{r=1}^{n} c_r^* e^{-b_r^* z}, \tag{2.32}$$

$$\dot{z} = g(s)\dot{s}, \tag{2.33}$$

$$\dot{s} = \sqrt{\dot{\mathbf{E}} \cdot \mathcal{P}[\dot{\mathbf{E}}]}. \tag{2.34}$$

Equations (2.30) – (2.34) represent the linear isotropic version of the so-called simple endochronic theory of plasticity. This constitutive model was postulated by VALANIS [28] (for a brief review of the simple endochronic theory of plasticity see also VALANIS and LEE [29]) in order to describe plastic material properties without using a yield surface as well as loading conditions. It must be emphasized that Valanis introduced this theory by replacing the natural time t in the linear viscoelasticity by the arc length z defined through (2.33), (2.34). In the present paper, it is shown that the simple endochronic theory of plasticity can be interpreted as a limiting case of the class of plasticity laws represented in Table 1 ([1]). In particular, this was the reason for defining the arc lengths s and z as in (2.10) – (2.14), which is typical for the endochronic theory of plasticity.

3. The postulate of Il'iushin for arbitrary strain cycles

For a fixed material particle, let us consider a strain cycle, which begins at time t_0 and ends at time t_e. Such strain cycles are denoted by $C[t_0, t_e]$. Il'iushin's postulate (strong form) requires the work to be non-negative in any strain cycle:

$$I(t_0, t_e) = \int_{t_0}^{t_e} \mathbf{T}(t) \cdot \dot{\mathbf{E}}(t)\, dt \geq 0 \qquad \text{for every} \qquad C[t_0, t_e]. \tag{3.1}$$

Evidently, a strain cycle does not generally imply a cycle for other process variables, if irreversible deformations are involved. To emphasize this fact, ŻYCZKOWSKI [31, p. 119] introduced the term quasi-cycle. However, in dealing with Il'iushin's postulate, it is customary (see e.g. LUBLINER [19, p. 122]) to denote closed trajectories in strain space simply as strain cycles, a notion which is adopted in the present paper as well.

We now proceed to examine compatibility of the plasticity laws given in Sec. 2.1 with the inequality (3.1). Especially, we will show that these plasticity laws can contradict inequality (3.1) for some special strain cycles, and thus in general. To this end, it suffices to consider one-dimensional strain-controlled processes involving tension and compression loading. The constitutive relations governing the material response are given by (2.22) – (2.27). Specifically, we consider strain cycles of the form $A'C'A'$ (see Fig. 1) to which the (ε, σ)-paths of the kind $ABCDE$ correspond. In the following, we denote by $(\,)^P$ the value of $(\,)$ at the point P.

Plastic flow first occurs during compression at point B and during tension at point D. We may parameterize the (ε, σ)-path e.g. by using the time t as a parameter. We denote by t_A, t_B, t_C, t_D, t_E, $(t_A < t_B < t_C < t_D < t_E)$, the times belonging to the (ε, σ)-points A, B, C, D, E, respectively. Thus, plastic

([1]) Actually, we deal only with the linear isotropic version of the simple endochronic theory here, but it is not difficult to extend the above discussion to the non-isotropic case.

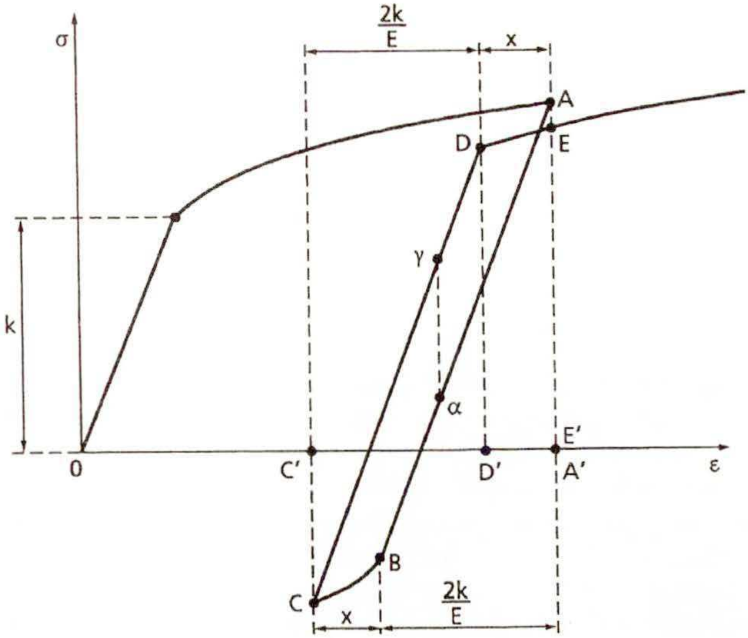

FIG. 1. Uniaxial strain cycle. To the cycle $A'C'A'$ in the strain axis corresponds the (ε, σ)-path $ABCDE$.

deformations are only involved during the time intervals $[t_B, t_C]$ and $[t_D, t_E]$, while pure elastic deformations take place during the time intervals $[t_A, t_B]$ and $[t_C, t_D]$. Since kinematic hardening is only incorporated, the strain differences between E and D, and between B and C, are equal:

$$\varepsilon^E - \varepsilon^D = \varepsilon^B - \varepsilon^C =: x \,. \tag{3.2}$$

The strain differences between A and B, and between D and C, satisfy the relations (see Fig. 1)

$$\varepsilon^A - \varepsilon^B = \varepsilon^D - \varepsilon^C = \frac{2k}{E} \,. \tag{3.3}$$

The work of the stress power for the time interval $[t_A, t_E]$ is

$$I_{AE} = \int_{t_A}^{t_E} \sigma(t)\, \dot{\varepsilon}(t)\, dt \,, \tag{3.4}$$

or, by (2.22), (2.23),

$$I_{AE} = E \int_{t_A}^{t_E} \varepsilon(t)\, \dot{\varepsilon}(t)\, dt - E \int_{t_A}^{t_E} \varepsilon_{\mathrm{p}}(t)\, \dot{\varepsilon}(t)\, dt \,. \tag{3.5}$$

Recalling that $\varepsilon^A = \varepsilon^E$, the last equation reduces to

$$I_{AE} = -E \int_{t_A}^{t_E} \varepsilon_p(t)\,\dot{\varepsilon}(t)\,dt\,, \tag{3.6}$$

or

$$I_{AE} = 2k\left(\varepsilon_p^A - \varepsilon_p^C\right) - E\left(\int_{t_B}^{t_C} \varepsilon_p(t)\,\dot{\varepsilon}(t)\,dt + \int_{t_D}^{t_E} \varepsilon_p(t)\,\dot{\varepsilon}(t)\,dt\right), \tag{3.7}$$

in view of the fact that $\varepsilon_p(t) = \varepsilon_p^A = \text{const}$ for $t \in [t_A, t_B]$ and $\varepsilon_p(t) = \varepsilon_p^C = \text{const}$ for $t \in [t_C, t_D]$. Now fix t_A (and therefore ε^A, ε^B too) and change the integration variable from t to ε. This is possible, since the strain ε is a monotonic function of t for $t \in [t_B, t_C]$ and $t \in [t_D, t_E]$, respectively. Further, suppose that the values of ε_p and ξ are known at t_A, and therefore at t_B too. Then, for $t \in [t_B, t_C]$, the solutions of Eqs. (2.22) – (2.26), with $\sigma - \xi = -k$ and $\dot{\varepsilon}_p < 0$, define ε_p and ξ as functions of the strain $\bar{\varepsilon}_p(\varepsilon)$ and $\bar{\xi}(\varepsilon)$, respectively. Hence the values of ε_p and ξ at t_C, and therefore at t_D as well, are determined as a function of x, respectively, where x is defined by (3.2). We may use these values as initial conditions in (2.22) – (2.26), with $\sigma - \xi = k$ and $\dot{\varepsilon}_p > 0$, to obtain ε_p and ξ for $t \in [t_D, t_E]$ as functions of the strain $\bar{\bar{\varepsilon}}_p(\varepsilon; x)$ and $\bar{\bar{\xi}}(\varepsilon; x)$, respectively:

$$(\varepsilon_p(t), \xi(t)) = \begin{cases} (\bar{\varepsilon}_p(\varepsilon), \bar{\xi}(\varepsilon)) & \text{if} \quad t \in [t_B, t_C], \\ (\bar{\bar{\varepsilon}}_p(\varepsilon; x), \bar{\bar{\xi}}(\varepsilon; x)) & \text{if} \quad t \in [t_D, t_E]. \end{cases} \tag{3.8}$$

Clearly,

$$\varepsilon_p^A = \varepsilon_p^B = \bar{\varepsilon}_p(\varepsilon^B) = \bar{\bar{\varepsilon}}_p(\varepsilon^A; 0), \tag{3.9}$$

$$\xi^A = \xi^B = \bar{\xi}(\varepsilon^B) = \bar{\bar{\xi}}(\varepsilon^A; 0). \tag{3.10}$$

Then, (3.7) can be written as

$$I_{AE}(x) = \varphi_1(x) - \varphi_2(x) - \varphi_3(x) \tag{3.11}$$

with

$$\varphi_1(x) = 2k\left(\varepsilon_p^A - \bar{\varepsilon}_p(\varepsilon^B - x)\right), \tag{3.12}$$

$$\varphi_2(x) = E \int_{\varepsilon^B}^{\varepsilon^B - x} \bar{\varepsilon}_p(\varepsilon)\,d\varepsilon\,, \tag{3.13}$$

$$\varphi_3(x) = E \int_{\varepsilon^A - x}^{\varepsilon^A} \bar{\bar{\varepsilon}}_p(\varepsilon; x)\,d\varepsilon\,. \tag{3.14}$$

For our purpose it is convenient to expand $I_{AE}(x)$ into a Taylor's series about $x = 0$:

$$(3.15)\qquad I_{AE}(x) = [\varphi_1(x) - \varphi_2(x) - \varphi_3(x)]_{x=0} + x\left[\frac{d}{dx}(\varphi_1(x) - \varphi_2(x) - \varphi_3(x))\right]_{x=0} + \frac{1}{2}x^2\left[\frac{d^2}{dx^2}(\varphi_1(x) - \varphi_2(x) - \varphi_3(x))\right]_{x=0} + O(x^3)$$

with O denoting the usual order symbol. The derivatives of the functions in (3.12) – (3.14) are given by

$$(3.16)\qquad \frac{d\varphi_1(x)}{dx} = 2k\left[\frac{d\bar{\varepsilon}_{\mathrm{p}}(\varepsilon)}{d\varepsilon}\right]_{\varepsilon=\varepsilon^B-x},$$

$$(3.17)\qquad \frac{d^2\varphi_1(x)}{dx^2} = -2k\left[\frac{d^2\bar{\varepsilon}_{\mathrm{p}}(\varepsilon)}{d\varepsilon^2}\right]_{\varepsilon=\varepsilon^B-x},$$

$$(3.18)\qquad \frac{d\varphi_2(x)}{dx} = -E\bar{\varepsilon}_{\mathrm{p}}(\varepsilon^B - x),$$

$$(3.19)\qquad \frac{d^2\varphi_2(x)}{dx^2} = E\left[\frac{d\bar{\varepsilon}_{\mathrm{p}}(\varepsilon)}{d\varepsilon}\right]_{\varepsilon=\varepsilon^B-x},$$

$$(3.20)\qquad \frac{d\varphi_3(x)}{dx} = E\bar{\bar{\varepsilon}}_{\mathrm{p}}(\varepsilon^A - x; x) + E\int_{\varepsilon^A-x}^{\varepsilon^A}\frac{\partial\bar{\bar{\varepsilon}}_{\mathrm{p}}(\varepsilon; x)}{\partial x}d\varepsilon,$$

$$(3.21)\qquad \frac{d^2\varphi_3(x)}{dx^2} = E\left[2\frac{\partial\bar{\bar{\varepsilon}}_{\mathrm{p}}(\varepsilon; x)}{\partial x} - \frac{\partial\bar{\bar{\varepsilon}}_{\mathrm{p}}(\varepsilon; x)}{\partial\varepsilon}\right]_{\varepsilon=\varepsilon^A-x} + E\int_{\varepsilon^A-x}^{\varepsilon^A}\frac{\partial^2\bar{\bar{\varepsilon}}_{\mathrm{p}}(\varepsilon; x)}{\partial x^2}d\varepsilon.$$

Before proceeding to calculate the values of these functions for $x = 0$, it is first necessary to establish some results concerning the derivatives of the functions in (3.8). From (2.27) and (3.10) we obtain

$$(3.22)\qquad \left[\frac{d\bar{\varepsilon}_{\mathrm{p}}(\varepsilon)}{d\varepsilon}\right]_{\varepsilon=\varepsilon^B} = \left[\frac{E}{E+\frac{3c}{2}+b\bar{\xi}(\varepsilon)}\right]_{\varepsilon=\varepsilon^B} = \frac{E}{E+\frac{3c}{2}+b\xi^B},$$

$$(3.23)\qquad \left[\frac{\partial\bar{\bar{\varepsilon}}_{\mathrm{p}}(\varepsilon; x)}{\partial\varepsilon}\right]_{\varepsilon=\varepsilon^A,\ x=0} = \left[\frac{E}{E+\frac{3c}{2}-b\bar{\bar{\xi}}(\varepsilon; x)}\right]_{\varepsilon=\varepsilon^A,\ x=0} = \frac{E}{E+\frac{3c}{2}-b\xi^B}.$$

Also, if $\varepsilon^D \le \varepsilon \le \varepsilon^E = \varepsilon^A$, we may diffenrentiate the identity

$$(3.24) \qquad \bar{\bar{\varepsilon}}_{\rm p}(\varepsilon;x) = \bar{\varepsilon}_{\rm p}(\varepsilon^B - x) + \int\limits_{\varepsilon^A - x}^{\varepsilon} \frac{\partial \bar{\bar{\varepsilon}}_{\rm p}(u;x)}{\partial u} du$$

with respect to x to obtain

$$(3.25) \qquad \frac{\partial \bar{\bar{\varepsilon}}_{\rm p}(\varepsilon;x)}{\partial x} = -\left[\frac{d\bar{\varepsilon}_{\rm p}(u)}{du}\right]_{u=\varepsilon^B - x} + \left[\frac{\partial \bar{\bar{\varepsilon}}_{\rm p}(u;x)}{\partial u}\right]_{u=\varepsilon^A - x} + \int\limits_{\varepsilon^A - x}^{\varepsilon} \frac{\partial^2 \bar{\bar{\varepsilon}}_{\rm p}(u;x)}{\partial u \partial x} du \,.$$

This implies

$$(3.26) \qquad \left[\frac{\partial \bar{\bar{\varepsilon}}_{\rm p}(\varepsilon;x)}{\partial x}\right]_{\varepsilon=\varepsilon^A,\ x=0} = -\left[\frac{d\bar{\varepsilon}_{\rm p}(\varepsilon)}{d\varepsilon}\right]_{\varepsilon=\varepsilon^B} + \left[\frac{\partial \bar{\bar{\varepsilon}}_{\rm p}(\varepsilon;x)}{\partial \varepsilon}\right]_{\varepsilon=\varepsilon^A,\ x=0},$$

the variable u in the two first terms on the right-hand side of (3.25) being replaced with ε. Using (3.12) – (3.14), (3.16) – (3.21), as well as (3.8), (3.9) and (3.26), it is now straightforward to deduce that

$$(3.27) \qquad \varphi_1(0) = 0 \,,$$

$$(3.28) \qquad \left[\frac{d\varphi_1(x)}{dx}\right]_{x=0} = 2k \left[\frac{d\bar{\varepsilon}_{\rm p}(\varepsilon)}{d\varepsilon}\right]_{\varepsilon=\varepsilon^B},$$

$$(3.29) \qquad \left[\frac{d^2\varphi_1(x)}{dx^2}\right]_{x=0} = -2k \left[\frac{d^2\bar{\varepsilon}_{\rm p}(\varepsilon)}{d\varepsilon^2}\right]_{\varepsilon=\varepsilon^B},$$

$$(3.30) \qquad \varphi_2(0) = 0 \,,$$

$$(3.31) \qquad \left[\frac{d\varphi_2(x)}{dx}\right]_{x=0} = -E\varepsilon_{\rm p}^A \,,$$

$$(3.32) \qquad \left[\frac{d^2\varphi_2(x)}{dx^2}\right]_{x=0} = E \left[\frac{d\bar{\varepsilon}_{\rm p}(\varepsilon)}{d\varepsilon}\right]_{\varepsilon=\varepsilon^B},$$

$$(3.33) \qquad \varphi_3(0) = 0 \,,$$

$$(3.34) \qquad \left[\frac{d\varphi_3(x)}{dx}\right]_{x=0} = E\varepsilon_{\rm p}^A \,,$$

$$(3.35) \qquad \left[\frac{d^2\varphi_3(x)}{dx^2}\right]_{x=0} = -2E \left[\frac{d\bar{\varepsilon}_{\rm p}(\varepsilon)}{d\varepsilon}\right]_{\varepsilon=\varepsilon^B} + E \left[\frac{\partial \bar{\bar{\varepsilon}}_{\rm p}(\varepsilon;x)}{\partial \varepsilon}\right]_{\varepsilon=\varepsilon^A,\ x=0}$$

Thus, by substituting (3.27) – (3.35) into (3.15) and rearranging the terms appropriately, $I_{AE}(x)$ takes the more specific form

$$(3.36)\qquad I_{AE}(x) = 2k\left\{x\left[\frac{d\bar{\varepsilon}_{\mathrm{p}}(\varepsilon)}{d\varepsilon}\right]_{\varepsilon=\varepsilon^B} - \frac{1}{2}x^2\left[\frac{d^2\bar{\varepsilon}_{\mathrm{p}}(\varepsilon)}{d\varepsilon^2}\right]_{\varepsilon=\varepsilon^B}\right\} + \frac{E}{2}x^2\left\{\left[\frac{d\bar{\varepsilon}_{\mathrm{p}}(\varepsilon)}{d\varepsilon}\right]_{\varepsilon=\varepsilon^B} - \left[\frac{\partial\bar{\bar{\varepsilon}}_{\mathrm{p}}(\varepsilon;x)}{\partial\varepsilon}\right]_{\varepsilon=\varepsilon^A,\ x=0}\right\} + O(x^3).$$

Now, we are in a position to show that the sign of $I_{AE}(x)$ can become negative. To this end, suppose $\xi^B > 0$, so that, from (3.22), (3.23),

$$(3.37)\qquad \left[\frac{d\bar{\varepsilon}_{\mathrm{p}}(\varepsilon)}{d\varepsilon}\right]_{\varepsilon=\varepsilon^B} - \left[\frac{\partial\bar{\bar{\varepsilon}}_{\mathrm{p}}(\varepsilon;x)}{\partial\varepsilon}\right]_{\varepsilon=\varepsilon^A,\ x=0} < 0\,.$$

Then, in view of (3.36), we recognize that there exist strain cycles with sufficiently small x, such that $I_{AE}(x)$ can be approximated by second-order terms in x. Further, by virtue of (3.37), $I_{AE}(x)$ can become negative if k is sufficiently small (but in general not infinitesimally small).

It must be noted that, as mentioned in the Introduction (Sec. 1), a negative work during strain cycles for the simple endochronic theory was calculated by SANDLER [26] and RIVLIN [25]. However, it was shown in Sec. 2.2 that constitutive models of the simple endochronic theory do not possess a finite elastic range, even not an elastic part of deformation. Roughly speaking, the qualitative difference between our work and the calculations of Sandler and Rivlin is that here we deal with plastic materials exhibiting finite elastic range.

4. The postulate of Il'iushin for small strain cycles

In the previous section, we have shown with reference to the well-established plasticity laws given in Sec. 2.1, that if the hardening parameters, and in particular the yield stress, are required to be only non-negative, a physically plausible assumption, then inequality (3.1) may be violated. It is then of interest to know if these plasticity laws satisfy some weaker forms of material stability(2) than (3.1). In the following, we will show that indeed they satisfy a weaker form of the postulate of Il'iushin, in which the integral of the stress power is required to be non-negative only for special, so-called small strain cycles.

We observe from Fig. 1 that during the loading segment corresponding to the time interval $(t_B, t_C]$, the initial strain state (point A') always lies outside the elastic ranges surrounded by the yield surfaces in the strain axis (strain space).

(2) The terms strong and weak forms of work postulate are introduced by MARTIN [21, Sec. 2.4] in relation to various inequalities imposing restrictions on constitutive relations.

For example, point A' is out of the line segment $C'D'$, which represents the one-dimensional elastic range assigned to the point C. Following LUCCHESI and SILHAVY [20] we denote strain cycles which always contain the initial strain state as small strain cycles. It can be seen that during a (simple closed) small strain cycle, generation of plastic deformation occurs only either due to compression or due to tension. For example, with regard to the small strain cycle corresponding to the strain-stress-path $\alpha BC\gamma$ (see Fig. 1), $\dot{\varepsilon}_p(t) \neq 0$ is satisfied only in the time interval $[t_B, t_C]$. In this case, regardless of the value of k, we have $(t_A < t_B < t_C < t_\gamma)$

$$
\begin{aligned}
(4.1) \qquad I_{\alpha\gamma} &= \int_{t_\alpha}^{t_\gamma} \sigma(t)\,\dot{\varepsilon}(t)\,dt = -E\int_{t_\alpha}^{t_\gamma} \varepsilon_p(t)\,\dot{\varepsilon}(t)\,dt \\
&= E\left(\varepsilon_p^\alpha(\varepsilon^\alpha - \varepsilon^B) + \int_{\varepsilon^C}^{\varepsilon^B} \bar{\varepsilon}_p(\varepsilon)\,d\varepsilon - \varepsilon_p^C(\varepsilon^\alpha - \varepsilon^C)\right) \\
&\geq E\left[\varepsilon^\alpha(\varepsilon^\alpha - \varepsilon^B) + \varepsilon_p^C(\varepsilon^B - \varepsilon^C) - \varepsilon_p^C(\varepsilon^\alpha - \varepsilon^C)\right] \\
&= E(\varepsilon_p^\alpha - \varepsilon_p^C)(\varepsilon^\alpha - \varepsilon^B) > 0\,.
\end{aligned}
$$

Hence, in the case of uniaxial loading, the plasticity laws given in Sec. 2.1 are stable in the sense of Il'iushin for all small cycles.

In order to extend this result to the three-dimensional case, it is convenient to introduce the notion of small strain cycles within a general framework for plastic constitutive relations. Suppose that the yield surface is represented in strain space by an equation of the form

$$
(4.2) \qquad \widetilde{\mathsf{G}}(\mathbf{E}, \mathbf{E}_p, \mathbf{q}) = 0\,,
$$

where $\mathbf{q}$ represents a set of plastic internal variables q_i, $1 \leq i \leq m$, which are scalar-valued or components of tensors. The internal variables q_i are supposed to change only when plastic flow occurs. We consider strain cycles which satisfy the following condition. During the cyclic process, the initial strain state is always included within or lies on the boundary of every yield surface corresponding to the process. In other words, the initial strain state always lies in the intersection of all the elastic ranges surrounded by the yield surfaces during the process. Generalizing the one-dimensional definition, we denote such strain cycles as small cycles, and write $C_s[t_0, t_e]$ for a small cycle which begins at time t_0 and ends at time t_e. Then, a material is defined to satisfy the postulate of Il'iushin for small cycles (weak form of the postulate of Il'iushin), if

$$
(4.3) \qquad I(t_0, t_e) = \int_{t_0}^{t_e} \mathbf{T}(t)\cdot\dot{\mathbf{E}}(t)\,dt \geq 0 \qquad \text{for every} \qquad C_s[t_0, t_e].
$$

In Appendix A, it is shown in a general context, that if the elasticity law is independent of the plastic deformations, then the weak form of Il'iushin's postulate (4.2) is equivalent to the two following conditions: i) the yield surface in stress space is convex, and ii) the plastic strain rate $\dot{\mathbf{E}}_{\mathrm{p}}$ is directed along the outward normal to the yield surface in stress space (normality rule). Since the plasticity laws of Sec. 2.1 satisfy these convexity and normality conditions, it follows that they satisfy (4.3) as well. This proves, for the three-dimensional case, stability in the sense of the weak form of Il'iushin's postulate for the plasticity laws given in Sec. 2.1.

Until now we have introduced the stability condition (4.3) in relation to specific constitutive models. However, from a physical point of view on a phenomenological level, it seems that there is no experimental evidence to exclude from a general theory of plasticity such material response which is concordant with the stability condition (4.3). Thus, regarding inequality (3.1) as being too restrictive for the material response, we may assume the weaker inequality (4.3) as a general principle which is in agreement with the observed behaviour of various materials. Note that such an assumption is in harmony with the statement by PALMER, MAIER and DRUCKER [24, p. 468], that "the degree of stability to be required of a material or a structural element is a matter of matching the observed or calculated behaviour of a selected portion of the real world with a sufficiently simple model".

On the other hand, it can be seen (see e.g. Appendix A), that if the elasticity law is independent of the plastic deformations, then (4.3) is equivalent to the so-called principle of maximum plastic dissipation. The latter was also derived from considerations of crystal plasticity by BISHOP and HILL [2]. This motivates from a physical point of view on a microstructural level, the use of the stability condition (4.3) in a general theory on plasticity.

It must be emphasized that, in the framework of the assumptions made in the present paper, (4.3) is the isothermal version of a general dissipation postulate, which has been proposed by LUCCHESI and SILHAVY [20] as a non-isothermal generalization of Il'iushin's inequality (3.1). However, it is of interest to note that the condition that the cycles should be small was imposed by Lucchesi and Silhavy in order to make Il'iushin's postulate "derivable from some sufficient conditions (the normality rule)". This is rather a mathematical point of view, while the condition of small cycles is assumed in the present work in order to obtain a stability condition for material response which is not too restrictive when modeling the observed behaviour of various materials. This is rather a physical point of view.

5. Consequences for the case of vanishing elastic range

In order to give a general treatment for the case of vanishing elastic range it is necessary to introduce a distance between two elements in the considered vector space of second-order tensors. Since we deal with normed vector spaces, we define the distance between two tensors to be the norm of their difference. Further, we denote by δ_{G} the supremum of the diameters of all the yield surfaces during the strain cycle (the diameter of a yield surface in a strain space representation is defined to be the supremum of the distances between two points on the yield surface). We assume that the yield surface is a closed hypersurface, at least in the space of deviatoric strain tensors. If the yield surface is a closed hypersurface in the space of strain tensors, then δ_{G} should be understood to be defined in the space of strain tensors. Otherwise, δ_{G} should be understood to be defined in the space of deviatoric strain tensors. In the latter case, plastic incompressibility is supposed to apply as well.

Next, suppose the yield surface to be given in a strain space representation as defined by (4.2) and consider the small strain cycle $ABCD$ displayed in Fig. 3 (see Appendix A). Let us denote by x the supremum of the distances between the strain points along BC and the initial yield surface $\widetilde{\mathsf{G}}(\mathbf{E}, \mathbf{E}_{\mathrm{p}}^{B}\mathbf{q}^{B}) = 0$ (the distance between a strain point and a surface is defined to be the infimum of all the distances between the strain point considered and arbitrary points on the surface). Since $ABCD$ is supposed to be a small cycle, the inequality

$$\delta_{\mathsf{G}} \geq x \tag{5.1}$$

applies. The case of vanishing elastic range is equivalent to

$$\delta_{\mathsf{G}} = 0. \tag{5.2}$$

Indeed, in view of the definition for δ_{G}, (5.2) implies that the yield surface shrinks to a point (it does not degenerate to some line segment).

Clearly, taking into account (5.1) and that $x \geq 0$, Eq. (5.2) implies

$$x = 0\,. \tag{5.3}$$

This result states that, in the case of vanishing elastic range, there are no small cycles, apart from the trivial cycle $E = E^{A} = E^{B} = E^{C} = E^{D}$. That is, concerning materials with vanishing elastic range, only trivial cycles satisfy the conditions for small cycles, and hence, (4.3) is satisfied trivially.

As a consequence, now we obtain compatibility between the simple endochronic theory of plasticity and the version (4.3) of Il'iushin's postulate. To see this, it is necessary to transform the yield condition (2.5) – (2.7),

$$\bar{\mathsf{F}}(\mathbf{T}, \boldsymbol{\xi}) = \sqrt{\frac{3}{2}(\mathbf{T} - \boldsymbol{\xi})^{D} \cdot (\mathbf{T} - \boldsymbol{\xi})^{D} + \alpha(\mathrm{tr}\,(\mathbf{T} - \boldsymbol{\xi}))^{2}} - k = 0 \tag{5.4}$$

into a strain space formulation. We denote by $\boldsymbol{\Delta}_{\mathrm{e}}$ the elastic strain with the property

$$\boldsymbol{\xi} = [\mathbf{T}]_{\mathbf{E}_{\mathrm{e}} = \boldsymbol{\Delta}_{\mathrm{e}}} = 2\mu\boldsymbol{\Delta}_{\mathrm{e}} + \lambda(\mathrm{tr}\,\boldsymbol{\Delta}_{\mathrm{e}})\mathbf{1}\,. \tag{5.5}$$

Using this relation, as well as the elasticity law (2.2), and the decomposition of strain (2.1), we may rewrite (5.4) as

$$\bar{\mathsf{G}}(\mathbf{E}, \boldsymbol{\Delta}) = \sqrt{\frac{3}{2}(\mathbf{E}-\boldsymbol{\Delta})^D \cdot (\mathbf{E}-\boldsymbol{\Delta})^D + \alpha\left(\frac{2\mu+3\lambda}{2\mu}\right) 2(\operatorname{tr}(\mathbf{E}-\boldsymbol{\Delta}))^2} - \frac{k}{2\mu} = 0, \tag{5.6}$$

(see Fig.2), where

$$\boldsymbol{\Delta} = \boldsymbol{\Delta}_{\mathrm{e}} + \mathbf{E}_{\mathrm{p}}\,. \tag{5.7}$$

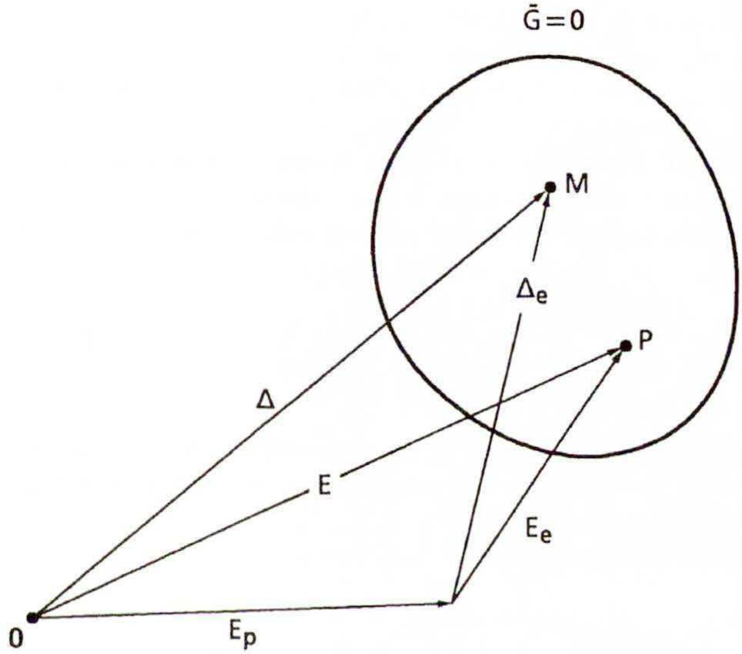

FIG. 2. Schematic representation of the yield surface in strain space: M denotes the "center" of the yield surface while P denotes an arbitrary strain state in the elastic range.

Equation (5.6) is a representation of the yield surface in strain space formulation. Evidently, if $k = 0$, then $\mathbf{E} = \boldsymbol{\Delta}$, and the yield surface shrinks to a point. Consequently, Il'iushin's postulate in the form (4.3) is satisfied trivially. Of course, this also applies to the case where, in addition,

$$\mu,\ \lambda \to \infty\,, \tag{5.8}$$

which in turn leads to the simple endochronic theory of plasticity, as shown in Sec. 2.2.

Appendix A

There are many investigations concerning integral inequalities like that obtained by Il'iushin's postulate as e.g. the works of IL'IUSHIN [16], PALMER, MAIER and DRUCKER [24], HILL [13, 14], HILL and RICE [15], MARTIN [21, Sec. 2.4, 2.5], DAFALIAS [7], PALGEN and DRUCKER [23], CASEY and TSENG [3], LUBLINER [19], CARROLL [4], LIN and NAGHDI [18], LUCCHESI and SILHAVY [20], HAVNER [12, Sec. 3.6], FOSDICK and VOLKMANN [10]. Apart from technical details, all these investigations utilize a common method for deriving consequences from the integral inequalities. This is now briefly described for the particular case that the elasticity law is independent of the plastic deformations, but otherwise a general theory with plastic internal variables $\mathbf{q}$ is assumed to apply.

Suppose that the stress tensor $\mathbf{T}$ satisfies an elasticity law of the form $\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{E}_e)$, which possesses an inverse $\mathbf{E}_e = \widehat{\mathbf{E}}_e(\mathbf{T})$, so that $\mathbf{E} = \mathbf{E}_p + \widehat{\mathbf{E}}_e(\mathbf{T})$. We may use the last relation in Eq. (4.2) to obtain a representation of the yield surface in stress space: $\widetilde{\mathsf{F}}(\mathbf{T}, \mathbf{E}_p, \mathbf{q}) = 0$. Further, it can be shown that a necessary and sufficient condition for the validity of (4.3), the latter being considered for pure elastic processes, is the existence of a scalar-valued function $\Psi = \widehat{\Psi}_e(\mathbf{E}_e) + \Psi_p = \bar{\Psi}_e(\mathbf{E}, \mathbf{E}_p) + \Psi_p$ such that the potential relation

$$\text{(A.1)} \qquad \mathbf{T} = \widehat{\mathbf{T}}(\mathbf{E}_e) = \frac{\partial \widehat{\Psi}_e(\mathbf{E}_e)}{\partial \mathbf{E}_e} = \frac{\partial \bar{\Psi}_e(\mathbf{E}, \mathbf{E}_p)}{\partial \mathbf{E}} = -\frac{\partial \bar{\Psi}_e(\mathbf{E}, \mathbf{E}_p)}{\partial \mathbf{E}_p}$$

holds. Ψ_p may depend on $\mathbf{E}_p$ and $\mathbf{q}$. We assume that (A.1) applies during plastic loading as well, and consider a small strain cycle $ABCD$ as in Fig. 3, which is parameterized by time t. We denote by t_A, t_B, t_C, t_D, $(t_A < t_B < t_C < t_D)$, the times related to the points A, B, C, D, respectively.

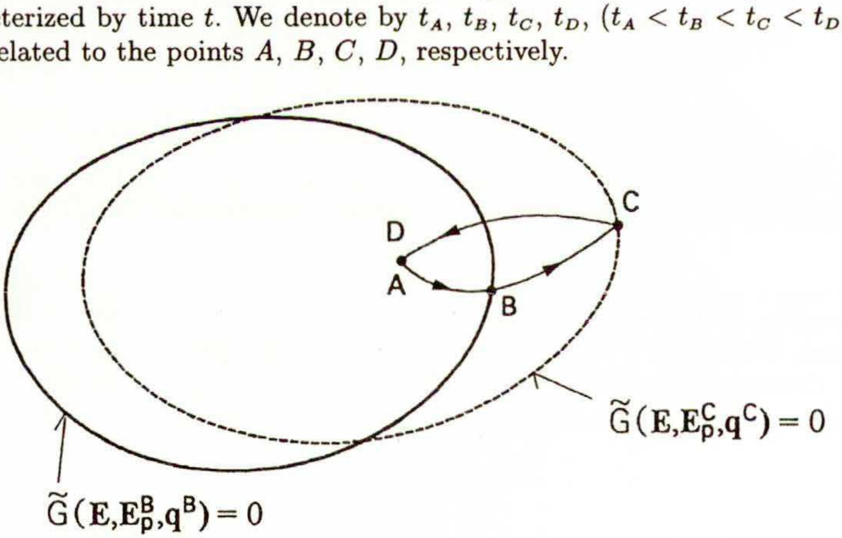

FIG. 3. During the small strain cycle $ABCD$ plastic flow occurs between B and C only.

The strain cycle begins and ends at $E = E^A = E^D$. Plastic flow occurs between B and C only. It follows from (4.3) that

$$(A.2)\qquad I(t_A,t_D) = \int_{t_A}^{t_D} \mathbf{T}(t)\cdot\dot{\mathbf{E}}(t)\,dt = \int_{t_A}^{t_D} \frac{\partial\bar{\Psi}_e(\mathbf{E}(t),\mathbf{E}_p(t))}{\partial\mathbf{E}(t)}\cdot\dot{\mathbf{E}}(t)\,dt$$

$$= \bar{\Psi}_e(\mathbf{E}^A,\mathbf{E}_p^C) - \bar{\Psi}_e(\mathbf{E}^A,\mathbf{E}_p^B) - \int_{t_B}^{t_C} \frac{\partial\bar{\Psi}_e(\mathbf{E}(t),\mathbf{E}_p(t))}{\partial\mathbf{E}(t)}\cdot\dot{\mathbf{E}}_p(t)\,dt$$

$$= \int_{t_B}^{t_C}\left[\frac{\partial\bar{\Psi}_e(\mathbf{E}^A,\mathbf{E}_p(t))}{\partial\mathbf{E}_p(t)} - \frac{\partial\bar{\Psi}_e(\mathbf{E}(t),\mathbf{E}_p(t))}{\partial\mathbf{E}(t)}\right]\cdot\dot{\mathbf{E}}_p(t)\,dt \geq 0.$$

We may apply Taylor's theorem to establish the relation

$$(A.3)\qquad \lim_{t_C\to t_B}\frac{I(t_A,t_D)}{(t_C-t_B)} = \left[\frac{\partial\bar{\Psi}_e(\mathbf{E}^A,\mathbf{E}_p(t))}{\partial\mathbf{E}_p(t)}\cdot\dot{\mathbf{E}}_p(t) - \frac{\partial\bar{\Psi}_e(\mathbf{E}(t),\mathbf{E}_p(t))}{\partial\mathbf{E}(t)}\cdot\dot{\mathbf{E}}_p(t)\right]_{t=t_B} \geq 0.$$

Thus, dropping t_B in (A.3), we obtain as a necessary condition for (4.3) the inequality

$$(A.4)\qquad -\frac{\partial\bar{\Psi}_e(\mathbf{E},\mathbf{E}_p)}{\partial\mathbf{E}_p}\cdot\dot{\mathbf{E}}_p \geq -\frac{\partial\bar{\Psi}_e(\mathbf{E}^A,\mathbf{E}_p)}{\partial\mathbf{E}_p}\cdot\dot{\mathbf{E}}_p\,,$$

with $\mathbf{E}$ denoting a strain state on the yield surface $\tilde{\mathsf{G}} = 0$, $\mathbf{E}_p$ the corresponding plastic strain, and $\mathbf{E}^A$ a strain state on or inside the yield surface $\tilde{\mathsf{G}} = 0$. Conversely, (A.4) constitutes a sufficient condition for (4.3). To see this, we take the integral of (A.4) along a strain cycle as shown in Fig. 3. For (A.4) to remain valid during this strain cycle, $\mathbf{E}^A$ must always lie in the intersection of all the elastic ranges during the strain cycle, which in turn implies that the strain cycle ABCD must be small. Then, following the steps similar to those used in (A.2), it is a straightforward matter to arrive at (4.3). Note that, with equation (A.1) in mind, we may introduce the definition

$$(A.5)\qquad \mathbf{T}^A = -\frac{\partial\bar{\Psi}_e(\mathbf{E}^A,\mathbf{E}_p)}{\partial\mathbf{E}_p}\,,$$

where $\mathbf{T}^A$ is a stress tensor on or inside the yield surface in stress space. Then, inequality (A.4) is equivalent to

$$(A.6)\qquad \mathbf{T}\cdot\dot{\mathbf{E}}_p \geq \mathbf{T}^A\cdot\dot{\mathbf{E}}_p\,,$$

where $\mathbf{T}$ is a stress tensor on the yield surface in stress space $\widetilde{\mathsf{F}} = 0$. Inequality (A.6) is known as the principle of maximum plastic dissipation and is equivalent to the following two conditions (for a proof of this fact see e.g. LUBLINER [19, Ch. 3.2.2]):

i) the yield surface $\widetilde{\mathsf{F}}(\mathbf{T}, \mathbf{E}_p, \mathbf{q}) = 0$ is convex, and

ii) $\dot{\mathbf{E}}_p$ is normal to $\widetilde{\mathsf{F}}(\mathbf{T}, \mathbf{E}_p, \mathbf{q}) = 0$ (normality rule).

Apart from notational differences, a large part of the analysis above follows closely that given by LUBLINER [19, Ch. 3.2]. Lubliner's analysis, however, deals with the strong form of Il'iushins's postulate and therefore (A.6) represents only a necessary condition for (4.3). Also, it must be mentioned that LUCCHESI and SILHAVY [20] have developed a finite plasticity theory which is based on the theory of materials with elastic range, and is formulated in a general mathematical framework. They discussed several interrelationships between inequalities of the work type, including the main results of the present Appendix.

References

1. P.J. ARMSTRONG and C.O. FREDERICK, *A mathematical representation of the multiaxial Bauschinger effect*, CEGB, Report RD/B/N 731, 1966.
2. J.F. BISHOP and R. HILL, *A theory of plastic distortion of a polycrystalline aggregate under combined stresses*, Philos. Mag., **42**, pp. 414–427, 1951.
3. J. CASEY and M. TSENG, *A constitutive restriction related to convexity of yield surfaces in plasticity*, J. Appl. Math. and Phys. (ZAMP), **35**, pp. 478–496, 1984.
4. M.M. CARROLL, *A rate-independent constitutive theory for finite inelastic deformation*, J. Appl. Mech., **54**, pp. 15–21, 1987.
5. J.L. CHABOCHE, K. DANG VAN and G. CORDIER, *Modelisation of the strain memory effect on the cyclic hardening of 316 stainless steel*, SMIRT-S, L11/3, Berlin 1979.
6. J.L. CHABOCHE, *Time-independent constitutive theories for cyclic plasticity*, Int. J. Plasticity, **2**, pp. 149–188, 1986.
7. Y.F. DAFALIAS, *Il'iushin's postulate and resulting thermodynamic conditions on elasto-plastic coupling*, Int. J. Solids Struct., **13**, pp. 239–251, 1977.
8. D.C. DRUCKER, *A more fundamental approach to plastic stress-strain relations*, Proc. of the first U.S. National Congress of Applied Mechanics, ASME, pp. 487–491, 1951.
9. D.C. DRUCKER, *On the postulate of stability of materials in the mechanics of continua*, J. Mécanique, **3**, pp. 235–249, 1964.
10. R. FOSDICK and E. VOLKMANN, *Normality and convexity of the yield surface in nonlinear plasticity*, Quart. Appl. Math., **LI**, pp. 117–127, 1993.
11. P. HAUPT, M. KAMLAH and CH. TSAKMAKIS, *Continuous representation of hardening properties in cyclic plasticity*, Int. J. Plasticity, **8**, pp. 803–817, 1993.
12. K.S. HAVNER, *Finite plastic deformation of crystalline solids*, Cambridge University Press, 1992.
13. R. HILL, *On constitutive inequalities for simple materials – I*, J. Mech. Phys. Solids, **16**, pp. 229–242, 1968.
14. R. HILL, *On constitutive inequalities for simple materials – II*, J. Mech. Phys. Solids, **16**, pp. 315–322, 1968.

15. R. HILL and J.R. RICE, *Elastic potentials and the structure of inelastic constitutive laws*, SIAM J. Appl. Math., **25**, pp. 448–461, 1973.

16. A.A. IL'IUSHIN, *On the postulate of plasticity*, PMM, **25**, pp. 746–752, 1961.

17. M. KORZEN, *Beschreibung des inelastischen Materialverhaltens im Rahmen der Kontinuumsmechanik: Vorschlag einer Materialgleichung vom viskoelastisch-plastischen Typ*, Dissertation, Technische Hochschule Darmstadt, Institut für Mechanik, 1988.

18. H.C. LIN and P.M. NAGHDI, *Necessary and sufficient conditions for the validity of a work inequality in finite plasticity*, Quart. J. Mech. Appl. Math., **42**, pp. 13–21, 1989.

19. J. LUBLINER, *Plasticity theory*, Macmillan Publishing Company, New York 1990.

20. M. LUCCHESI and M. SILHAVY, *Il'iushin's conditions in non-isothermal plasticity*, Arch. Rational Mech. Anal., **113**, pp. 121–163, 1991.

21. J.B. MARTIN, *Plasticity: Fundamentals and general results*, The MIT Press, Cambridge, Massachusetts and London, England 1975.

22. R.W. OGDEN, *Non-linear elastic deformations*, Ellis Hordwood Limited Publishers, England 1984.

23. L. PALGEN and D.C. DRUCKER, *The structure of stress-strain relations in finite elasto-plasticity*, Int. J. Solids Struct., **19**, pp. 519–531, 1983.

24. A.C. PALMER, G. MAIER and D.C. DRUCKER, *Normality relations and convexity of yield surfaces for unstable materials or structural elements*, Transactions of the ASME, J. Appl. Mech., **24**, pp. 464–470, 1967.

25. R.S. RIVLIN, *Some comments on the endochronic theory of plasticity*, Int. J. Solids Struct., **17**, pp. 231–248, 1981.

26. I.S. SANDLER, *On the uniqueness and stability of endochronic theories of material behavior*, Transactions of the ASME, J. Appl. Mech., **45**, pp. 263–266, 1978.

27. H.P. SHRIVASTAVA, Z. MROZ and R.N. DUBEY, *Yield criterion and the hardening rule for a plastic solid*, Z. Angew. Math. Mech., **53**, pp. 625–633, 1973.

28. K.C. VALANIS, *A theory of viscoplasticity without a yield surface. Part I. General theory*, Arch. Mech., **23**, pp. 517–533, 1971.

29. K.C. VALANIS and C.F. LEE, *Some recent developments in the endochronic theory with application to cyclic histories*, [in:] Nonlinear Constitutive Relations for High Temperature Applications, NASA Conference Publication, 2271, 1982.

30. C.-C. WANG and C. TRUESDELL, *Introduction to rational elasticity*, Noordhoff International Publishing Leyden, The Netherlands 1973.

31. M. ŻYCZKOWSKI, *Combined loadings in the theory of plasticity*, PWN-Polish Scientific Publishers, 1981.

FORSCHUNGSZENTRUM KARLSRUHE,
INSTITUT FÜR MATERIALFORSCHUNG II,
Postfach 3640, 76021 Karlsruhe, Germany.

Received May 20, 1996; new version December 30, 1996.