



## A new method of finding strong approximation to solutions to some IBVPs

Z. TUREK (WARSZAWA)

IT HAS BEEN PROVED that every solution to a 1D initial boundary value problem (IBVP) represented by a uniformly convergent series in some domain can be approximated by a Fourier cosine series. The new series is also uniformly convergent in that domain. The strong approximation to the heat-conduction problem subject to any boundary conditions with the application of the *Fourier cosine series* is found. It is the Fourier cosine series approximation to the exact solution to the problem under consideration. Its coefficients form an infinite set of ordinary differential equations (ISODE). Numerical results presented for heat conduction problems show – in comparison with solutions derived by the method of separation of variables – that relatively small number of terms of the Cosine Series approximate very well the exact solutions.

### 1. Introduction

IN [6] A NEW METHOD of finding approximate solutions to the heat conduction equation in one dimension subject to mixed boundary conditions has been presented. From the results obtained we could see that the solution to the problem derived by the *Fourier cosine series* approximated well the solution to the same problem derived by the method of separation of variables. The boundary conditions for the problem were not satisfied. Paper [7], however, applies the new method to a certain class of partial differential equations of engineering and physics subject to non-Dirichlet boundary conditions, considering the problem of boundary conditions as well. In that paper we solved two initial boundary-value problems with non-Dirichlet boundary conditions without solving the eigenvalue problems. The new approach was applied to the equation describing the heat conduction subject to non-Dirichlet boundary conditions and the vibrations of a rod also subject to non-Dirichlet boundary conditions. The numerical results showed that the new solutions also approximated well the solutions derived by the method of separation of variables. For the heat equation, however, even the boundary conditions at the initial instant of time were satisfied. Analysing the boundary conditions of the vibrating rod for a given initial displacement of the rod we came to the conditions on the Fourier cosine coefficients at  $t = 0$ . They were expressed as convergent series of the Fourier cosine coefficients mentioned above. They did not tend to zero which meant that the new method solution did not satisfy the prescribed boundary conditions even at  $t = 0$ . The classical method of solution did not satisfy the prescribed boundary conditions either since the initial condition for the problem did not satisfy the given boundary conditions.

The aim of the present paper is to give mathematical grounds to the new method. So it has first been proved that every solution to a one-dimensional IBVP represented by a uniformly convergent series in  $[0, L] \times [0, t_e)$  can be approximated by the Fourier cosine series which is uniformly convergent in that domain. It has been proved that the Fourier cosine series is a strong approximation to the problem under consideration, which is a solution to the so-called integro-differential-boundary equations (IDBE) [6, 7]. It has been found out that solutions to the corresponding ISODE form the Fourier cosine coefficients for the strong solution (*satisfying given equations and conditions*) of the heat-conduction problem (*this is why we call our approximation strong approximation*). In the paper we present the new approach to the heat conduction problem for all kinds of homogeneous boundary conditions. But the method can be applied to other boundary-value problems as well [5].

## 2. Fourier cosine series representation for a certain function of two variables

In this section we are going to show a Fourier cosine series representation for a function of two variables which is a sum of a uniformly convergent series. To this end we need some results concerning a Fourier cosine representation for a function of one variable.

LEMMA 1 [4]. Every function  $X$ , continuous in the interval  $[0, L]$  whose derivative  $X'$  is piecewise continuous in that interval can be expressed by a uniformly convergent series

$$X(x) = \frac{\gamma_0}{2} + \sum_{n=1}^{\infty} \gamma_n \cos \frac{n\pi x}{L},$$

in that interval, where

$$(2.1) \quad \gamma_n = \frac{2}{L} \int_0^L \cos \frac{n\pi x}{L} X(x) dx, \quad n = 0, 1, 2, \dots$$

Proof. Let  $\bar{X}$  be an even extension of  $X$ . Then  $\bar{X}$  is an even continuous function whose derivative  $\bar{X}'$  is piecewise continuous on  $[-L, L]$ . The Fourier coefficients for  $\bar{X}$  are

$$(2.2) \quad c_n = \frac{1}{L} \int_{-L}^L \bar{X}(x) \cos \frac{n\pi x}{L} dx = \frac{1}{\pi n} \left[ \bar{X}(x) \sin \frac{n\pi x}{L} \right]_{-L}^L - \frac{1}{\pi n} \int_{-L}^L \bar{X}'(x) \sin \frac{n\pi x}{L} dx = -\frac{1}{\pi n} \int_{-L}^L \bar{X}'(x) \sin \left( \frac{n\pi}{L} x \right) dx =: -\frac{L}{\pi n} \beta_n$$

and

$$b_n = \frac{1}{L} \int_{-L}^L \overline{X}(x) \sin \frac{n \pi x}{L} dx = 0,$$

as the function  $\overline{X}$  is even. The Fourier Coefficients  $\beta_n$  for the derivative  $\overline{X}'$  of the function  $\overline{X}$  exist since  $\overline{X}'$  is piecewise continuous and the series

$$(2.3) \quad \sum_{n=1}^{\infty} \beta_n^2$$

is convergent. Let us now consider the formula

$$\left( |\beta_n| - \frac{L}{n \pi} \right)^2 = \beta_n^2 - \frac{2L |\beta_n|}{n \pi} + \left( \frac{L}{n \pi} \right)^2 \geq 0,$$

from which we have

$$\frac{L |\beta_n|}{n \pi} \leq \frac{1}{2} \beta_n^2 + \frac{1}{2} \left( \frac{L}{n \pi} \right)^2 \quad (n = 1, 2, \dots).$$

The right-hand side of the above inequality consists of the elements of a convergent series. Then the series

$$\sum_{n=1}^{\infty} \frac{L |\beta_n|}{n \pi}$$

is convergent and so is the series

$$(2.4) \quad \sum_{n=1}^{\infty} |c_n|.$$

As the series in (2.4) is convergent, then the series

$$\frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos \frac{n \pi x}{L}$$

is absolutely and uniformly convergent and its sum is  $\overline{X}(x)$  [1]. But the function  $\overline{X}$  is an even extension of  $X$  and coincides with  $X$  in the fundamental interval, then

$$c_n = \frac{1}{L} \int_{-L}^L \overline{X}(x) \cos \frac{n \pi x}{L} dx = \frac{2}{L} \int_0^L X(x) \cos \frac{n \pi x}{L} dx = \gamma_n.$$

So the Lemma has been proved. □

Now we will use a known result [1] concerning the uniform convergence of an infinite series whose terms are products of functions of a certain class. This results in verifying formal solutions to boundary value problems represented in the following form:

$$(2.5) \quad S(x, t) = \sum_{i=1}^{\infty} X_i(x)T_i(t).$$

**Abel's test for uniform convergence** [1]

The series (2.5) converges uniformly with respect to the two variables  $x$  and  $t$  together in a region  $D$  of the  $x - t$  plane provided that

a) the series

$$\sum_{i=1}^{\infty} X_i(x)$$

converges uniformly with respect to  $x$  for all  $x$  such that  $(x, t)$  is in  $D$ , and

b) the functions  $T_i$  are uniformly bounded and monotonic with respect to  $i$  ( $i = 1, 2, \dots$ ) for all  $t$  such that  $(x, t)$  is in  $D$ .  $\square$

In establishing the way for the Fourier cosine series representation of the series in (2.5) we will mostly depend on the important fact stated in the following lemma which is an extension of Lemma 1 for a function with a parameter  $t$ .

**LEMMA 2.** Any continuous function

$$u(\cdot, t) : [0, L] \rightarrow \mathcal{R}$$

piecewise  $C^1$  in  $[0, L]$  for every  $t \in [0, t_e)$  can be represented by the Fourier cosine series

$$(2.6) \quad \frac{c_0(t)}{2} + \sum_{n=1}^{\infty} c_n(t) \cos \frac{n \pi x}{L},$$

that is uniformly convergent in  $[0, L]$  for all  $t \in [0, t_e)$  with coefficients

$$(2.7) \quad c_n(t) = \frac{2}{L} \int_0^L u(x, t) \cos \frac{n \pi x}{L} dx.$$

The proof of this fact is exactly the same as that of Lemma 1.  $\square$

The fundamental fact leading to the Fourier cosine representation for the series (2.5) is expressed by the following theorem.

**THEOREM 1.** Let  $X_i$  satisfy, for each  $i = 0, 1, 2, \dots$  the conditions stated in Lemma 1 and additionally let  $X_i$  and  $T_i$  satisfy for each  $i = 0, 1, 2, \dots$  the conditions of the Abel's test for uniform convergence.

If the sum  $S(x, t)$  of the series (2.5) satisfies the conditions stated in Lemma 2 in  $D := [0, L] \times [0, t_e)$ , then the series

$$(2.8) \quad \frac{c_0(t)}{2} + \sum_{n=1}^{\infty} c_n(t) \cos \frac{n \pi x}{L},$$

with the coefficients

$$(2.9) \quad c_n(t) = \sum_{i=1}^{\infty} \gamma_{ni} T_i(t) \quad n = 0, 1, 2, \dots,$$

where  $\gamma_{ni}$  are defined by (2.1) for  $X \equiv X_i$ , converges uniformly to the sum  $S(x, t)$  of the series (2.5) in  $D$ .

**P r o o f.** Notice that

$$c_n(t) = \sum_{i=1}^{\infty} \gamma_{ni} T_i(t) = \frac{2}{L} \sum_{i=1}^{\infty} \int_0^L \cos \frac{n \pi x}{L} X_i(x) T_i(t) dx.$$

Using the theorem on integration term by term [2] we come to

$$c_n(t) = \frac{2}{L} \int_0^L \cos \frac{n \pi x}{L} \sum_{i=1}^{\infty} X_i(x) T_i(t) dx = \frac{2}{L} \int_0^L \cos \frac{n \pi x}{L} S(x, t) dx,$$

for  $n = 0, 1, 2, \dots$ . These coefficients are Fourier cosine series coefficients for the sum  $S(x, t)$  of the series (2.5). So the series (2.8) is the Fourier cosine series for the sum  $S(x, t)$  which satisfies the conditions of Lemma 2, and that ends the proof of the theorem.  $\square$

### 3. A new approach to strong approximation to a solution to the heat conduction equation

From Theorem 1 we know that every series  $\sum_{i=1}^{\infty} X_i(x) T_i(t)$  uniformly convergent in  $[0, L] \times [0, t_e)$  whose sum  $S(x, t)$  satisfies the conditions of Lemma 2, can be represented by a Fourier cosine series that is uniformly convergent in that domain. This means that every solution  $u(x, t)$  to an IBVP represented by a series satisfying the conditions mentioned above can be expanded in the cosine series

$$(3.1) \quad u(x, t) = \frac{c_0(t)}{2} + \sum_{n=1}^{\infty} c_n(t) \cos \frac{n \pi x}{L}$$

with

$$(3.2) \quad c_n(t) = \sum_{i=1}^{\infty} \gamma_{ni} T_i(t),$$

where

$$\gamma_{ni} = \frac{2}{L} \int_0^L X_i(x) \cos \frac{n\pi x}{L} dx$$

with  $X_i(x)$  being eigenfunctions of the problem under consideration and  $T_i(t)$  satisfying the corresponding uncoupled infinite set of ordinary differential equations.

The above representation will have a practical application if we find a method of calculating Fourier coefficients other than those presented by (3.2). Such a method exists, however, and has been described in [5, 6, 7]. The method leads first to the IDBE and then to the ISODE for the problem. The corresponding IDBE and ISODE are derived for each boundary-value problem separately. In the present paper we are going to demonstrate the approach for the heat conduction problem.

### 3.1. IDBE and ISODE for the heat conduction problem with non-Dirichlet boundary conditions

Let us consider the equation

$$(3.3) \quad \frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} = 0 \quad \text{for } (x, t) \in (0, L) \times (0, t_e),$$

subject to the boundary conditions

$$(3.4) \quad \begin{aligned} \frac{\partial U}{\partial x} - hU &= 0 & \text{for } x = 0, \\ \frac{\partial U}{\partial x} + gU &= 0 & \text{for } x = L, \end{aligned}$$

for all  $t \in [0, t_e]$ , and the initial condition

$$(3.5) \quad U(x, 0) = U_0(x),$$

for all  $x \in [0, L]$ , where  $h$  and  $g$  are constants. The IDBE for the problem is the following one [6, 7]:

$$(3.6) \quad \begin{aligned} \frac{d}{dt} \int_0^L U(x, t) \cos \frac{n\pi x}{L} dx + \alpha_n^2 \int_0^L U(x, t) \cos \frac{n\pi x}{L} dx &= F_n, \\ F_n = -hU(0, t) - gU(L, t)(-1)^n, \quad \alpha_n = \pi n/L, \quad n = 0, 1, 2, \dots \end{aligned}$$

Before we find the ISODE let us introduce a notion

DEFINITION. *A function*

$$U(\cdot, \cdot) : [0, L] \times [0, t_e) \rightarrow \mathcal{R}$$

is a strong approximation to a solution to the initial boundary-value problem (3.3) – (3.5) if it satisfies the IDBE in (3.6), i.e.  $U$  satisfies almost everywhere the following infinite set of integro-differential boundary equations

$$(3.7) \quad \frac{d}{dt} \int_0^L U(x, t) \cos \frac{n \pi x}{L} dx + \alpha_n^2 \int_0^L U(x, t) \cos \frac{n \pi x}{L} dx = F_n, \\ F_n = -h U(0, t) - g U(L, t)(-1)^n$$

for  $n = 0, 1, 2, \dots$ . □

The corresponding ISODE for the problem in (3.3)–(3.5) appears in the following lemma which we give without a proof.

LEMMA 3. Any function  $U(\cdot, \cdot) : [0, L] \times [0, t_e) \rightarrow \mathcal{R}$  satisfying the conditions stated in Lemma 2 is a strong approximation to a solution to the IBV problem (3.3)–(3.5) and can be represented by the series

$$(3.8) \quad U(x, t) = \frac{c_0(t)}{2} + \sum_{n=1}^{\infty} c_n(t) \cos \frac{n \pi x}{L}$$

with  $c_n$  computed from the so-called ISODE for the problem (3.3)–(3.5) in the following form

$$(3.9) \quad \dot{c}_n + \alpha_n^2 c_n = \frac{2}{L} G_n, \\ G_n := -\frac{c_0}{2} [h + g(-1)^n] - \sum_{k=1}^{\infty} c_k [h + g(-1)^{k+n}], \\ c_n(0) = \frac{2}{L} \int_0^L U_0(x) \cos \frac{n \pi x}{L} dx,$$

for  $n = 0, 1, 2, \dots$  and  $G_n$  were derived from  $F_n$  using the boundary conditions in (3.4) □

Solutions to the ISODE in (3.9) follow from the following theorem.

THEOREM 2. *If the series*

$$(3.10) \quad U(x, t) = \sum_{i=1}^{\infty} X_i(x) T_i(t)$$

is a solution to (3.3) – (3.5) derived by the method of separation of variables, then

$$(3.11) \quad c_n(t) = \sum_{i=1}^{\infty} \gamma_{ni} T_i(t), \quad n = 0, 1, 2, \dots$$

with

$$\gamma_{ni} = \frac{2}{L} \int_0^L \cos \frac{n \pi x}{L} X_i(x) dx$$

being solutions to the ISODE (3.9).

**P r o o f.** We have to consider two cases;  $n \neq 0$  and  $n = 0$ .

In the first case (for  $n \neq 0$ ) let us consider the second term at the left-hand side of (3.9)<sub>1</sub>

$$(3.12) \quad \alpha_n^2 c_n = \alpha_n^2 \sum_{i=1}^{\infty} \gamma_{ni} T_i = \sum_{i=1}^{\infty} \alpha_n^2 \gamma_{ni} T_i.$$

Using now the expression for  $\gamma_{ni}$  and integrating twice by parts we get for each term in (3.12) the following expression

$$(3.13) \quad \begin{aligned} \alpha_n^2 \gamma_{ni} T_i &= T_i \alpha_n^2 \frac{2}{L} \int_0^L \cos \frac{n \pi x}{L} X_i(x) dx \\ &= -\frac{2}{L} [g X_i(L) T_i (-1)^n + h X_i(0) T_i] - \dot{T}_i \gamma_{ni}. \end{aligned}$$

In the above formula we have used the boundary conditions (3.4) and the fact that  $X_i'' + \omega_i^2 X_i = 0$  and  $\dot{T}_i + \omega_i^2 T_i = 0$ , where  $\omega_i$  is the  $i$ -th eigenvalue for the problem (3.3)–(3.5).

As the terms of (3.13) contain terms of the series (3.10), we can add up both sides of (3.13), to get

$$\alpha_n^2 \sum_{i=1}^{\infty} \gamma_{ni} T_i = -\frac{2}{L} \left[ g(-1)^n \sum_{i=1}^{\infty} X_i(L) T_i + h \sum_{i=1}^{\infty} X_i(0) T_i \right] - \sum_{i=1}^{\infty} \dot{T}_i \gamma_{ni},$$

or using the fact that

$$\sum_{i=1}^{\infty} X_i(L) T_i(t) = U(L, t), \quad \sum_{i=1}^{\infty} X_i(0) T_i(t) = U(0, t), \quad \sum_{i=1}^{\infty} \dot{T}_i(t) \gamma_{ni} = \dot{c}_n$$

we have

$$\alpha_n^2 \sum_{i=1}^{\infty} \gamma_{ni} T_i = -\frac{2}{L} [gU(L, t)(-1)^n + hU(0, t)] - \dot{c}_n.$$



Now we can exploit the above formula at the left-hand side of (3.9)<sub>1</sub>

$$\begin{aligned} \dot{c}_n + \alpha_n^2 c_n &= \dot{c}_n + \alpha_n^2 \sum_{i=1}^{\infty} \gamma_{ni} T_i = \dot{c}_n - \frac{2}{L} [gU(L, t)(-1)^n + hU(0, t)] - \dot{c}_n \\ &= -\frac{2}{L} [gU(L, t)(-1)^n + hU(0, t)] = \frac{2}{L} G_n \end{aligned}$$

and that ends the proof for  $n \neq 0$ .

In the second case (for  $n = 0$ ), Eq.(3.9)<sub>1</sub> has the form

$$\dot{c}_0 = \frac{2}{L} G_0.$$

Consider the left-hand side of the above equation. Proceeding in the same way as for  $n \neq 0$  we get

$$\begin{aligned} \mathcal{L} \equiv \dot{c}_0 &= \sum_{i=1}^{\infty} \gamma_{0i} \dot{T}_i = -\frac{2}{L} \sum_{i=1}^{\infty} T_i(t) \int_0^L \omega_i^2 X_i(x) dx \\ &= \frac{2}{L} \sum_{i=1}^{\infty} T_i(t) \int_0^L X_i''(x) dx = -\frac{2}{L} \sum_{i=1}^{\infty} T_i(t) [gX_i(L) + hX_i(0)] \\ &= -\frac{2}{L} [gU(L, t) + hU(0, t)] = \frac{2}{L} G_0 \end{aligned}$$

and that ends the proof for  $n = 0$ .

We also have to prove the initial condition (3.9)<sub>3</sub> to be true. Consider Eq.(3.11) for  $t = 0$ . Then

$$\begin{aligned} c_n(0) &= \frac{2}{L} \sum_{i=1}^{\infty} \int_0^L \cos \frac{n\pi x}{L} X_i(x) T_i(0) dx \\ &= \frac{2}{L} \int_0^L \cos \frac{n\pi x}{L} \sum_{i=1}^{\infty} X_i(x) T_i(0) dx = \frac{2}{L} \int_0^L U_0(x) \cos \frac{n\pi x}{L} dx \end{aligned}$$

which agrees with (3.9)<sub>3</sub>. □

### 3.2. ISODE for the heat conduction problem with any boundary conditions

This time we consider the heat conduction equation

$$(3.14) \quad \frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} = 0 \quad \text{for } (x, t) \in (0, L) \times (0, t_e),$$

subject to the boundary conditions

$$(3.15) \quad \begin{aligned} \beta \frac{\partial U}{\partial x} + \alpha U &= 0 & \text{for } x = 0, \\ \delta \frac{\partial U}{\partial x} + \gamma U &= 0 & \text{for } x = L, \end{aligned}$$

for all  $t \in [0, t_e]$ , and the initial condition

$$(3.16) \quad U(x, 0) = U_0(x),$$

for all  $x \in [0, L]$  where  $\alpha, \beta, \gamma, \delta$  are constants satisfying the conditions

$$\alpha^2 + \beta^2 \neq 0, \quad \gamma^2 + \delta^2 \neq 0.$$

This time the boundary conditions in (3.15) describe also the Dirichlet boundary conditions for the heat equation (3.14).

The IDBE for the heat conduction equation with these boundary conditions are the following:

$$(3.17) \quad \begin{aligned} \frac{d}{dt} \int_0^L U(x, t) \cos \frac{n\pi x}{L} dx + \alpha_n^2 \int_0^L U(x, t) \cos \frac{n\pi x}{L} dx &= Z_n, \\ Z_n &:= \frac{\partial U}{\partial x}(x, t) \cos \frac{n\pi x}{L} \Big|_0^L, \quad n = 0, 1, 2, \dots \end{aligned}$$

Although the functions  $Z_n$  in (3.17) are defined by the boundary values of spatial derivative of the function  $U(x, t)$  and, in general, case cannot be expressed by the given boundary conditions, we can also find an infinite set of ordinary differential equations for the coefficients  $c_n$ . In this case the ISODE is in the form

$$(3.18) \quad \begin{aligned} \dot{c}_n + \alpha_n^2 c_n &= \frac{2}{L} H_n(t), \\ c_n(0) &= \frac{2}{L} \int_0^L U_0(x) \cos \frac{n\pi x}{L} dx, \quad n = 0, 1, 2, \dots, \end{aligned}$$

where after exploiting the cosine series representation for the first derivative [5] in the function  $Z_n$ , we get  $H_n$  instead of  $Z_n$

$$(3.19) \quad H_n(t) = \frac{c_0^{(1)}}{2} [(-1)^n - 1] + \sum_{k=1}^{\infty} c_k^{(1)}(t) [(-1)^{k+n} - 1]$$

with

$$\begin{aligned}
 (3.20) \quad c_k^{(1)}(t) &= \frac{2}{L} \sum_{j=1}^{\infty} c_j(t) \left[ (-1)^{j+k} - \eta_{jk} - 1 \right], \\
 \eta_{jk} &= -\alpha_k \int_0^L \cos \frac{j \pi x}{L} \sin \frac{k \pi x}{L} dx.
 \end{aligned}$$

We can also prove that  $c_n$  are Fourier cosine coefficients for the function  $U$  which is the Fourier cosine approximation to a strong solution to (3.14), (3.15). Theorem 2 is also valid for this case.

#### 4. Some applications of the new approach

In [6] we have solved the heat conduction problem for mixed boundary conditions (3.4). In [7] we have solved two IBVPs with non-Dirichlet type boundary conditions for the heat equation and for the wave equation. Now we solve the heat conduction problem for other boundary conditions (*including also Dirichlet ones*) using the new method. Generally we solve

$$(4.1) \quad \frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} = 0 \quad \text{for } (x, t) \in (0, L) \times (0, t_e),$$

subject to the boundary conditions

$$\begin{aligned}
 (4.2) \quad \beta \frac{\partial U}{\partial x} + \alpha U &= 0 \quad \text{for } x = 0, \\
 \delta \frac{\partial U}{\partial x} + \gamma U &= 0 \quad \text{for } x = L,
 \end{aligned}$$

for all  $t \in [0, t_e]$  and initial condition

$$(4.3) \quad U(x, 0) = U_0(x),$$

for all  $x \in [0, L]$  where

$$\alpha^2 + \beta^2 \neq 0, \quad \gamma^2 + \delta^2 \neq 0.$$

In further calculations  $U_0(x) = 1 + \sin[2\pi(x - L/4)/L]$ ,  $L = 1$  for all the examples and  $Bi = 0.185$ . The corresponding approximate solutions

$$(4.4) \quad U_a(x, t) = \frac{c_0(t)}{2} + \sum_{k=1}^{N_a} c_k(t) \cos \frac{k \pi x}{L}$$

for  $N_a$  terms of the series (4.4) we compare to the corresponding classical solutions of the problem (4.1)–(4.3),

$$(4.5) \quad U_c(x, t) = \sum_{k=1}^{\infty} a_k \exp(-\omega_k^2 t) \psi_k(x),$$

for  $N_c$  terms of the series (4.5), where

$$a_k := \int_0^L U_0(x) \psi_k(x) dx / \|\psi_k(x)\|^2;$$

$\psi_k(x)$  and  $\omega_k$  are eigenfunctions and eigenvalues, respectively, calculated from a corresponding IBVP. We compute the corresponding ISODE using the Runge–Kutta method.

The form of the ISODE depends on the type of boundary conditions involved. In the case of *non-Dirichlet boundary conditons* (i.e.  $\beta\delta \neq 0$ ), the function  $Z_n$  (3.17) can be expressed by the boundary values of the function  $U$  itself (e.g. (3.6)) and consequently, by a single series in terms of the coefficients of  $U$  (e.g. (3.9)). The simplest case in this class is when  $\alpha^2 + \gamma^2 = 0$  where the new method solution agrees with the classical one. In this case the functions  $\cos[(n\pi x)/L]$  are eigenfunctions of the heat equation and the ISODE reduces to the infinite uncoupled set of ordinary differential equations for the time components of the Fourier series known from the method of separation of variables. Other examples are presented in Figs. 1–2.

$$\begin{aligned} \frac{\partial U}{\partial x} - \text{Bi}U &= 0 & \text{for } x = 0, & & \frac{\partial U}{\partial x} &= 0 & \text{for } x = 0, \\ \frac{\partial U}{\partial x} &= 0 & \text{for } x = L, & & \frac{\partial U}{\partial x} + \text{Bi}U &= 0 & \text{for } x = L. \end{aligned}$$

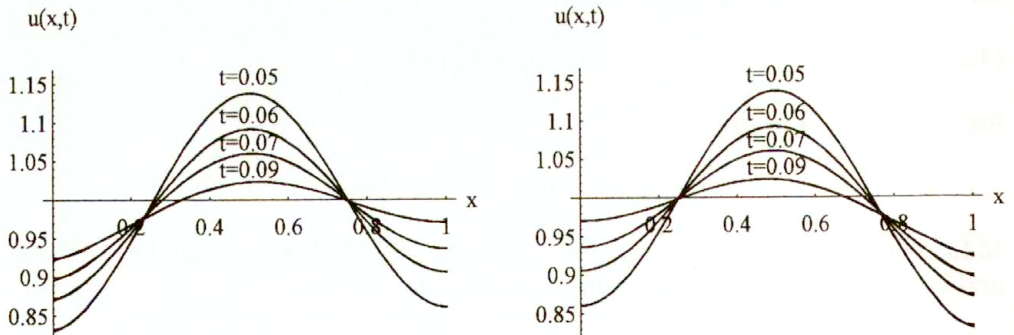


FIG. 1. Temperature field for some values of  $t$  for  $N_a = 10$  due to the new solution and for  $N_c = 5$  due to the classical solution (they cannot be distinguished).

$$\frac{\partial U}{\partial x} - Bi U = 0 \quad \text{for } x = 0,$$

$$\frac{\partial U}{\partial x} + Bi U = 0 \quad \text{for } x = L.$$

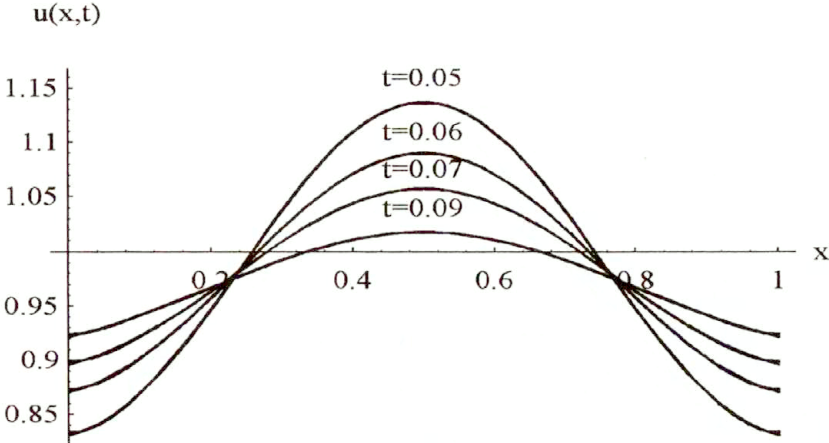


FIG. 2. Temperature field for some values of  $t$  for  $N_a = 10$  due to the new solution and for  $N_c = 5$  due to the classical solution (they cannot be distinguished).

$$U = 0 \quad \text{for } x = 0,$$

$$\frac{\partial U}{\partial x} + Bi U = 0 \quad \text{for } x = L.$$

$$\frac{\partial U}{\partial x} - Bi U = 0 \quad \text{for } x = 0,$$

$$U = 0 \quad \text{for } x = L.$$

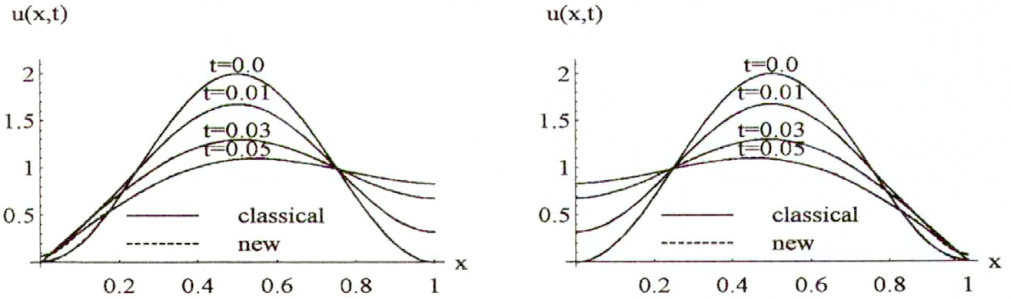


FIG. 3. Temperature field for some values of  $t$  for  $N_a = 15$  due to the new solution and for  $N_c = 10$  due to the classical solution.

For *Dirichlet-type boundary conditions* (i.e.  $\beta\delta = 0$ ), the function  $Z_n$  in (3.17) is expressed by a double series (3.19). In this class we consider three examples of boundary conditions. For each IBVP we solve the corresponding ISODE and numerical results are drawn in Figs. 3–4.

From the figures presented it will be seen how closely the new solutions approach the classical solutions right through the interval.

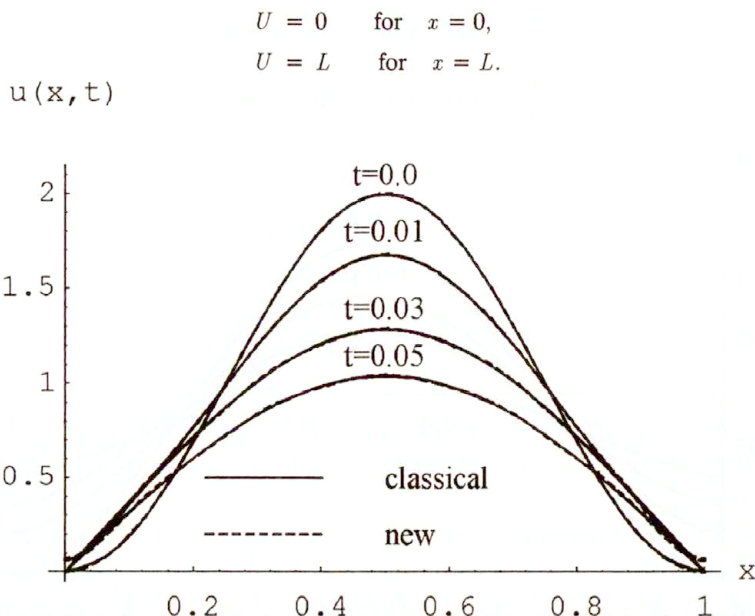


FIG. 4. Temperature field for some values of  $t$  for  $N_a = 15$  due to the new solution and for  $N_c = 10$  due to the classical solution.

## 5. Conclusions

From the mathematical considerations presented in the paper we conclude that the Fourier cosine series can be applied to many initial-boundary value problems without solving eigen-value problems. Computing relatively small number of terms of the cosine series, the new solutions called strong approximations approach very closely the exact solutions right through the interval. Since the new solutions are Fourier cosine series approximations to the exact solutions of such problems, the boundary conditions in general cannot be satisfied.

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02-352 Warszawa, Szczęśliwicka 2/26.

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