On attainability of Hashin-Shtrikman bounds
by iterative hexagonal layering
Plane elasticity problems

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THE PAPER presents a derivation of the Hashin–Shtrikman bounds for the plane elasticity problems and for the Kirchhoff problem of plates in bending. A two-dimensional counterpart of the method of FRANCFORT–MURAT [1] is applied. The method consists of three subsequent layerings along the directions of vertices of a unilateral triangle.

1. Introduction

FRANCFORT AND MURAT [1] showed how to mix in space two isotropic elastic components to obtain the stiffest possible isotropic material. The aim of the present paper is twofold. First we consider this problem in a two-dimensional setting, which means that both plane-stress and plane-strain elasticity problems are comprised. Secondly we deal with the plate bending problem of Kirchhoff.

FRANCFORT and MURAT [1] performed subsequent layerings in the directions of vertices of a regular icosahedron and arrived at an isotropic material of extremal properties. The present paper shows details of a similar but plane mixing process, using layerings in the directions of vertices of a unilateral triangle.

The Hashin – Shtrikman bounds for both the problems considered are similar due to the analogy between two-dimensional elasticity and plate bending problems. This paper shows the details of how to perform the passage between the bounds. It turns out that such a passage is feasible, but requires a change of the assumptions of ordering of the material phases, which makes it non-intuitive. Thus the independent derivations of these bounds, like that presented here for both the problems, seem indispensable, whether the analogy recalled applies here or not.

2. Plane elasticity problem. Laminate of first rank composed of two isotropic materials

Let us denote by $\mathbf{A}$ the tensor of elastic moduli for a plane-stress or plane-strain problem. We consider two isotropic materials of moduli $\mathbf{A}_\alpha$ ($\alpha = 1, 2$) and restrict our consideration to the ordered case: $\mathbf{A}_2 > \mathbf{A}_1$. These tensors are

determined by the pairs of moduli (k_α, μ_α) as follows

(2.1) $$\mathbf{A}_\alpha = 2k_\alpha \mathbf{a}_1 \otimes \mathbf{a}_1 + 2\mu_\alpha(\mathbf{a}_2 \otimes \mathbf{a}_2 + \mathbf{a}_3 \otimes \mathbf{a}_3),$$

where

(2.2) $$\begin{aligned}
\mathbf{a}_1 &= \frac{1}{\sqrt{2}}(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2), \\
\mathbf{a}_2 &= \frac{1}{\sqrt{2}}(\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2), \\
\mathbf{a}_3 &= \frac{1}{\sqrt{2}}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1),
\end{aligned}$$

and $(\mathbf{e}_1, \mathbf{e}_2)$ are versors of the orthogonal Cartesian coordinate system.

Let us stack both materials to make an in-plane laminate directed along versor $\mathbf{n}$ with area fractions θ_α. The tensor of effective moduli of the laminate is given by the formula of FRANCFORT and MURAT [1], truncated here to the plane problem, see also KOHN [3]

(2.3) $$\theta_1(\mathbf{A}_2 - \mathbf{A}_h)^{-1} = (\mathbf{A}_2 - \mathbf{A}_1)^{-1} - \frac{\theta_2}{\mu_2}\,\boldsymbol{\Psi}^{(\mathbf{n})},$$

where

(2.4) $$\boldsymbol{\Psi}^{(\mathbf{n})} = \Psi^{\mathbf{n}}_{\alpha\beta\gamma\delta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta \otimes \mathbf{e}_\gamma \otimes \mathbf{e}_\delta,$$

(2.5) $$\Psi^{\mathbf{n}}_{\alpha\beta\gamma\delta} = \frac{1}{4}\left(\delta_{\beta\gamma}n_\alpha n_\delta + \delta_{\beta\delta}n_\alpha n_\gamma + \delta_{\alpha\gamma}n_\beta n_\delta + \delta_{\alpha\delta}n_\beta n_\gamma\right) - \frac{k_2}{k_2 + \mu_2}n_\alpha n_\beta n_\delta n_\gamma$$

with $n_\alpha = \mathbf{n} \cdot \mathbf{e}_\alpha$. The formula (2.3) holds true if $\mathbf{A}_1$ is non-isotropic, which will be utilised later.

3. Layering of second rank

Let us form the subsequent laminate by stacking the layers of material 2 with the first-rank laminate constructed in Sec. 2, along direction of versor $\mathbf{m} = (m_\alpha)$ with area fractions α_2 (material 2) and α_1 (composite material of moduli $\mathbf{A}_h$). Since Eq. (2.3) holds even if the first material is anisotropic, we apply this formula to arrive at the following implicit formula for the effective stiffness tensor $\mathbf{A}_{hh}$:

(3.1) $$\alpha_1(\mathbf{A}_2 - \mathbf{A}_{hh})^{-1} = (\mathbf{A}_2 - \mathbf{A}_h)^{-1} - \frac{\alpha_2}{\mu_2}\,\boldsymbol{\Psi}^{(\mathbf{m})}.$$

By combining (3.1) and (2.3) one finds

(3.2) $$\alpha_1\theta_1(\mathbf{A}_2 - \mathbf{A}_{hh})^{-1} = (\mathbf{A}_2 - \mathbf{A}_1)^{-1} - \frac{\theta_2}{\mu_2}\,\boldsymbol{\Psi}^{(\mathbf{n})} - \frac{\alpha_2\theta_1}{\mu_2}\,\boldsymbol{\Psi}^{(\mathbf{m})}.$$

One can prove that $\mathbf{A}_2 > \mathbf{A}_{hh}$, hence the first term of Eq. (3.2) makes sense.

4. Laminate of third rank

Now we stack the material obtained by the second layering with material (2), in layers along versor $\mathbf{p}$, with area fractions β_1 (material of modulus $\mathbf{A}_{hh}$) and β_2 (material of modulus $\mathbf{A}_2$) to obtain a laminate of moduli $\mathbf{A}_{hhh}$. Again $\mathbf{A}_2 > \mathbf{A}_{hhh}$ and by applying (2.3) one finds

$$(4.1) \qquad \beta_1(\mathbf{A}_2 - \mathbf{A}_{hhh})^{-1} = (\mathbf{A}_2 - \mathbf{A}_{hh})^{-1} - \frac{\beta_2}{\mu_2}\boldsymbol{\Psi}^{(\mathbf{p})}.$$

On combining (4.1) and (3.2) one arrives at

$$(4.2) \qquad \alpha_1\theta_1\beta_1(\mathbf{A}_2 - \mathbf{A}_{hhh})^{-1} = (\mathbf{A}_2 - \mathbf{A}_1)^{-1} - \frac{1}{\mu_2}\boldsymbol{\Psi},$$

$$(4.3) \qquad \boldsymbol{\Psi} = \theta_2\boldsymbol{\Psi}^{(\mathbf{n})} + \theta_1\alpha_2\boldsymbol{\Psi}^{(\mathbf{m})} + \beta_2\alpha_1\theta_1\boldsymbol{\Psi}^{(\mathbf{p})}.$$

5. Hexagonal lamination

Let us choose

$$(5.1) \qquad \mathbf{n} = (1,0), \qquad \mathbf{m} = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \qquad \mathbf{p} = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right),$$

hence end points of versors $\mathbf{n}$, $\mathbf{m}$, $\mathbf{p}$ are vertices of a unilateral triangle. A direct computation yields

$$(5.2) \qquad \Psi^{(\mathbf{n})}_{1111} = 1 - b, \qquad \Psi^{(\mathbf{n})}_{1212} = \frac{1}{4}, \qquad \Psi^{(\mathbf{n})}_{\alpha\alpha\beta\beta} = 0,$$

$$\alpha = 2, \qquad \beta = 1 \text{ or } 2, \qquad b = \frac{k_2}{k_2 + \mu_2};$$

$$(5.3) \qquad
\begin{aligned}
&\Psi^{(\mathbf{m})}_{1111} = \frac{1}{4} - \frac{b}{16}, && \Psi^{(\mathbf{m})}_{2222} = \frac{3}{4} - \frac{9}{16}b, \\
&\Psi^{(\mathbf{m})}_{1122} = -\frac{3b}{16}, && \Psi^{(\mathbf{m})}_{1112} = -\frac{\sqrt{3}}{4}\left(\frac{1}{2} - \frac{b}{4}\right), \\
&\Psi^{(\mathbf{m})}_{2221} = -\frac{\sqrt{3}}{4}\left(\frac{1}{2} - \frac{3b}{4}\right), && \Psi^{(\mathbf{m})}_{1212} = \frac{1}{4} - \frac{3b}{16};
\end{aligned}$$

$$(5.4) \qquad
\begin{aligned}
&\Psi^{(\mathbf{p})}_{1111} = \frac{1}{4} - \frac{b}{16}, && \Psi^{(\mathbf{p})}_{1212} = \frac{1}{4} - \frac{3b}{16}, \\
&\Psi^{(\mathbf{p})}_{1122} = -\frac{3}{16}b, && \Psi^{(\mathbf{p})}_{2221} = \frac{\sqrt{3}}{4}\left(\frac{1}{2} - \frac{3}{4}b\right), \\
&\Psi^{(\mathbf{p})}_{1112} = \frac{\sqrt{3}}{8}\left(1 - \frac{b}{2}\right), && \Psi^{(\mathbf{p})}_{2222} = \frac{3}{4} - \frac{9}{16}b.
\end{aligned}$$

Now we stipulate that Ψ given by Eq. (4.3) is isotropic. Conditions $\Psi_{2221} = 0$, $\Psi_{1112} = 0$ yield $\beta_2 = \alpha_2/\alpha_1$. The condition $\Psi_{1111} = \Psi_{2222}$ implies $\alpha_2 = \theta_2/\theta_1$. Thus we have

(5.5)
$$\Psi_{1111} = \theta_2 \left(\frac{3}{2} - \frac{9}{8}b\right), \qquad \Psi_{2222} = \Psi_{1111},$$
$$\Psi_{1212} = \theta_2 \left(\frac{3}{4} - \frac{3}{8}b\right), \qquad \Psi_{1122} = -\frac{3}{8}b\theta_2$$

and finally we conclude that Ψ can be put in the form similar to (2.1)

(5.6) $\quad \Psi = \frac{3}{2}\theta_2 \left[(1 - b)\mathbf{a}_1 \otimes \mathbf{a}_1 + \left(1 - \frac{b}{2}\right)(\mathbf{a}_2 \otimes \mathbf{a}_2 + \mathbf{a}_3 \otimes \mathbf{a}_3)\right].$

Let us compute

(5.7) $\quad (\mathbf{A}_2 - \mathbf{A}_1)^{-1} - \frac{1}{\mu_2}\Psi = \left[\frac{1}{2\Delta k} - \frac{3(1-b)}{2\mu_2}\theta_2\right]\mathbf{a}_1 \otimes \mathbf{a}_1$
$$+ \left[\frac{1}{2\Delta \mu} - \frac{3}{2\mu_2}\left(1 - \frac{b}{2}\right)\theta_2\right](\mathbf{a}_2 \otimes \mathbf{a}_2 + \mathbf{a}_3 \otimes \mathbf{a}_3),$$
$$\Delta f = f_2 - f_1, \qquad f = k \text{ or } \mu$$

and equate this expression to

(5.8) $\quad \alpha_1\theta_1\beta_1(\mathbf{A}_1 - \mathbf{A}_{hhh})^{-1}$
$$= \frac{\alpha_1\beta_1\theta_1}{2(k_2 - \overline{k}_{hhh})}\mathbf{a}_1 \otimes \mathbf{a}_1 + \frac{\alpha_1\beta_1\theta_1}{2(\mu_2 - \overline{\mu}_{hhh})}(\mathbf{a}_2 \otimes \mathbf{a}_2 + \mathbf{a}_3 \otimes \mathbf{a}_3),$$

where
$$\mathbf{A}_{hhh} = 2\overline{k}_{hhh}\mathbf{a}_1 \otimes \mathbf{a}_1 + 2\overline{\mu}_{hhh}(\mathbf{a}_2 \otimes \mathbf{a}_2 + \mathbf{a}_3 \otimes \mathbf{a}_3),$$

which gives

(5.9)
$$\frac{\theta_1\alpha_1\beta_1}{k_2 - \overline{k}_{hhh}} = \frac{1}{\Delta k} - \frac{3\theta_2(1-b)}{\mu_2},$$
$$\frac{\theta_1\alpha_1\beta_1}{\mu_2 - \overline{\mu}_{hhh}} = \frac{1}{\Delta \mu} - \frac{3\theta_2(1-b/2)}{\mu_2}.$$

Assume that the area fraction of the first material is fixed and equals m_1, or

(5.10) $\qquad\qquad m_1 = \alpha_1\beta_1\theta_1.$

We can express θ_2, α_2, β_2 in terms of m_1 as follows

(5.11) $\qquad \theta_2 = \frac{m_2}{3}, \qquad \alpha_2 = \frac{m_2}{3 - m_2}, \qquad \beta_2 = \frac{m_2}{3 - 2m_2}$

and Eqs. (5.9) assume the form

$(5.12)_1$
$$\frac{m_1}{k_2 - \overline{k}_{hhh}} = \frac{1}{k_2 - k_1} - \frac{m_2}{k_2 + \mu_2},$$
$$\frac{m_1}{\mu_2 - \overline{\mu}_{hhh}} = \frac{1}{\mu_2 - \mu_1} - \frac{m_2(k_2 + 2\mu_2)}{2\mu_2(k_2 + \mu_2)}.$$

The moduli $(\overline{k}_{hhh}, \overline{\mu}_{hhh})$ determined by Eq. (5.12) are just upper estimates of Hashin and Shtrikman, see CHERKAEV and GIBIANSKI [2]. Formulae (5.12) can be written as follows

$(5.12)_2$
$$\overline{k}_{PE} = F(m; k_1, k_2, \mu_2),$$
$$\overline{\mu}_{PE} = G_{PE}(m; \mu_1, \mu_2, k_2); \qquad m = (m_1, m_2);$$

PE means "plane elasticity problem", the functions F and G_{PE} being determined by Eq. (5.12)$_1$; here $\overline{k}_{PE} = \overline{k}_{hhh}$, $\overline{\mu}_{PE} = \overline{\mu}_{hhh}$.

REMARK 1. The laminate of 3rd rank constructed by subsequent layerings discussed here is the stiffest among all isotropic composites made from two given isotropic materials with given area fractions. Let us stress that both bounds for μ and k are attained simultaneously.

REMARK 2. In a similar way one can construct the softest isotropic composite. To this end one should mix subsequent laminates with the softest material to get

$(5.13)_1$
$$\frac{m_2}{k_1 - \underline{k}_{hhh}} = \frac{1}{k_1 - k_2} - \frac{m_1}{k_1 + \mu_1},$$
$$\frac{m_2}{\mu_1 - \underline{\mu}_{hhh}} = \frac{1}{\mu_1 - \mu_2} - \frac{m_1(k_1 + 2\mu_1)}{2\mu_1(k_1 + \mu_1)}$$

or

$(5.13)_2$
$$\underline{k}_{PE} = F(\widetilde{m}; k_2, k_1, \mu_1),$$
$$\underline{\mu}_{PE} = G_{PE}(\widetilde{m}; \mu_2, \mu_1, k_1), \qquad \widetilde{m} = (m_2, m_1),$$

where hhh is replaced with PE, which indicates that the plane elasticity problem is considered. We recognize that $\underline{k}_{hhh} = \underline{k}_{PE}$, $\underline{\mu}_{hhh} = \underline{\mu}_{PE}$ are lower bounds of Hashin and Shtrikman, see CHERKAEV and GIBIANSKI [2]. These lower bounds are attained simultaneously.

6. Kirchhoff bending problem. A rank-1 Kirchhoff's in-plane laminate

Let us consider now a bending problem of thin transversely symmetric plates. Given two isotropic plate materials of bending/torsion stiffness tensors $\mathbf{D}_\alpha$,

$\alpha = 1, 2; \mathbf{D}_2 - \mathbf{D}_1 > 0$, we construct an in-plane laminate (we use the term *laminate* to stress similarities with the layering construction of the previous sections; the term *ribbed plate* would be more appropriate) in the $\mathbf{n}$ direction, $\mathbf{n} = (n_1, n_2)$, with area fractions θ_α, respectively. Let $\mathbf{e}_\alpha$ be versors of Cartesian axes. Let us decompose

$$(6.1) \qquad \mathbf{D}_\lambda = D_\lambda^{\alpha\beta\gamma\delta} \mathbf{e}_\alpha \otimes \mathbf{e}_\beta \otimes \mathbf{e}_\gamma \otimes \mathbf{e}_\delta,$$

$$(6.2) \qquad D_\alpha^{1111} = D_\alpha^{2222} = k_\alpha + \mu_\alpha, \qquad D_\alpha^{1122} = k_\alpha - \mu_\alpha, \qquad D_\alpha^{1212} = \mu_\alpha,$$

k_α, μ_α being Kelvin and Kirchhoff moduli, respectively.

Within the Kirchhoff's framework the effective stiffness tensor $\mathbf{D}_\lambda$ of the in-plane laminate considered is uniquely determined. The dispersed in the literature contributions by G. Duvaut, R.V. Kohn, A.V. Cherkaev, K.A. Lur'e, G. Francfort, F. Murat, G. Milton, R. Lipton and L. Tartar lead to the formula

$$(6.3) \qquad \theta_1 (\mathbf{D}_2 - \mathbf{D}_h)^{-1} = (\mathbf{D}_2 - \mathbf{D}_1)^{-1} - \frac{\theta_2}{s_2} \boldsymbol{\Gamma}^{(\mathbf{n})},$$

$$(6.4) \qquad s_2 = k_2 + \mu_2, \qquad \boldsymbol{\Gamma}^{(\mathbf{n})} = \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n},$$

$\mathbf{D}_1$ being not necessarily isotropic. A derivation of Eq. (6.3) can be found in LIPTON [4].

7. A rank-two Kirchhoff laminate

Let us envelop the rank-1 laminate around the strongest phase (2), the area fractions of $\mathbf{D}_h$, $\mathbf{D}_2$ being α_1 and α_2, respectively. Thus we stack materials $\mathbf{D}_h$, $\mathbf{D}_2$ in layers orthogonal to a versor $\mathbf{m}$. To find effective stiffness tensor $\mathbf{D}_{hh}$ one can apply (6.3) again to find

$$(7.1) \qquad \alpha_1 (\mathbf{D}_2 - \mathbf{D}_{hh})^{-1} = (\mathbf{D}_2 - \mathbf{D}_h)^{-1} - \frac{\alpha_2}{s_2} \boldsymbol{\Gamma}^{(\mathbf{m})}.$$

On combining (6.3) and (7.1) one finds

$$(7.2) \qquad \theta_1 \alpha_1 (\mathbf{D}_2 - \mathbf{D}_{hh})^{-1} = (\mathbf{D}_2 - \mathbf{D}_1)^{-1} - \frac{\theta_2}{s_2} \boldsymbol{\Gamma}^{(\mathbf{n})} - \frac{\theta_1 \alpha_2}{s_2} \boldsymbol{\Gamma}^{(\mathbf{m})},$$

the quantity $\theta_1 \alpha_1$ being the resulting area fraction of material (1).

8. A rank-three Kirchhoff's laminate

Let us stack the materials of stiffnesses $\mathbf{D}_{hh}$, $\mathbf{D}_2$ together thus building an in-plane laminate along versor $\mathbf{p}$, with area fractions β_1, β_2, respectively. Applying

(6.3) once again we have

$$(8.1) \qquad \beta_1(\mathbf{D}_2 - \mathbf{D}_{hhh})^{-1} = (\mathbf{D}_2 - \mathbf{D}_{hh})^{-1} - \frac{\beta_2}{s_2}\mathbf{\Gamma}^{(\mathbf{p})}.$$

On combining (8.1) with (7.2) one finds

$$(8.2) \qquad \beta_1\theta_1\alpha_1(\mathbf{D}_2 - \mathbf{D}_{hhh})^{-1} = (\mathbf{D}_2 - \mathbf{D}_1)^{-1} - \frac{1}{s_2}\mathbf{\Gamma},$$

$$(8.3) \qquad \mathbf{\Gamma} = \theta_2\mathbf{\Gamma}^{(\mathbf{n})} + \theta_1\alpha_2\mathbf{\Gamma}^{(\mathbf{m})} + \theta_1\alpha_1\beta_2\mathbf{\Gamma}^{(\mathbf{p})}.$$

9. Hexagonal lamination

Let us take versors **n**, **m**, **p** such that their vertices form an unilateral triangle. Thus

$$(9.1) \qquad \begin{aligned} \mathbf{m} &= \left(-\frac{1}{2}n_1 - \frac{\sqrt{3}}{2}n_2, \ -\frac{1}{2}n_2 + \frac{\sqrt{3}}{2}n_1\right), \\ \mathbf{p} &= \left(-\frac{1}{2}n_1 + \frac{\sqrt{3}}{2}n_2, \ -\frac{1}{2}n_2 - \frac{\sqrt{3}}{2}n_1\right). \end{aligned}$$

We shall prove that the above choice of **m** and **p** implies isotropy of $\mathbf{\Gamma}$. Assume for simplicity that $\mathbf{n} = (1, 0)$, which is not a restriction. Let us compute

$$(9.2) \qquad \begin{aligned} &\Gamma_{1111} = \theta_2 + \frac{\theta_1}{16}(\alpha_2 + \beta_2\alpha_1), \qquad\qquad \Gamma_{2222} = \left(\frac{\sqrt{3}}{2}\right)^4 \theta_1(\alpha_2 + \beta_2\alpha_1), \\ &\Gamma_{1122} = \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)^2 \theta_1(\alpha_2 + \beta_2\alpha_1), \qquad \Gamma_{1112} = \frac{\sqrt{3}}{16}\theta_1(-\alpha_2 + \beta_2\alpha_1), \\ &\Gamma_{2221} = \Gamma_{1112}, \qquad\qquad\qquad\qquad\quad \Gamma_{1212} = \frac{3}{16}\theta_1(\alpha_2 + \beta_2\alpha_1). \end{aligned}$$

We stipulate

$$(9.3) \qquad \Gamma_{1222} = \Gamma_{1112} = 0, \qquad \Gamma_{1111} = \Gamma_{2222},$$

hence

$$(9.4) \qquad \beta_2 = \alpha_2/\alpha_1, \qquad \theta_2 = \theta_1\alpha_2,$$

and consequently make the tensor $\mathbf{\Gamma}$ isotropic

$$(9.5) \qquad \mathbf{\Gamma} = \frac{3}{4}\theta_2[2\mathbf{a}_1 \otimes \mathbf{a}_1 + (\mathbf{a}_2 \otimes \mathbf{a}_2 + \mathbf{a}_3 \otimes \mathbf{a}_3)].$$

Let us find now $\bar{k}_{hhh}$, $\bar{\mu}_{hhh}$ involved in the representation

(9.6) $$\mathbf{D}_{hhh} = 2\bar{k}_{hhh}\mathbf{a}_1 \otimes \mathbf{a}_1 + 2\bar{\mu}_{hhh}(\mathbf{a}_2 \otimes \mathbf{a}_2 + \mathbf{a}_3 \otimes \mathbf{a}_3).$$

Since

(9.7) $$(\mathbf{D}_2 - \mathbf{D}_{hhh})^{-1} = \frac{1}{2(k_2 - \bar{k}_{hhh})}\mathbf{a}_1 \otimes \mathbf{a}_1 + \frac{1}{2(\mu_2 - \bar{\mu}_{hhh})}(\mathbf{a}_2 \otimes \mathbf{a}_2 + \mathbf{a}_3 \otimes \mathbf{a}_3)$$

and

(9.8) $$(\mathbf{D}_2 - \mathbf{D}_1)^{-1} - \frac{1}{s_2}\boldsymbol{\Gamma}$$

$$= \left[\frac{1}{2(k_2 - k_1)} - \frac{\frac{3}{2}\theta_2}{s_2}\right]\mathbf{a}_1 \otimes \mathbf{a}_1 + \left[\frac{1}{2(\mu_2 - \mu_1)} - \frac{\frac{3}{4}\theta_2}{s_2}\right](\mathbf{a}_2 \otimes \mathbf{a}_2 + \mathbf{a}_3 \otimes \mathbf{a}_3),$$

the formula (4.2) implies

(9.9) $$\frac{\theta_1\alpha_1\beta_1}{k_2 - \bar{k}_{hhh}} = \frac{1}{k_2 - k_1} - \frac{3\theta_2}{k_2 + \mu_2},$$

$$\frac{\theta_1\alpha_1\beta_1}{\mu_2 - \bar{\mu}_{hhh}} = \frac{1}{\mu_2 - \mu_1} - \frac{3\theta_2}{2(k_2 + \mu_2)}.$$

Denote the resulting area fraction of phase 1 by m_1. Then $m_1 = \theta_1\alpha_1\beta_1$, $m_2 = 1 - m_1$. The previous results imply

(9.10) $$\theta_2 = \frac{m_2}{3}, \qquad \alpha_2 = \frac{m_2}{3 - m_2}, \qquad \beta_2 = \frac{m_2}{3 - 2m_2}.$$

By (9.9), (9.10) we find finally

(9.11) $$\frac{m_1}{k_2 - \bar{k}_{hhh}} = \frac{1}{k_2 - k_1} - \frac{m_2}{k_2 + \mu_2},$$

$$\frac{m_1}{\mu_2 - \bar{\mu}_{hhh}} = \frac{1}{\mu_2 - \mu_1} - \frac{m_2}{2(k_2 + \mu_2)}.$$

REMARK 3. Functions $\bar{k}_{hhh}(m_2)$, $\bar{\mu}_{hhh}(m_2)$ grow monotonically from k_1 to k_2 and from μ_1 to μ_2, respectively, if m_2 varies from 0 to 1.

REMARK 4. It turns out that the resulting isotropic plate of stiffness $\mathbf{D}_{hhh}$ given by (9.6) with $\bar{k}_{hhh}$, $\bar{\mu}_{hhh}$ given by (9.11) is just the stiffest possible plate among all plates formed from phases (1) and (2) with given area fractions m_1, m_2. Thus equations (9.11) provide the upper Hashin–Shtrikman bounds for both k and μ.

REMARK 5. To find the softest plate one should envelop the homogenized material around the softest one. In the same manner one arrives at the lower Hashin–Shtrikman estimates for k and μ.

10. Plane elasticity versus plate bending results

One can show a remarkable correspondence between Hashin–Shtrikman bounds for Kirchhoff plates and plane elasticity problems. For Kirchhoff plate model the upper Hashin–Shtrikman bounds $\overline{k}_K, \overline{\mu}_K$ are solutions to the following equations, cf. Eq. (9.11)

$$(10.1) \qquad \overline{k}_K = F(m; k_1, k_2, \mu_2),$$
$$(10.2) \qquad \overline{\mu}_K = G_K(m; \mu_1, \mu_2, k_2).$$

Index K refers to the Kirchhoff plate model; function G_K is determined by (9.11) and function F – defined by Eq. (5.12). Let us note that

$$(10.3) \qquad F(m; k_1, k_2, \mu_2) = \left[F\left(m; (k_1)^{-1}, (k_2)^{-1}, (\mu_2)^{-1}\right)\right]^{-1},$$
$$G_K(m; \mu_1, \mu_2, k_2) = \left[G_{PE}\left(m; (\mu_1)^{-1}, (\mu_2)^{-1}, (k_2)^{-1}\right)\right]^{-1},$$

which can be proved by algebraic manipulations. Thus the link between plane elasticity bounds

$$(10.4) \qquad \begin{aligned} F(\widetilde{m}; k_2, k_1, \mu_1) &\leq k_{PE} \leq F(m; k_1, k_2, \mu_2), \\ G_{PE}(\widetilde{m}; \mu_2, \mu_1, k_1) &\leq \mu_{PE} \leq G_{PE}(m; \mu_1, \mu_2, k_2), \end{aligned}$$

and Kirchhoff plate bounds

$$(10.5) \qquad \begin{aligned} F(\widetilde{m}; k_2, k_1, \mu_1) &\leq k_K \leq F(m; k_1, k_2, \mu_2), \\ G_K(\widetilde{m}; \mu_2, \mu_1, k_1) &\leq \mu_K \leq G_K(m; \mu_1, \mu_2, k_2), \end{aligned}$$

can be explained as follows.

Assume that $k_1 \geq k_2$, $\mu_1 \geq \mu_2$. Then the bounds for the plate moduli assume the form

$$(10.6) \qquad \begin{aligned} F(m; k_1, k_2, \mu_2) &\leq k_K \leq F(\widetilde{m}; k_2, k_1, \mu_1), \\ G_K(m; \mu_1, \mu_2, k_2) &\leq \mu_K \leq G_K(\widetilde{m}; \mu_2, \mu_1, k_1), \end{aligned}$$

since $\widetilde{\widetilde{m}} = m$. Hence by (10.3) we find the bounds for the flexibilities

$$\begin{aligned} F\left(\widetilde{m}; (k_2)^{-1}, (k_1)^{-1}, (\mu_1)^{-1}\right) &\leq (k_K)^{-1} \leq F\left(m; (k_1)^{-1}, (k_2)^{-1}, (\mu_2)^{-1}\right), \\ (10.7) \quad G_{PE}\left(\widetilde{m}; (\mu_2)^{-1}, (\mu_1)^{-1}, (k_1)^{-1}\right) &\leq (\mu_K)^{-1} \\ &\leq G_{PE}\left(m; (\mu_1)^{-1}, (\mu_2)^{-1}, (k_2)^{-1}\right), \end{aligned}$$

similar to elasticity bounds (10.4). We see the analogy

$$(10.8) \qquad k_{PE} \Leftrightarrow (k_K)^{-1}, \qquad \mu_{PE} \Leftrightarrow (\mu_K)^{-1}, \qquad k_\alpha \Leftrightarrow (k_\alpha)^{-1}, \qquad \mu_\alpha \Leftrightarrow (\mu_\alpha)^{-1}.$$

Let us stress that inequalities (10.4) are valid for

$$(10.9) \qquad\qquad\qquad k_2 > k_1, \qquad \mu_2 > \mu_1,$$

while inequalities (10.7) are valid if

$$(10.10) \qquad\qquad (k_2)^{-1} > (k_1)^{-1}, \qquad (\mu_2)^{-1} > (\mu_1)^{-1},$$

which is compatible with analogy (10.8).

Let us prove a correspondence between (10.4) and (10.7).

The analogy to be proved follows from the following homogenization formulae

a) the homogenized plane elasticity tensor $\mathbf{A}_h$ is given by

$$(10.11) \qquad E_{\alpha\beta} A_h^{\alpha\beta\lambda\mu} E_{\lambda\mu} = \min\left\{ \left\langle \varepsilon_{\alpha\beta} A^{\alpha\beta\lambda\mu} \varepsilon_{\lambda\mu} \right\rangle \mid \varepsilon_{\alpha\beta} \text{ are kinematically admissible:} \right.$$

$$\left. \varepsilon_{11,22} + \varepsilon_{22,11} - 2\varepsilon_{12,12} = 0, \qquad Y\text{-periodic and such that } \langle \boldsymbol{\varepsilon} \rangle = \mathbf{E} \right\}.$$

$\langle \cdot \rangle$ means averaging over periodicity cell Y; $(\ )_{,\alpha} = \partial/\partial y_\alpha$, $y = (y_1, y_2) \in Y$.

b) the homogenized tensor $\mathbf{C}_h$ of Kirchhoff's plate flexibilities is given by

$$(10.12) \qquad M^{\alpha\beta} C^h_{\alpha\beta\lambda\mu} M^{\lambda\mu} = \min\left\{ \left\langle m^{\alpha\beta} C_{\alpha\beta\lambda\mu} m^{\lambda\mu} \right\rangle \mid m^{\alpha\beta} \text{ are statically admissible:} \right.$$

$$\left. m^{\alpha\beta}{}_{,\alpha\beta} = 0, \qquad Y\text{-periodic and such that } \langle \mathbf{m} \rangle = \mathbf{M} \right\}.$$

Let us denote

$$(10.13) \qquad\qquad \varepsilon_{22} = n^{11}, \qquad \varepsilon_{11} = n^{22}, \qquad \varepsilon_{12} = -n^{12}.$$

Then

$$(10.14) \qquad\qquad n^{\alpha\beta}{}_{,\alpha\beta} = 0, \qquad \langle n^{\alpha\beta} \rangle = e_{\alpha\sigma} e_{\beta\gamma} E_{\sigma\gamma} = \tilde{E}_{\alpha\beta},$$

where $e_{\alpha\alpha} = 0$, $e_{12} = -e_{21} = 1$. Thus formula (10.11) assumes the form

$$(10.15) \qquad \tilde{E}_{\alpha\beta} \tilde{A}_h^{\alpha\beta\lambda\mu} \tilde{E}_{\lambda\mu} = \min\left\{ \left\langle n^{\gamma\sigma} \tilde{A}_{\gamma\delta\sigma\kappa} n^{\sigma\kappa} \right\rangle \right.$$

$$\left. \mid n^{\alpha\beta}{}_{,\alpha\beta} = 0; \quad n^{\alpha\beta} \text{ are } Y\text{-periodic}; \quad \langle \mathbf{n} \rangle = \tilde{\mathbf{E}} \right\},$$

where

$$(10.16) \qquad\qquad \tilde{A}_{\gamma\delta\sigma\kappa} = e_{\alpha\gamma} e_{\beta\delta} e_{\lambda\sigma} e_{\mu\kappa} A^{\alpha\beta\lambda\mu},$$

and $\tilde{\mathbf{A}}_h$ is defined similarly. Transformation (10.16) changes indices (1,2) into (2,1), which is unimportant if we estimate the energy by isotropic quadratic forms, which is the case here. Hence the upper/lower estimates for strain energy (10.15) assume the form of upper/lower estimates of complementary energy of Kirchhoff's plate, cf. Eq. (10.12). Therefore estimates (10.4) have formally the same form as estimates (10.7) for the plate flexibilities. Note, however, that the applicability ranges of both estimates are complementary.

Acknowledgment

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