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Duality based solution of contact problem with Coulomb friction

Z. DOSTÁL and V. VONDRÁK (OSTRAVA)

NUMERICAL SOLUTION of quasi-variational inequalities that describe the equilibrium of elastic bodies in contact with friction is presented. The problem is first reduced to a sequence of well conditioned problems with given friction that are reformulated by means of duality as quadratic programming problems with box constraints. Then the algorithm for the solution of quadratic programming problems with proportioning and projections is applied to the solution of the resulting contact problem with Coulomb friction. The characteristic feature of this active set-based algorithm is that it accepts approximate solutions of auxiliary problems and that it is able to drop and add many constraints whenever the active set is changed. The results of our numerical experiments indicate that the algorithms presented are efficient. The algorithm may prove to be useful in parallel implementation.

1. Introduction

THE DUAL SCHUR complement domain decomposition method introduced recently by FARHAT and ROUX [5] turned out to be an efficient algorithm for parallel solution of self-adjoint elliptic partial differential equations. Recently, we have combined this method with our results [2] on quadratic programming with simple bounds, in order to develop an efficient algorithm for the solution of variational inequalities that describe the conditions of equilibrium of a system of elastic bodies in frictionless contact [3]. The results of [2] turned out to be closely related to the results of FRIEDLANDER and MARTÍNEZ [7] and were further extended in [4].

In this paper, we extend this approach to the solution of unilateral contact problems of linear elasticity with Coulomb friction. The main feature of our new algorithm for the solution of coercive problems is that it accepts approximate solutions of auxiliary minimization problems, that it is able to drop and add many constraints whenever the active set is changed, and that it treats the bodies independently of each other, so that parallel implementation is possible. The application of the duality theory to a discrete problem may be considered as an implementation of the reciprocal formulation of [8]. The performance of the algorithm is demonstrated on the solution of a model problem.

2. Discretized contact problem with given friction

We shall start our exposition from the discretized contact problem. Suppose that **K** is the stiffness matrix of the order n resulting from the finite element discretization of a system of elastic bodies $\Omega_1, \ldots, \Omega_p$ with enhanced bilateral boundary conditions. With suitable numbering of nodes, we can achieve that $\mathbf{K} = \mathrm{diag}(\mathbf{K}_1, \ldots, \mathbf{K}_p)$, where each $\mathbf{K}_i$ denotes a band matrix which may be identified with the stiffness matrix of the body Ω_i. We assume that $\mathbf{K}$ is positive definite.

Let m denote the number of nodes in contact. The linearized conditions of contact with given friction are supposed to be defined by the $m \times n$ matrices $\mathbf{N} = (n_{ij})$, $\mathbf{T} = (t_{ij})$, by the $m \times m$ non-negative diagonal matrix $\mathbf{\Gamma} = \mathrm{diag}(\gamma_i)$, and by the m-vector $\mathbf{c} = (c_i)$. The rows $\mathbf{n}_{i*}$ of $\mathbf{N}$ are vectors defined by unit outer normals that enable us to evaluate the change of the normal distance $c_i \geq 0$ between two potential contact surfaces. The formula for the displacement $\mathbf{x}$ is $\mathbf{n}_{i*}\mathbf{x}$. The matrix $\mathbf{N}$ is sparse as non-zero entries of $\mathbf{n}_{i*}$ may be only in positions of nodal variables which correspond to the nodes involved in some constraint. The diagonal entries γ_i of the diagonal matrix $\mathbf{\Gamma}$ define what we can call nodal given friction that corresponds to a couple of points in contact. In analogy to $\mathbf{n}_{i*}$, the row vectors t_{i*} of the matrix $\mathbf{T}$ are defined by the tangential vectors in these contact points. The matrix $\mathbf{T}$ is also sparse. We shall use $\mathbf{T}$ for evaluation of the tangential part of the relative displacement of the contact surfaces.

With these notations, solution of the discretized contact problem with given friction amounts to the solution of the problem

$$\min_{\mathbf{x} \in C} \max_{\mu \in M} f(\mathbf{x}, \mu), \tag{2.1}$$

where

$$f(\mathbf{x}, \mu) = \frac{1}{2}\mathbf{x}^T\mathbf{K}\mathbf{x} - \mathbf{b}^T\mathbf{x} + \mu^T\mathbf{\Gamma}\mathbf{T}\mathbf{x}, \quad M = \left\{\mu \middle| \, |\mu| \leq 1\right\} \quad \text{and} \quad C = \left\{\mathbf{x} \middle| \, \mathbf{N}\mathbf{x} \leq \mathbf{c}\right\}.$$

In the last equation and in what follows, all the vector inequalities should be read pointwise. Similarly, $|\mu|$ denotes a vector with entries μ_i. More details about formulation and discretization of contact problems with friction may be found in Refs. [8, 9].

3. Reciprocal formulation

First observe that

$$\min_{\mathbf{x} \in C} \max_{\mu \in M} f(\mathbf{x}, \mu) = \min_{\mathbf{x}} \max_{\mu \in M} \max_{\lambda \geq 0} L(\mathbf{x}, \mu, \lambda), \tag{3.1}$$

where

$$\begin{aligned} L(\mathbf{x}, \mu, \lambda) &= f(\mathbf{x}, \mu) + \lambda^T(\mathbf{N}\mathbf{x} - \mathbf{c}) \\ &= \frac{1}{2}\mathbf{x}^T\mathbf{K}\mathbf{x} - \mathbf{b}^T\mathbf{x} + \mu^T\mathbf{\Gamma}\mathbf{T}\mathbf{x} + \lambda^T(\mathbf{N}\mathbf{x} - \mathbf{c}). \end{aligned} \tag{3.2}$$

For fixed μ and λ, the Lagrange function (3.2) is strictly convex in the first variable, and the gradient argument shows that any minimizer of $L(\cdot, \mu, \lambda)$ satisfies

$$\mathbf{K}\mathbf{x} - \mathbf{b} + \mathbf{T}^T\mathbf{\Gamma}^T\mu + \mathbf{N}^T\lambda = \mathbf{o}. \tag{3.3}$$

This equation has a solution for any $\mathbf{b}$, $\mathbf{T}$, $\mathbf{\Gamma}$, μ, $\mathbf{N}$, λ because the matrix $\mathbf{K}$ is positive definite. Simple computation shows that (3.3) is equivalent to

$$\mathbf{x} = \mathbf{K}^{-1}\left(\mathbf{b} - \mathbf{T}^T\mathbf{\Gamma}^T\mu - \mathbf{N}^T\lambda\right). \tag{3.4}$$

After substituting (3.4) into the Lagrange function (3.2) and after some simplifications that exploit the structure of the matrices, we get for fixed μ and λ the problem to find

$$\min_{\substack{\lambda\geq 0\\ |\mu|\leq 1}} \frac{1}{2}\left(\lambda^T, \mu^T\right)\begin{pmatrix}\mathbf{N}\\ \mathbf{\Gamma}\mathbf{T}\end{pmatrix}\mathbf{K}^{-1}\left(\mathbf{N}^T, \mathbf{T}^T\mathbf{\Gamma}^T\right)\begin{pmatrix}\lambda\\ \mu\end{pmatrix} - \left(\lambda^T, \mu^T\right)\begin{pmatrix}\mathbf{N}\mathbf{K}^{-1}\mathbf{b} - \mathbf{c}\\ \mathbf{\Gamma}\mathbf{T}\mathbf{K}^{-1}\mathbf{b}\end{pmatrix}. \tag{3.5}$$

The latter is the quadratic programming problem

$$\min_{\mathbf{z}\in S} F(\mathbf{z}), \tag{3.6}$$

where

$$F(\mathbf{z}) = \frac{1}{2}\mathbf{z}^T\mathbf{Q}\mathbf{z} - \mathbf{z}^T\mathbf{h}, \quad \mathbf{z} = \begin{pmatrix}\lambda\\ \mu\end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix}\mathbf{N}\\ \mathbf{\Gamma}\mathbf{T}\end{pmatrix}\mathbf{K}^{-1}\left(\mathbf{N}^T, \mathbf{T}^T\mathbf{\Gamma}^T\right),$$
$$\mathbf{h} = \begin{pmatrix}\mathbf{N}\mathbf{K}^{-1}\mathbf{b} - \mathbf{c}\\ \mathbf{\Gamma}\mathbf{T}\mathbf{K}^{-1}\mathbf{b}\end{pmatrix} \quad \text{and} \quad S = \left\{\mathbf{z} = \begin{pmatrix}\lambda\\ \mu\end{pmatrix}\middle|\ \lambda \geq 0,\ \ |\mu| \leq 1\right\}. \tag{3.7}$$

4. Proportioning

We shall consider here the problem (3.6) with a general choice of $S = \{\mathbf{z}|\ \mathbf{l} \leq \mathbf{z} \leq \mathbf{u}\}$. The only solution $\mathbf{z}$ of this problem satisfies the Kuhn–Tucker contact conditions

$$\begin{aligned} r_i &= 0 \quad \text{for} \quad l_i < z_i < u_i\,, \\ r_i^- &= 0 \quad \text{for} \quad z_i = l_i\,, \\ r_i^+ &= 0 \quad \text{for} \quad z_i = u_i\,, \end{aligned} \tag{4.1}$$

where $\mathbf{r} = \mathbf{Q}\mathbf{z} - \mathbf{h}$, $r_i^- = \min\{r_i, 0\}$ and $r_i^+ = \max\{r_i, 0\}$. Let us recall that the *active set* $\mathbf{A}(\mathbf{z}) = \{i|\ z_i = l_i \vee z_i = u_i\}$ and the *free set* $\mathbf{F}(\mathbf{z}) = \{i|\ l_i < z_i < u_i\}$. The *chopped gradient* $\beta(\mathbf{z})$ and *reduced gradient* $\varphi(\mathbf{z})$ are defined by

$$\begin{gathered} \varphi_i(\mathbf{z}) = r_i(\mathbf{z}) \quad \text{for} \quad i \in \mathbf{F}(\mathbf{z}) \qquad \text{and} \qquad \varphi_i(\mathbf{z}) = 0 \quad \text{for} \quad i \in \mathbf{A}(\mathbf{z}), \\ \beta_i(\mathbf{z}) = 0 \quad \text{for} \quad i \in \mathbf{F}(\mathbf{z}), \\ \beta_i(\mathbf{z}) = r_i^- \quad \text{for} \quad z_i = l_i \quad \text{and} \quad \beta_i(\mathbf{z}) = r_i^+ \quad \text{for} \quad z_i = u_i\,. \end{gathered} \tag{4.2}$$

Hence the conditions (4.1) are equivalent to $\varphi(\mathbf{z}) = \beta(\mathbf{z}) = \mathbf{o}$, so that $\mathbf{z}$ is the solution of our problem iff the *projected gradient* $\nu(\mathbf{z}) = \beta(\mathbf{z}) + \varphi(\mathbf{z}) = \mathbf{o}$.

The algorithm that we proposed in [2] is a modification of the Polyak algorithm that controls the precision of the solution of auxiliary problems by the norm of violation of the Kuhn–Tucker contact condition in each iteration. If for $G > 0$ the inequality $\|\beta(\mathbf{z}^i)\|_\infty \le G\|\varphi(\mathbf{z}^i)\|_2$ holds, we shall call $\mathbf{z}^i$ *proportional*. The algorithm explores the face $W_I = \left\{\mathbf{y}|\ y_i = l_i \text{ or } y_i = u_i \text{ for } i \in I\right\}$ with a given active set I as long as the iterations are proportional. If $\mathbf{z}^i$ is not proportional, we generate $\mathbf{z}^{i+1}$ by means of decrease direction $\mathbf{d}^i = -\beta(\mathbf{z}^i)$ in a process that we call *proportioning* and we continue by exploring the new face defined by $I = \mathbf{A}(\mathbf{z}^{i+1})$. The class of algorithms driven by proportioning may be defined as follows.

ALGORITHM GPS (General proportioning scheme).

Let $\mathbf{z}^0 \in S$ and $G > 0$ be given. For $k > 0$, choose $\mathbf{z}^{k+1}$ by the following rules:

(i) If $\mathbf{z}^i$ is not proportional, define $\mathbf{z}^{i+1}$ by proportioning.

(ii) If $\mathbf{z}^i$ is proportional, choose feasible $\mathbf{z}^{i+1}$ so that $F(\mathbf{z}^{k+1}) \le F(\mathbf{z}^k)$ and $\mathbf{z}^{i+1}$ satisfies at least one of the conditions $\mathbf{A}(\mathbf{z}^k) \subset \mathbf{A}(\mathbf{z}^{k+1})$, $\mathbf{z}^{i+1}$ is not proportional, or $\mathbf{z}^{i+1}$ minimizes $F(\xi)$ subject to $\xi \in W_I$, $I = \mathbf{A}(\mathbf{z}^k)$.

The symbol $\subset$ denotes proper subset. The basic theoretical results have been proved in [2].

THEOREM. *Let $\mathbf{z}^k$ denote an infinite sequence generated by algorithm* GPS *with given $\mathbf{z}^0$, let $S = \{\mathbf{z}|\ \mathbf{l} \le \mathbf{z} \le \mathbf{u}\}$, and let $F(\mathbf{z})$ denote a strictly convex quadratic function. Then the following statements are true:*

(i) *$\mathbf{z}^k$ converges to the solution $\bar{\mathbf{z}}$ of* (3.6).

(ii) *If the problem* (3.6) *is not degenerate, then there is k such that $\bar{\mathbf{z}} = \mathbf{z}^k$.*

(iii) *If $G \ge \sqrt{\kappa(\mathbf{Q})}$, where $\kappa(\mathbf{Q})$ is the spectral condition number of $\mathbf{Q}$, then there is k such that $\bar{\mathbf{z}} = \mathbf{z}^k$.*

Step (ii) of algorithm GPS may be implemented by means of the conjugate gradient method. The implementations differ in stopping rules for the solution of auxiliary problems and in application of projectors. In the following numerical experiments, we used so-called *monotone proportioning* [2] which, starting from $\mathbf{v}^0 = \mathbf{z}^k$, generates the conjugate gradient iterations for minimization of (3.6) on current face until $F(P\mathbf{v}^{i+1}) > F(P\mathbf{v}^i)$, where P denotes the projection to S. If the conjugate gradient iterations are interrupted on this condition, then a new iteration is defined by $\mathbf{z}^{k+1} = P\mathbf{v}^i$ or by some backtracking strategy. More details may be found in [2].

5. Solution of the friction problem

Now assume that the given friction γ_i is given by $\gamma_i = \Phi_i|T_i^n(\mathbf{x})|$, $i = 1, \ldots, m$, where $\Phi = (\Phi_i)$ is the vector of friction coefficients and $\mathbf{T}^n = (T_i^n(\mathbf{x}))$ is the vector of normal stresses on the contact surface that correspond to the displacement $\mathbf{x}$. Denoting by $\mathbf{x}(\gamma)$ the solution of the contact problem (2.1) with given friction $\gamma = (\gamma_i)$ and by $\mathbf{T}^n(\mathbf{x}(\gamma))$ the corresponding normal stress, the solution of the contact problem with friction amounts to finding the fixed point of the mapping $\Psi : \gamma \mapsto \mathbf{T}^n(\mathbf{x}(\gamma))$. Existence results of the fixed point for sufficiently small friction coefficients may be found in [8, 12]. Hence we can find the solution of the contact problem with friction by

$$\tag{5.1} \begin{cases} \text{initial } \gamma^0, \\ \gamma^{n+1} = \Psi(\gamma^n). \end{cases}$$

Notice that λ corresponds to the normal stress $\mathbf{T}^n$ and the vector $\mathbf{\Gamma}\mu$ corresponds to the tangential stress $\mathbf{T}^t$ on the contact interface.

6. Numerical experiments

We have tested our algorithm on a problem of contact of two bodies of Fig. 1 that was discretized by a grid with 169 nodes, so that the discretized problem had 338 primal and 42 dual unknowns, respectively. The problem was solved for two

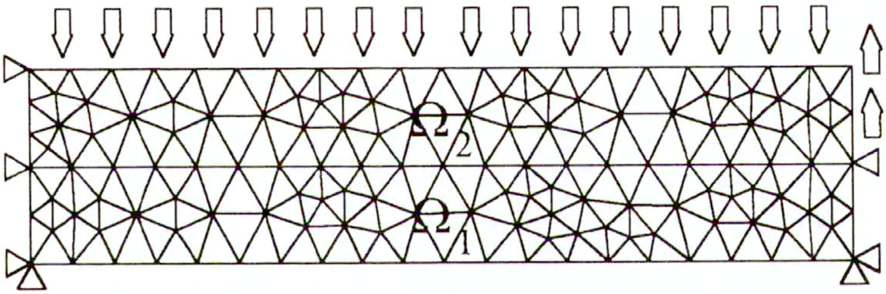

FIG. 1. Test problem.

combinations of elastic constants with the friction coefficient $\Phi = 0.3$. We used the value $G = 1$, so that we interrupted the conjugate gradient iterations when the chopped gradient began to dominate the reduced gradient. Relative precision of the solution was 10^{-3}. The solution turned out to be quite sensitive to the relative precision ε of the solution of the inner problems with given friction. The performance of the algorithm for various ε is given in Table 1 (Fig. 2, 3). The number of the conjugate gradient iterations seems to be relatively low, which also indicates that the distribution of the spectrum of the Hessian of $F(\mathbf{z})$ is favourable. Indeed, Fig. 4 shows that there are very few eigenvalues in both ends of the spectrum. This observations extend the experimental results reported by F.-X. ROUX [14] for the dual Schur complement. All numerical experiments were carried out in Mathworks Matlab with PDE Toolbox.

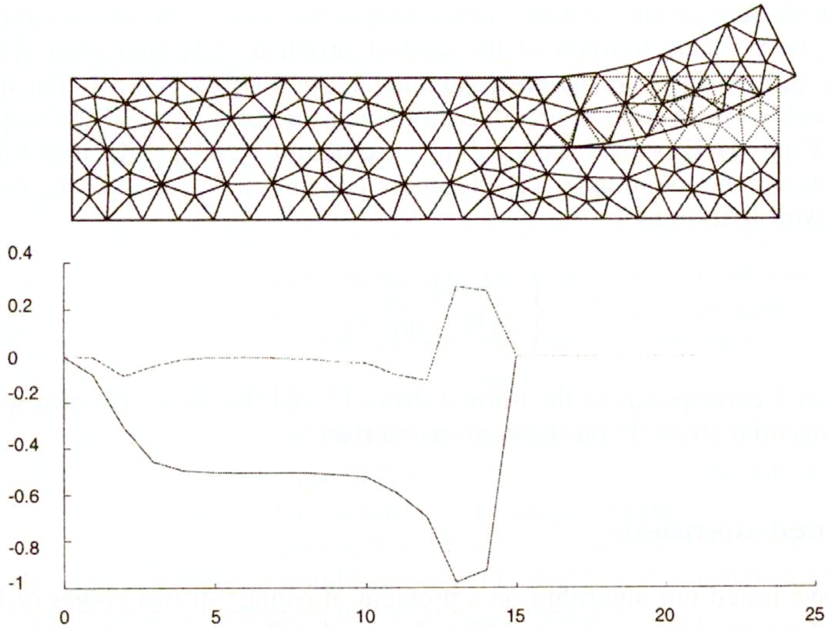

FIG. 2. Distribution of contact stresses for problem 1 ($\varepsilon = 10^{-7}$). Solid line – normal contact stress, dashed line – tangential contact stress.

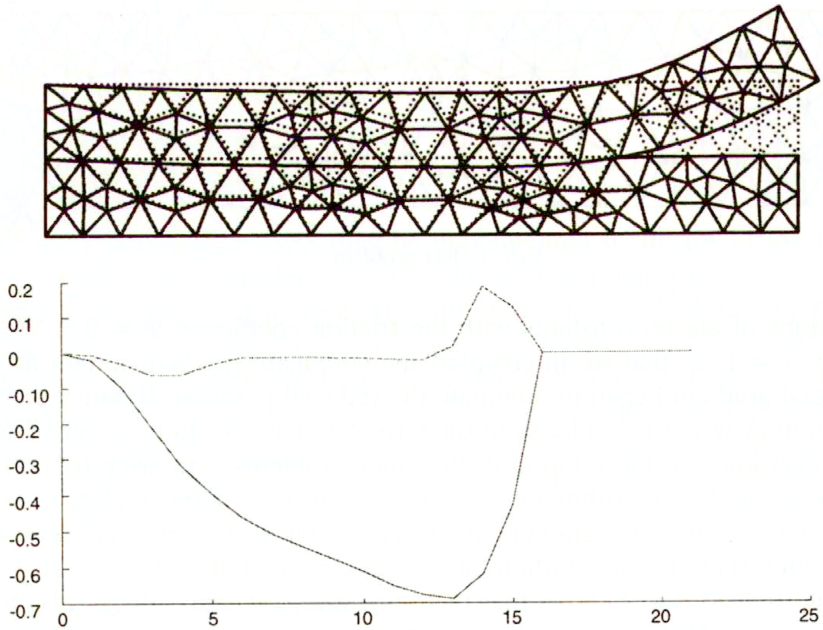

FIG. 3. Displacement and contact stresses for problem 2 ($\varepsilon = 10^{-7}$). Solid line – normal contact stress, dashed line – tangential contact stress.

Table 1. Performance of the algorithm.

Problem 1 ($E_1 = 10^6$ MPa, $E_2 = 10^3$ MPa)				Problem 1 ($E_1 = 10^2$ MPa, $E_2 = 10^3$ MPa)			
iterations				iterations			
number of outer	number of inner (cg steps)			number of outer	number of inner (cg steps)		
	$\varepsilon = 1e-5$	$\varepsilon = 1e-6$	$\varepsilon = 1e-7$		$\varepsilon = 1e-5$	$\varepsilon = 1e-6$	$\varepsilon = 1e-7$
1	28	29	38	1	58	67	75
2	28	36	55	2	84	108	142
3	4	10	15	3	2	9	13
4	3	1	6				
	63	76	114		144	184	230

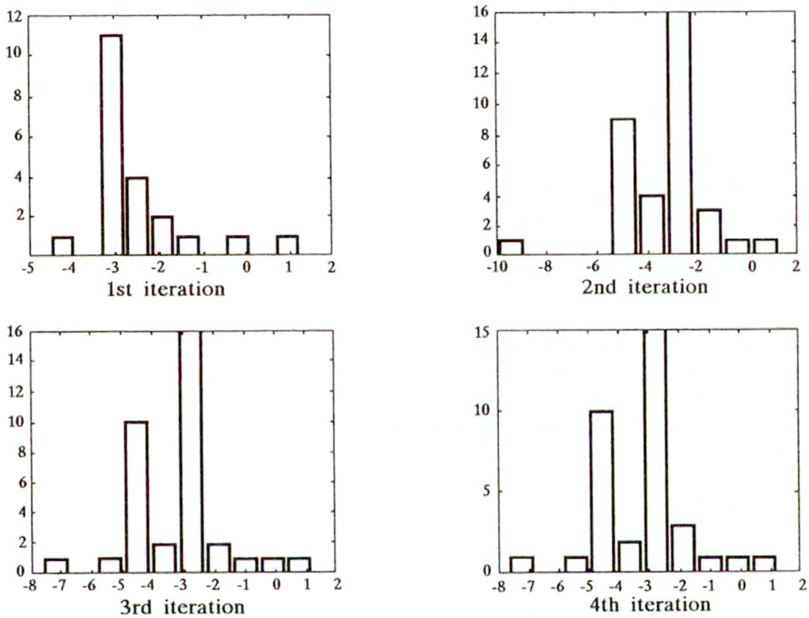

FIG. 4. Distribution of the spectrum of the Hessian.

7. Comments and conclusions

We applied a variant of our new algorithm [2] for the solution of quadratic programming problems with simple bounds to the solution of static coercive contact problem of elasticity with Coulomb friction in reciprocal formulation. A new feature of the presented algorithm is the adaptive precision control of the solution of auxiliary problems and application of projections, so that both the contact and slip interfaces may be identified in a small number of iterations. Theoretical results are reported that grant the convergence of the algorithm. The algorithm also demonstrates the usefulness of the duality theory for practical computation.

Both theoretical results and results of numerical experiments indicate that there are problems which may be solved very efficiently by the algorithm presented. If applied to the solution of a contact problem that involves several bodies, then the algorithm may be considered as Neumann – Neumann type domain decomposition algorithm which may be useful in parallel environment. Using the results [4], the algorithm may be extended to the solution of semicoercive problems.

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DEPARTMENT OF APPLIED MATHEMATICS,
TECHNICAL UNIVERSITY IN OSTRAVA, OSTRAVA-PORUBA, CZECH REPUBLIC
e-mail: zdenek.dostal@vsb.cz
e-mail: vit.vondrak@vsb.cz

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