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Internal variables in dynamics of composite solids with periodic microstructure

Dedicated to Professor Franz Ziegler on the occasion of His 60th birthday

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A NEW UNIFIED micromechanical approach to dynamics of micro-periodic composite solids is formulated. The proposed approach introduces the concept of internal variables in order to describe the effect of the microstructure size on the global body behaviour. It is shown that the evolution equation for internal variables can be obtained without any specification of the material properties of the composite.

1. Introduction

IT IS KNOWN that the behaviour of the composite solids with periodic microstructure can be examined on two levels. On the micro-level the interactions between constituents of a composite are detailed while the global body response is investigated in the framework of macromechanics. The passage from micro- to macromechanics is realized by so-called micromechanical approaches, [1], leading to various mathematical models of the composite solid on the macro-level. The best known ones are those based on the concept of homogeneous equivalent body where the micro-heterogeneous composite is modelled as made of a certain "homogenized" material. The above models can be obtained by some special procedures, [1, 12], derived by means of the asymptotic methods, [3, 5, 11], by the Fourier expansions, [23], or using so-called micro-local parameters, [15, 32]. However, following the concept of a homogeneous equivalent body we neglect the effect of the microstructure size on the global body behaviour. This effect plays an important role mainly in the vibration and wave propagation analysis. In order to describe dynamic problems in the framework of macromechanics, a number of mathematical models, mainly based on the concept of the continuum with extra local degrees of freedom, or obtained by finding the higher-order terms of the asymptotic expansions, was proposed, [2, 9, 14, 15, 22, 24]. Models of this kind have a rather complicated analytical form and, applied to the investigation of boundary-value problems, often lead to a large number of boundary conditions which may be not well motivated from the physical viewpoint. Between the models using the concept of a homogeneous equivalent body and those applying the continua with extra local degrees of freedom, are situated the models of refined macrodynamics, [32]. The effect of periodic microstructure size on the dynamic body behaviour in the framework of refined macrodynamics is described by certain unknown fields, called macro-internal variables (MIV). These variables, being governed by ordinary differential equations involving time derivatives, do not enter the boundary conditions. So far, the internal variables were mainly used in formulations of the constitutive relations, [7]. Applications of this concept to the micromechanical approach in dynamics of periodic composite solids were recently investigated in a series of papers [4, 6, 8, 10, 13, 16, 18–21, 26–30, 33–44].

In the aforementioned papers the micromechanical approach to macromechanics, using the macro-internal variables, was based on certain heuristic assumptions related to the specification of materials and the expected motions of the body. The aim of this contribution is to derive the governing equations for models with MIV without those assumptions. The main result is that the evolution equations for MIV can be obtained without any specification of the material properties of a composite solid. The considerations in Secs. 1–5 are restricted to the periodic composites. Certain generalization of the MIV-model, describing microstructures which may be non-periodic in some directions, are proposed in Sec. 6, where two kinds of what are called the quasi-internal variable models (QIV-models) are introduced. The investigations are related to composites with perfectly bonded constituents and are carried out in the framework of the small displacement gradient theory.

Throughout the paper all capital Roman superscripts run over $1, \ldots, N$ (summation convention holds unless otherwise stated). Points of the physical space E are denoted by $\mathbf{x}$, $\mathbf{y}$ or $\mathbf{z}$ and their distance by $\|\mathbf{x}-\mathbf{y}\|$. The letter t stands for the time coordinate and $t \in [t_0, t_f]$. By $|\ \ |$ we define both the absolute value of a real number and the length of a vector. It is assumed that all introduced functions satisfy the regularity conditions required in the subsequent analysis.

2. Analytical preliminaries

Let Ω be a region in the Euclidean 3-space E occupied by the composite solid in the reference configuration. Setting $V := (-l_1/2, l_1/2) \times (-l_2/2, l_2/2) \times (-l_3/2, l_3/2)$ we assume that the solid in this configuration has the V-periodic heterogeneous structure (is V-periodic) and that the microstructure length parameter defined by $l := \sqrt{l_1^2 + l_2^2 + l_3^2}$ is negligibly small as compared to the smallest characteristic length dimension L_Ω of Ω. We shall use the notation $V(\mathbf{x}) = \mathbf{x} + V$; if $V(\mathbf{x}) \subset \Omega$ then $V(\mathbf{x})$ will be called the cell or the volume element of Ω. The set $\Omega_0 := \{\mathbf{x} \in \Omega;\ \ V(\mathbf{x}) \subset \Omega\}$ is said to be the macro-interior of Ω. For an arbitrary integrable function $f(\cdot)$, defined almost everywhere on

Ω, we define the averaged value of $f(\cdot)$ on $V(\mathbf{x})$ by means of

$$(2.1)\qquad < f(\mathbf{z}) > (\mathbf{x}) = \frac{1}{l_1 l_2 l_3} \int\limits_{V(\mathbf{x})} f(\mathbf{z})\, dv(\mathbf{z}), \qquad \mathbf{x} \in \Omega_0 .$$

If $f(\cdot)$ is a V-periodic function then $< f(\mathbf{z}) > (\mathbf{x})$ is a constant which will be denoted by $< f >$. Now we shall recall two auxiliary concepts which will be used in the subsequent analysis, [32].

Let $\Phi(\cdot)$ be a real-valued function defined on Ω, which represents a certain scalar field. Let us assume that the values of this field in the problem under consideration have to be calculated and/or measured up to a certain tolerance determined by the tolerance parameter ε_Φ, $\varepsilon_\Phi > 0$. It means that an arbitrary real number Φ_0 satisfying condition

$$|\Phi(\mathbf{x}) - \Phi_0| < \varepsilon_\Phi$$

can be also treated as describing with sufficient accuracy the value of this field at the point $\mathbf{x}$. The triple $(\Phi(\cdot), \varepsilon_\Phi, l)$ will be called the *ε-macrofunction* (related to the region Ω) if the following condition holds

$$(\forall(\mathbf{x}, \mathbf{y}) \in \Omega^2)\ [||\mathbf{x} - \mathbf{y}|| < l \ \Rightarrow\ |\Phi(\mathbf{x}) - \Phi(\mathbf{y})| < \varepsilon_\Phi].$$

Roughly speaking, from both the calculation and measurement viewpoints, every ε-macrofunction restricted to an arbitrary cell $V(\mathbf{x})$, $\mathbf{x} \in \Omega_0$, can be treated as constant. Now assume that $\Phi(\cdot, t)$, $t \in [t_0, t_f]$, for every t is a differentiable function defined on Ω, having piecewise continuous time derivatives. Moreover, let Ψ stand for Φ as well as for an arbitrary derivative of Φ and assume that the value of Ψ has to be calculated and/or measured up to a certain tolerance given by the tolerance parameter ε_Ψ. If every triple $(\Psi(\cdot), \varepsilon_\Psi, l)$ is the ε-macrofunction, then the n-tuple $(\Phi(\cdot), \varepsilon_\Phi, \varepsilon_{\nabla\Phi}, \varepsilon_{\dot{\Phi}}, \ldots, l)$ is said to be the regular ε-macrofunction (related to the region Ω). In the sequel we shall tacitly assume that all tolerance parameters ε_Φ, $\varepsilon_{\nabla\Phi}$, $\varepsilon_{\dot{\Phi}}$, ..., as well as the microstructure length parameter l are known and hence $\Phi(\cdot)$ will be referred to as the regular ε-macrofunction. This concept will be also extended to vector and tensor functions by assuming that all their components in an arbitrary coordinate system are regular ε-macrofunctions.

To the concept of ε-macrofunction certain approximations are strictly related which will be used in this contribution. Let $f(\cdot)$ be an integrable function defined almost everywhere on Ω and $\Phi(\cdot)$ stand for an arbitrary ε-macrofunction (the tolerance parameter ε_Φ as well as the microstructure length parameter l are assumed to be known). Denote by $\mathcal{O}(\varepsilon_\Phi)$ a set of possible local increments $\Delta\Phi$ of Φ such that $|\Delta\Phi| < \varepsilon_\Phi$. Due to the meaning of the ε-macrofunction in calculations of integrals of the form

$$\int\limits_{V(\mathbf{x})} f(\mathbf{z})[\Phi(\mathbf{z}) + \mathcal{O}(\varepsilon_\Phi)]dv, \qquad \mathbf{x} \in \Omega_0 ,$$

terms $\mathcal{O}(\varepsilon_\Phi)$ can be neglected. This statement will be called the *Macro-Averaging Approximation* (MAA). Using the MAA we assign to every $f(\cdot)$ the tolerance relation $\approx$ (i.e., the binary relation which is reflexive and symmetric) defined on a set of integrals over $V(\mathbf{x})$ and given by

$$\int\limits_{V(\mathbf{x})} f(\mathbf{z})[\Phi(\mathbf{z}) + \mathcal{O}(\varepsilon_\Phi)]dv \approx \int\limits_{V(\mathbf{x})} f(\mathbf{z})\Phi(\mathbf{z})\,dv, \qquad \mathbf{x} \in \Omega_0. \tag{2.2}$$

Since

$$\int\limits_{V(\mathbf{x})} f(\mathbf{z})dv\Phi(\mathbf{x}) = \int\limits_{V(\mathbf{x})} f(\mathbf{z})[\Phi(\mathbf{z}) + \mathcal{O}(\varepsilon_\Phi)]dv,$$

where now $\mathcal{O}(\varepsilon_\Phi) = \Phi(\mathbf{x}) - \Phi(\mathbf{z})$ for every $\mathbf{z} \in V(\mathbf{x})$, then (2.1) yields

$$\int\limits_{V(\mathbf{x})} f(\mathbf{z})\Phi(\mathbf{z})dv \approx \int\limits_{V(\mathbf{x})} f(\mathbf{z})dv\Phi(\mathbf{z}), \qquad \mathbf{x} \in \Omega_0. \tag{2.3}$$

It has to be emphasized that terms $\mathcal{O}(\varepsilon_\Phi)$ will be neglected only in the course of averaging procedure, i.e., only in the tolerance relations of the form (2.2). Using the notation $\approx$ in (2.2) and (2.3), we have tacitly assumed that every tolerance relation $\approx$ is assigned to a certain integrable function $f(\cdot)$ and is not transitive. It means that in the formula

$$\int\limits_{V(\mathbf{x})} f(\mathbf{z})\Phi_1(\mathbf{z})\Phi_2(\mathbf{z})dv \approx \int\limits_{V(\mathbf{x})} f(\mathbf{z})\Phi_1(\mathbf{z})dv\Phi_2(\mathbf{x}) \approx \int\limits_{V(\mathbf{x})} f(\mathbf{z})dv\Phi_1(\mathbf{x})\Phi_2(\mathbf{x}), \tag{2.4}$$

where $\Phi_1(\cdot)$, $\Phi_2(\cdot)$ are ε-macrofunctions, symbol $\approx$ stands for two different tolerance relations.

In order to introduce the second auxiliary concept used in the subsequent analysis, define by $h^A(\cdot)$, $A = 1, 2, \ldots$, the system of linear independent continuous V-periodic functions (and hence defined on E) having continuous first-order derivatives. Let the above functions satisfy conditions

$$< h^A >= 0, \qquad < h^A h^B >= \delta^{AB} l^2, \qquad (\forall \mathbf{x}) \left[|h^A(\mathbf{x})| \leq l \right]$$

and constitute a basis in the space of sufficiently regular functions defined on an arbitrary cell $V(\mathbf{x})$ and having on $V(\mathbf{x})$ the averaged values equal to zero. We also assume that for every $h^A(\cdot)$ there exists a V-periodic lattice Λ_A of points in E such that $h^A/_{\partial V(\mathbf{x})} = 0$ for every $\mathbf{x} \in \Lambda_A$. It means that $(\forall h^A)((\exists \mathbf{x})[h^A/_{\partial V(\mathbf{x})} = 0]$. Under the aforementioned conditions the system $h^A(\cdot)$, $A = 1, 2, \ldots$, will be called *the local oscillation basis*.

The concepts of the regular ε-macrofunction and the local oscillation basis as well as the macro-averaging approximation (MAA) formulated above constitute the fundamental tools of the micromechanical approach to the macrodynamics of composites which will be proposed in Secs. 3–5 of this contribution.

In Sec. 6 the aforementioned concepts and definitions will be adapted to the cases in which the composite solids have the periodic heterogeneous structures only in one or two directions. Setting $\Omega = \Pi \times (0, H)$, where Π stands for the plane region, we shall deal in Sec. 6 with ε-macrofunctions related to Π or $(0, H)$. Similarly, the functions $h^A(\cdot)$ will be defined either on $\mathbf{R}^2$ or on $\mathbf{R}$, and the averaging operation (2.1) will be restricted either to the area element or to the straight-line element.

3. Foundations of kinematics

Let $\mathbf{u}(\cdot, t)$ stand for a displacement field defined on Ω for every instant t. Define on Ω_0 the averaged displacement field by means of

$$\mathbf{U}(\mathbf{x}, t) := <\mathbf{u}(\mathbf{z}, t)>(\mathbf{x}), \qquad \mathbf{x} \in \Omega_0 . \tag{3.1}$$

By the local displacement oscillations we shall mean the vector functions $\mathbf{w}_\mathbf{x}(\cdot, t)$ defined independently on every $V(\mathbf{x})$, $\mathbf{x} \in \Omega_0$, such that

$$\mathbf{w}_\mathbf{x}(\mathbf{y}, t) = \mathbf{u}(\mathbf{y}, t) - \mathbf{U}(\mathbf{y}, t) + \mathbf{r}_\mathbf{x}(\mathbf{y}, t), \qquad \mathbf{y} \in V(\mathbf{x}), \tag{3.2}$$

where $\mathbf{r}_\mathbf{x}(\cdot)$ satisfy condition

$$<\mathbf{r}_\mathbf{x}(\mathbf{z}, t)> (\mathbf{y}) = <\mathbf{U}(\mathbf{z}, t)>(\mathbf{x}) - \mathbf{U}(\mathbf{x}, t),$$

and will be specified at the end of this section. It can be seen that

$$<\mathbf{w}_\mathbf{x}(\mathbf{z}, t)>(\mathbf{x}) = 0, \qquad \mathbf{x} \in \Omega_0 ,$$

and hence, under the known regularity conditions, every function $\mathbf{w}_\mathbf{x}(\cdot, t)$ can be represented by the Fourier series in the local oscillation basis $h^A(\cdot)$, $A = 1, 2, \ldots$. Denoting the Fourier coefficients by

$$\mathbf{W}^A(\mathbf{x}, t) := <\mathbf{w}_\mathbf{x}(\mathbf{z}, t) h^A(\mathbf{z})>(\mathbf{x}) l^{-2}, \qquad \mathbf{x} \in \Omega_0 , \tag{3.3}$$

we obtain

$$\mathbf{w}_\mathbf{x}(\mathbf{y}, t) = \sum_{A=1}^{\infty} \mathbf{W}^A(\mathbf{x}, t) h^A(\mathbf{y}), \qquad \mathbf{y} \in V(\mathbf{x}), \quad \mathbf{x} \in \Omega_0 . \tag{3.4}$$

The kinematics of the composites under consideration will be based on two assumptions.

Truncation Assumption (TA) states that the Fourier series (3.4) can be approximated by the sum of the first N terms for some $N \geq 1$, where N has to be specified in every problem under consideration.

From TA it follows that instead of (3.4) we assume

$$\mathbf{w}_{\mathbf{x}}(\mathbf{y},t) = \mathbf{W}^A(\mathbf{x},t)h^A(\mathbf{y}), \qquad \mathbf{y} \in V(\mathbf{x}), \quad \mathbf{x} \in \Omega_0, \tag{3.5}$$

where here and in the sequel the Roman superscripts run over $1, ..., N$ (summation convention holds). The functions $h^A(\cdot)$, $A = 1, 2, ..., N$ are called *the micro-shape functions*.

Kinematic Macro-Regularity Assumption (KRA) restricts the class of motions in every problem under consideration by assuming that fields $\mathbf{U}(\cdot,t)$, $\mathbf{W}^A(\cdot,t)$, $A = 1, 2, ..., N$, are regular ε-macrofunctions.

It can be seen that the formulation of KRA takes into account TA by means of which the number N of the micro-shape function is postulated in every problem under consideration. Under the KRA, fields $\mathbf{U}(\cdot,t)$, $\mathbf{W}^A(\cdot,t)$, $A = 1, 2, ..., N$, are said to be *the macrodisplacements and the macro-internal variables* (MIV), respectively. The meaning of the term MIV will be explained in Sec. 5. The ε-macrofunctions $\mathbf{U}(\cdot,t)$, $\mathbf{W}^A(\cdot,t)$ describe the kinematics of composites on the macro-level (macro-kinematics) and will constitute the basic kinematic unknowns in the framework of the proposed model. The results of this section are summarized by the following lemma.

LEMMA. Under TA the displacements on the micro-level are related to the macrodisplacements and the macro-internal variables by the formulae

$$\mathbf{u}(\mathbf{y},t) = \mathbf{U}(\mathbf{y},t) + h^A(\mathbf{y})\mathbf{W}^A(\mathbf{y},t). \tag{3.6}$$

The proof of the above lemma is based on the specification of fields $\mathbf{r}_{\mathbf{x}}(\cdot,t)$ in (3.2) to the form

$$\mathbf{r}_{\mathbf{x}}(\mathbf{y},t) = h^A(\mathbf{y})\left[\mathbf{W}^A(\mathbf{x},t) - \mathbf{W}^A(\mathbf{y},t)\right], \qquad \mathbf{y} \in V(\mathbf{x}), \quad A = 1, ..., N. \tag{3.7}$$

Substituting the right-hand sides of (3.7) into (3.2) and using (3.5) we arrive at (3.6), which ends the proof.

COROLLARY. From the above lemma it follows that the MIV are related to the displacements $\mathbf{u}(\cdot,t)$ by means of the system of equations

$$< h^A(\mathbf{z})h^B(\mathbf{z})\mathbf{W}^B(\mathbf{z},t) >(\mathbf{x}) = < [\mathbf{u}(\mathbf{z},t) - \mathbf{U}(\mathbf{z},t)]h^A(\mathbf{z}) >(\mathbf{x}), \qquad \mathbf{x} \in \Omega_0,$$

which under KRA and MAA can be replaced by

$$\mathbf{W}^A(\mathbf{x},t) > = < \mathbf{u}(\mathbf{z},t)h^A(\mathbf{z}) >(\mathbf{x}), \qquad \mathbf{x} \in \Omega_0. \tag{3.8}$$

Formula (3.8) yields the simple interpretation of the macro-internal variables as certain weighted averages of displacements.

4. From micro- to macrodynamics

Let $\mathbf{s}(\cdot, t)$ denote the Cauchy stress tensor field defined for every t on $\Omega \backslash \Gamma$ where Γ is a set of all interfaces between the components of the composite. Let us define on Ω_0 the following averaged stress fields

$$
\begin{aligned}
\mathbf{S}(\mathbf{x}, t) &:= <\mathbf{s}(\mathbf{z}, t)>(\mathbf{x}), \\
\mathbf{H}^A(\mathbf{x}, t) &:= <\mathbf{s}(\mathbf{z}, t) \cdot \nabla h^A(\mathbf{z})>(\mathbf{x}), \qquad \mathbf{x} \in \Omega_0 .
\end{aligned} \tag{4.1}
$$

In order to pass from micro- to macrodynamics, two extra assumptions will be required.

Stress Macro-Regularity Assumption (SRA) restricts the class of stress fields in the problem under consideration to that in which the fields $\mathbf{S}(\cdot, t)$, $\mathbf{H}^A(\cdot, t)$ are regular ε-macrofunctions.

Under SRA the fields defined by (4.1) will be called *the macrostresses* and *the micro-dynamic forces*, respectively. The meaning of the latter term will be explained at the end of this section.

Let $\varrho(\cdot)$ stand for the mass density field (which is the V-periodic function defined almost everywhere on Ω), and assume that the body force $\mathbf{b}$ is constant. Let us denote by $\mathbf{n}(\mathbf{y})$ the unit normal outward to $\partial V(\mathbf{x})$ at $\mathbf{y}$. The starting point of the proposed micromechanical procedure will be the weak form of equations of motion in micromechanics. Taking into account the symmetry of the stress tensor, these equations can be assumed in the form of conditions

$$
\begin{aligned}
\int\limits_{V(\mathbf{x})} \mathbf{s}(\mathbf{y}, t) : \nabla \overline{\mathbf{u}}(\mathbf{y})\, dv = \oint\limits_{\partial V(\mathbf{x})} [\mathbf{s}(\mathbf{y}, t) \cdot \mathbf{n}(\mathbf{y})] \cdot \overline{\mathbf{u}}(\mathbf{y})\, da \\
+ \int\limits_{V(\mathbf{x})} \varrho(\mathbf{y}) \left[\mathbf{b} - \ddot{\mathbf{u}}\, (\mathbf{y}, t)\right] \cdot \overline{\mathbf{u}}(\mathbf{y})\, dv
\end{aligned} \tag{4.2}
$$

which have to hold for every $\mathbf{x} \in \Omega_0$ and for an arbitrary test function $\overline{\mathbf{u}}(\cdot)$. In order to pass from micro- to macrodynamics we have to specify the set of test functions in (4.2). Taking into account (3.6) we assume that

$$
\overline{\mathbf{u}}(\mathbf{y}) = \overline{\mathbf{U}}(\mathbf{y}) + h^A(\mathbf{y}) \overline{\mathbf{W}}^A(\mathbf{y}), \tag{4.3}
$$

where $\overline{\mathbf{U}}(\cdot)$, $\overline{\mathbf{W}}^A(\cdot)$ are arbitrary linearly independent regular ε-macrofunctions defined on Ω.

Now we shall explain the meaning of the micro-shape functions in the modelling procedure. To this end denote $\Gamma(\mathbf{x}) := V(\mathbf{x}) \cap \Gamma$, $\mathbf{x} \in \Omega_0$. Hence $\Gamma(\mathbf{x})$ is a set of all interfaces in the volume element $V(\mathbf{x})$ across which the tensor $\mathbf{s}(\cdot, t)$ can suffer jump discontinuities. Let us also introduce the residual fields in every cell $V(\mathbf{x})$ given by

$$\begin{aligned} \mathbf{r}(\mathbf{y},t) &:= \varrho(\mathbf{y})\,\ddot{\mathbf{u}}\,(\mathbf{y},t) - \varrho(\mathbf{y})\mathbf{b} - \operatorname{Div}\mathbf{s}(\mathbf{y},t) && \text{if} \quad \mathbf{y}\in V(\mathbf{x})\backslash\Gamma(\mathbf{x}), \\ \mathbf{j}(\mathbf{y},t) &:= [\mathbf{s}](\mathbf{y},t)\cdot\mathbf{n}(\mathbf{y}) && \text{if} \quad \mathbf{y}\in\Gamma(\mathbf{x}), \quad \mathbf{x}\in\Omega_0, \end{aligned}$$

where $[\mathbf{s}]$ is a jump of the stress tensor across $\Gamma(\mathbf{x})$ in the direction of the unit normal $\mathbf{n}(\mathbf{y})$ to $\Gamma(\mathbf{x})$. Obviously, in the framework of micromechanics the residual fields are identically equal to zero. It can be verified that, under the above notations, Eqs. (4.2) can be written in the equivalent form of conditions

$$\int\limits_{V(\mathbf{x})} \mathbf{r}\cdot\overline{\mathbf{u}}\,dv + \int\limits_{\Gamma(\mathbf{x})} \mathbf{j}\cdot\overline{\mathbf{u}}\,da = 0, \qquad \mathbf{x}\in\Omega_0,$$

which have to hold for an arbitrary test function $\overline{\mathbf{u}}(\cdot)$ defined on Ω. In the framework of the proposed approach, we shall assume that the piecewise constant (discontinuous across Γ) distribution of heterogeneity is approximated in the vicinity of interfaces by the continuous one. Hence $\Gamma = \emptyset$ and the integrals over $\Gamma(\mathbf{x})$ drop out. Taking into account (4.3) and using MAA, the above conditions reduce to the following ones

$$\int\limits_{V(\mathbf{x})} \mathbf{r}\,dv = 0, \qquad \int\limits_{V(\mathbf{x})} \mathbf{r}h^A\,dv = 0, \qquad A = 1,\dots,N, \quad \mathbf{x}\in\Omega_0.$$

The second of the above formulae represents the interrelation between the residual field $\mathbf{r}$ and the form and number of the micro-shape functions. More general models will be presented in the separate paper.

Taking into account the above approximation we shall formulate the fundamental assertion of the micromechanical approach to macrodynamics proposed in this contribution. For the sake of simplicity we shall also assume that the micro-shape functions $h^A(\cdot)$ satisfy the extra conditions $<\varrho h^A>=0$, $A = 1,\dots,N$.

Fundamental Assertion. Under TA, KRA, SRA and in the framework of MAA, the equations of motion (4.2) imply the following interrelation between the macrodisplacements $\mathbf{U}(\cdot,t)$ and the macrostresses $\mathbf{S}(\cdot,t)$:

$$\operatorname{Div}\mathbf{S}(\mathbf{x},t) - <\varrho>\ddot{\mathbf{U}}\,(\mathbf{x},t) + <\varrho>\mathbf{b} = 0, \tag{4.4}$$

as well as between the macro-internal parameters $\mathbf{W}^A(\cdot,t)$ and the micro-dynamic forces $\mathbf{H}^A(\cdot,t)$:

$$<\varrho h^A h^B>\ddot{\mathbf{W}}^B(\mathbf{x},t) + \mathbf{H}^A(\mathbf{x},t) = 0, \qquad A = 1,\dots,N; \tag{4.5}$$

the above relations hold for every $\mathbf{x}\in\Omega_0$ and $t\in(t_0,t_f)$.

The relations given by (4.4), (4.5) will be called *the equations of motion* and *the dynamic evolution equations*, respectively. Since the moduli $< \varrho h^A h^B >$ are of the order $\mathcal{O}(l^2)$, then the above equations describe the effect of the microstructure size l on the body dynamic response. For this reason Eqs. (4.4), (4.5) are said to represent *the refined macrodynamics* of the composites under consideration, [30]. Let us observe that both in the quasi-stationary processes and for the problems in which the above effect can be neglected we obtain $\mathbf{H}^A(\mathbf{x}, t) = 0$. That is why the fields defined by the second of Eqs. (4.1) were called the micro-dynamic forces. It has to be emphasized that under the aforementioned assumptions, Eqs. (4.4), (4.5) are related to a composite solid with periodic microstructure made of arbitrary materials. Hence, the aforementioned equations represent the averaged laws of motion in the framework of the proposed refined macrodynamics.

At the end of this section we shall prove the fundamental assertion. To this end let us substitute the right-hand side of (4.3) into (4.2). Using (2.1) we obtain

$$
\begin{aligned}
&\int\limits_{V(\mathbf{x})} \mathbf{s}(\mathbf{y},t) : \nabla\overline{\mathbf{u}}(\mathbf{y})\,dv = \int\limits_{V(\mathbf{x})} \mathbf{s}(\mathbf{y},t) : \nabla\overline{\mathbf{U}}(\mathbf{y})\,dv \\
&\qquad\qquad + \int\limits_{V(\mathbf{x})} \left[\nabla h^A(\mathbf{y})\cdot\mathbf{s}(\mathbf{y},t)\right]\cdot\overline{\mathbf{W}}^A(\mathbf{y})\,dv, \\
&\oint\limits_{\partial V(\mathbf{x})} [\mathbf{s}(\mathbf{y},t)\cdot\mathbf{n}(\mathbf{y})]\cdot\overline{\mathbf{u}}(\mathbf{y})\,da \approx \oint\limits_{\partial V(\mathbf{x})} [\mathbf{s}(\mathbf{y},t)\cdot\mathbf{n}(\mathbf{y})]\cdot\overline{\mathbf{U}}(\mathbf{y})\,da \\
&\qquad\qquad + \oint\limits_{\partial V(\mathbf{x})} h^A(\mathbf{y})\,[\mathbf{s}(\mathbf{y},t)\cdot\mathbf{n}(\mathbf{y})]\cdot\overline{\mathbf{W}}^A(\mathbf{y})\,da, \\
&\int\limits_{V(\mathbf{x})} \varrho(\mathbf{y})\left[\mathbf{b}-\ddot{\mathbf{u}}(\mathbf{y},t)\right]\cdot\overline{\mathbf{u}}(\mathbf{y})\,dv \approx \int\limits_{V(\mathbf{x})} \varrho(\mathbf{y})\left[\mathbf{b}-\ddot{\mathbf{u}}(\mathbf{y},t)\right]\cdot\overline{\mathbf{U}}(\mathbf{y})\,dv \\
&\qquad\qquad + \int\limits_{V(\mathbf{x})} \varrho(\mathbf{y})h^A(\mathbf{y})\left[\mathbf{b}-\ddot{\mathbf{u}}(\mathbf{y},t)\right]\cdot\overline{\mathbf{W}}^A(\mathbf{y})\,dv.
\end{aligned}
\tag{4.6}
$$

Combining (4.6) and (4.2) and bearing in mind that $\overline{\mathbf{U}}(\cdot)$, $\overline{\mathbf{W}}^A(\cdot)$ are linearly independent ε-macrofunctions, we conclude that the equations of motion (4.2) imply the following conditions

$$
\begin{aligned}
&\int\limits_{V(\mathbf{x})} \mathbf{s}(\mathbf{y},t) : \nabla\overline{\mathbf{U}}(\mathbf{y})\,dv = \oint\limits_{\partial V(\mathbf{x})} [\mathbf{s}(\mathbf{y},t)\cdot\mathbf{n}(\mathbf{y})]\cdot\overline{\mathbf{U}}(\mathbf{y})\,da \\
&\qquad\qquad + \int\limits_{V(\mathbf{x})} \varrho(\mathbf{y})\left[\mathbf{b}-\ddot{\mathbf{u}}(\mathbf{y},t)\right]\cdot\overline{\mathbf{U}}(\mathbf{y})\,dv,
\end{aligned}
\tag{4.7}
$$

$$\int\limits_{V(\mathbf{x})} \left[\mathbf{s}(\mathbf{y},t)\cdot\nabla h^A(\mathbf{y})\right]\cdot\overline{\mathbf{W}}^A(\mathbf{y})\,dv = \oint\limits_{\partial V(\mathbf{x})} [\mathbf{s}(\mathbf{y},t)\cdot\mathbf{n}(\mathbf{y})]\cdot\overline{\mathbf{W}}^A(\mathbf{y})h^A(\mathbf{y})\,da + \int\limits_{V(\mathbf{x})} \varrho(\mathbf{y})\left[\mathbf{b}-\ddot{\mathbf{u}}(\mathbf{y},t)\right]\cdot\overline{\mathbf{W}}^A(\mathbf{y})h^A(\mathbf{y})\,dv, \qquad \mathbf{x}\in\Omega_0, \tag{4.7 [cont.]}$$

which have to hold for arbitrary regular ε-macrofunctions $\overline{\mathbf{U}}(\cdot)$, $\overline{\mathbf{W}}^A(\cdot)$ defined on Ω. By means of (2.2), (2.1) and bearing in mind the remark following (2.3), the integrals on the left-hand sides of (4.7) can be transformed as follows

$$\begin{aligned}
&\int\limits_{V(\mathbf{x})} \mathbf{s} : \nabla\overline{\mathbf{U}}(\mathbf{y})\,dv \approx \int\limits_{V(\mathbf{x})} \mathbf{s}\,dv : \nabla\overline{\mathbf{U}}(\mathbf{x}) = l_1l_2l_3\mathbf{S}(\mathbf{x},t) : \nabla\overline{\mathbf{U}}(\mathbf{x})\\
&\qquad\approx \int\limits_{V(\mathbf{x})} \mathbf{S}\,dv : \nabla\overline{\mathbf{U}}(\mathbf{x}) \approx \int\limits_{V(\mathbf{x})} \mathbf{S} : \nabla\overline{\mathbf{U}}\,dv\\
&\qquad= \oint\limits_{\partial V(\mathbf{x})} (\mathbf{S}\cdot\mathbf{n})\cdot\overline{\mathbf{U}}\,dv - \int\limits_{V(\mathbf{x})} \mathrm{Div}\,\mathbf{S}\cdot\overline{\mathbf{U}}\,dv,\\
&\int\limits_{V(\mathbf{x})} (\mathbf{s}\cdot\nabla h^A)\overline{\mathbf{W}}^A\,dv \approx \int\limits_{V(\mathbf{x})} (\mathbf{s}\cdot\nabla h^A)\,dv\cdot\overline{\mathbf{W}}^A(\mathbf{x}) \approx \int\limits_{V(\mathbf{x})} \mathbf{H}^A\cdot\overline{\mathbf{W}}^A\,dv.
\end{aligned} \tag{4.8}$$

For the sake of simplicity, in (4.8) and in the subsequent formulas the argument $\mathbf{y}\in V(\mathbf{x})$ is not specified in all integrands. Using (2.2) and the first of formulas (3.7) as well as the conditions $<\varrho h^A>= 0$, $<h^A>= 0$, the integrals over $V(\mathbf{x})$ on the right-hand sides of (4.7) can be given

$$\begin{aligned}
&\int\limits_{V(\mathbf{x})} \varrho(\mathbf{b}-\ddot{\mathbf{u}})\cdot\overline{\mathbf{U}}\,dv \approx \int\limits_{V(\mathbf{x})} \varrho\,dv\,\mathbf{b}\cdot\overline{\mathbf{U}}(\mathbf{x}) - \int\limits_{V(\mathbf{x})} \varrho(\ddot{\mathbf{U}}+h^A\,\ddot{\mathbf{W}}^A)\,dv\cdot\overline{\mathbf{U}}(\mathbf{x})\\
&\qquad\approx l_1l_2l_3\left[<\varrho>\mathbf{b}\cdot\overline{\mathbf{U}}(\mathbf{x}) - <\varrho>\ddot{\mathbf{U}}(\mathbf{x},t)\cdot\overline{\mathbf{U}}(\mathbf{x})\right]\\
&\qquad\approx \int\limits_{V(\mathbf{x})} <\varrho>(\mathbf{b}-\ddot{\mathbf{U}})\cdot\overline{\mathbf{U}}\,dv,\\
&\int\limits_{V(\mathbf{x})} \varrho(\mathbf{b}-\ddot{\mathbf{u}})\cdot\overline{\mathbf{W}}^A h^A\,dv \approx \int\limits_{V(\mathbf{x})} \varrho h^A\,dv\,\mathbf{b}\cdot\overline{\mathbf{W}}^A(\mathbf{x})\\
&\qquad- \int\limits_{V(\mathbf{x})} \varrho(\ddot{\mathbf{U}}+h^B\,\ddot{\mathbf{W}}^B)h^A\,dv\cdot\overline{\mathbf{W}}^A(\mathbf{x}) \approx \int\limits_{V(\mathbf{x})} \varrho h^A\,dv\;\ddot{\mathbf{U}}(\mathbf{x},t)\cdot\overline{\mathbf{W}}^A(\mathbf{x})\\
&\qquad- \int\limits_{V(\mathbf{x})} \varrho h^A h^B\,dv\;\ddot{\mathbf{W}}^B(\mathbf{x},t)\cdot\overline{\mathbf{W}}^A(\mathbf{x}) \approx \int\limits_{V(\mathbf{x})} <\varrho h^A h^B>\ddot{\mathbf{W}}^B\cdot\overline{\mathbf{W}}^A\,dv.
\end{aligned} \tag{4.9}$$

Taking into account (4.8) and (4.9) we shall represent (4.7) in the form

$$(4.10)\qquad \begin{aligned} &\int_{V(\mathbf{x})} \Big(\operatorname{Div}\mathbf{S} - <\varrho>\ddot{\mathbf{U}} + <\varrho>\mathbf{b}\Big)\cdot\overline{\mathbf{U}}\,dv - \oint_{\partial V(\mathbf{x})} (\mathbf{S}\cdot\mathbf{n} - \mathbf{s}\cdot\mathbf{n})\cdot\overline{\mathbf{U}}\,da = 0,\\ &\int_{V(\mathbf{x})} \Big(<\varrho h^A h^B>\ddot{\mathbf{W}}^{B} + \mathbf{H}^A\Big)\,\overline{\mathbf{W}}^A\,dv - \oint_{\partial V(\mathbf{x})} (\mathbf{s}\cdot\mathbf{n})h^A\cdot\overline{\mathbf{W}}^{\mathbf{A}}\,da = 0. \end{aligned}$$

The first of equations (4.10) has to hold for every $\overline{\mathbf{U}}(\cdot)$ and the second one for every $\overline{\mathbf{W}}^A(\cdot)$. Moreover, $\overline{\mathbf{U}}(\cdot)$, $\overline{\mathbf{W}}^A(\cdot)$, $\operatorname{Div}\mathbf{S}(\cdot,t)$, $\mathbf{H}^A(\cdot,t)$ and $\ddot{\mathbf{U}}\,(\cdot,t)$, $\ddot{\mathbf{W}}^B\,(\cdot,t)$ are ε-macrofunctions. It follows that using MAA, from Eqs. (4.10) we obtain

$$(4.11)\qquad \begin{aligned} &l_1l_2l_3\left[\operatorname{Div}\mathbf{S}(\mathbf{x},t) - <\varrho>\ddot{\mathbf{U}}\,(\mathbf{x},t) + <\varrho>\mathbf{b}\right]\cdot\overline{\mathbf{U}}(\mathbf{x})\\ &\qquad\qquad - \oint_{\partial V(\mathbf{x})} (\mathbf{S}\cdot\mathbf{n} - \mathbf{s}\cdot\mathbf{n})\cdot\overline{\mathbf{U}}\,da = 0,\\ &l_1l_2l_3\left[<\varrho h^A h^B>\ddot{\mathbf{W}}^{B}(\mathbf{x},t) + \mathbf{H}^A(\mathbf{x},t)\right]\cdot\overline{\mathbf{W}}^A(\mathbf{x})\\ &\qquad\qquad - \oint_{\partial V(\mathbf{x})} (\mathbf{s}\cdot\mathbf{n})\cdot\overline{\mathbf{W}}^A h^A\,da = 0. \end{aligned}$$

Introducing the local coordinate $\mathbf{p}\in V$ we also obtain

$$(4.12)\qquad \begin{aligned} \oint_{\partial V(\mathbf{x})} (\mathbf{S}\cdot\mathbf{n})\cdot\overline{\mathbf{U}}\,da &= \int_{V(\mathbf{x})} \operatorname{Div}(\mathbf{S}\cdot\overline{\mathbf{U}})\,dv \approx l_1l_2l_3\operatorname{Div}\left[\mathbf{S}(\mathbf{x},t)\cdot\overline{\mathbf{U}}(\mathbf{x})\right]\\ &\approx \operatorname{Div}\int_{V(\mathbf{x})} \mathbf{s}\cdot\overline{\mathbf{U}}\,dv = \operatorname{Div}_{\mathbf{x}}\int_V \mathbf{s}(\mathbf{x}+\mathbf{p},t)\cdot\overline{\mathbf{U}}(\mathbf{x}+\mathbf{p},t)\,dv(\mathbf{p})\\ &= \int_V \operatorname{Div}_{\mathbf{x}}\left[\mathbf{s}(\mathbf{x}+\mathbf{p},t)\cdot\overline{\mathbf{U}}(\mathbf{x}+\mathbf{p},t)\right]dv(\mathbf{p})\\ &= \int_{V(\mathbf{x})} \operatorname{Div}(\mathbf{s}\cdot\overline{\mathbf{U}})dv = \oint_{\partial V(\mathbf{x})} (\mathbf{s}\cdot\mathbf{n})\cdot\overline{\mathbf{U}}\,da. \end{aligned}$$

Hence, the surface integral in the first of equations (4.11) can be neglected and we arrive at (4.4). Bearing in mind that $\ddot{\mathbf{W}}^B(\cdot,t)$, $\overline{\mathbf{W}}^A(\cdot)$ and $\mathbf{H}^A(\cdot,t)$ are ε-macrofunctions, we conclude that the surface integrals in the second of equation (4.11) are values of a certain ε-macrofunction; at the same time $h^A = 0$ on $\partial V(\mathbf{x})$ for every $\mathbf{x}\in\Lambda_A$, cf. Sec. 2. It follows that for an arbitrary $\mathbf{x}\in\Omega_0$ the above surface integrals attain the values which in the framework of MAA can be neglected. Hence the second equation (4.11) reduces to (4.5), which ends the proof. An alternative proof of the above fundamental assertion can be found in [35].

5. MIV-model

In order to describe the dynamic response of the composite body in the framework of the MIV-model, we have to complete equations (4.4), (4.5) by introducing the constitutive equations for macrostresses $\mathbf{S}$ and micro-dynamic forces $\mathbf{H}^A$. Taking into account the definitions (4.1), the second of the formulae (3.7) and applying MAA to the integrals in (4.1), this can be done for arbitrary periodic composites the components of which are simple materials. To simplify the subsequent considerations we shall restrict ourselves to the linear visco-elastic materials governed by the constitutive equations of the form

$$\mathbf{s} = \mathbf{C}(\mathbf{z}) : \mathbf{e} + \mathbf{D}(\mathbf{z}) : \dot{\mathbf{e}}, \qquad \mathbf{e} := 0.5\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right], \tag{5.1}$$

where $\mathbf{C}(\cdot)$, $\mathbf{D}(\cdot)$ are V-periodic piecewise constant functions the values of which are the fourth order tensors of elastic and viscous moduli, respectively, for the component materials. Define the linearized macro-strain tensor by means of

$$\mathbf{E}(\mathbf{x}, t) := 0.5\left[\nabla\mathbf{U}(\mathbf{x}, t) + (\nabla\mathbf{U}(\mathbf{x}, t))^T\right]. \tag{5.2}$$

Now, we shall prove that the following formula holds for an arbitrary sufficiently regular V-periodic tensor field $\mathbf{F}(\cdot)$:

$$< \mathbf{F}\cdot\nabla(h^A\mathbf{W}^A) > (\mathbf{x}) \approx < \mathbf{F}\cdot\nabla h^A > \otimes\mathbf{W}^A(\mathbf{x}). \tag{5.3}$$

To this end let us observe that

$$<\mathbf{F}\cdot\nabla h^A> (\mathbf{x}) = -\frac{1}{|\mathbf{V}|}\int\limits_{\Gamma(\mathbf{x})} h^A[\mathbf{F}]\cdot\mathbf{n}\, da - <h^A \operatorname{Div}\mathbf{F}> (\mathbf{x}), \tag{5.4}$$

where $[\mathbf{F}]$ is a jump of $\mathbf{F}$ across all interfaces $\Gamma(\mathbf{x})$, oriented by a unit normal $\mathbf{n}$ in $V(\mathbf{x})$. At the same time, for every A we obtain (no summation over A!)

$$\begin{aligned} <\mathbf{F}\cdot\nabla(h^A\mathbf{W}^A)> (\mathbf{x}) &= <\operatorname{Div}(\mathbf{F}\otimes\mathbf{W}^A h^A)> (\mathbf{x}) - <\operatorname{Div}\mathbf{F}\otimes\mathbf{W}^A h^A> \\ &\approx \frac{1}{|\mathbf{V}|}\oint\limits_{\partial V(\mathbf{x})} h^A(\mathbf{F}\cdot\mathbf{n})\otimes\mathbf{W}^A\, da \\ &+ \left(\frac{1}{|\mathbf{V}|}\int\limits_{\Gamma(\mathbf{x})} h^A[\mathbf{F}]\cdot\mathbf{n}\, da - <h^A\operatorname{Div}\mathbf{F}> (\mathbf{x})\right)\otimes\mathbf{W}^A(\mathbf{x}, t). \end{aligned} \tag{5.5}$$

For every $\mathbf{x}\in\Lambda_A$ the value of the first integral on the right-hand side of (5.5) is equal to zero. At the same time, this integral represents a certain ε-macrofunction defined on Ω_0 (since $\mathbf{W}^A(\cdot, t)$ is the ε-macrofunction) and hence bearing in mind

(5.4), we conclude that (5.3) holds true, which ends the proof. Using this result we also have

$$
(5.6)\qquad <\mathbf{F}\cdot\nabla h^A>\cdot\mathbf{W}^A(\mathbf{x},t)+<\mathbf{F}h^A>:\nabla\mathbf{W}^A(\mathbf{x},t)\approx<\mathbf{F}\cdot\nabla h^A>\cdot\mathbf{W}^A(\mathbf{x},t).
$$

Substituting (5.1) into definitions (4.1), taking into account the second of equations (3.7) and using MAA, by means of (5.6) we obtain

$$
(5.7)\qquad
\begin{aligned}
\mathbf{S}(\mathbf{x},t) = &<\mathbf{C}>:\mathbf{E}(\mathbf{x},t)+<\mathbf{C}\cdot\nabla h^A>\cdot\mathbf{W}^A(\mathbf{x},t)\\
&+<\mathbf{D}>:\dot{\mathbf{E}}(\mathbf{x},t)+<\mathbf{D}\cdot\nabla h^A>\cdot\dot{\mathbf{W}}^A(\mathbf{x},t),\\
\mathbf{H}^A(\mathbf{x},t) = &<\nabla h^A\cdot\mathbf{C}>:\mathbf{E}(\mathbf{x},t)+<\nabla h^A\cdot\mathbf{C}\cdot\nabla h^B>\cdot\mathbf{W}^B(\mathbf{x},t)\\
&+<\nabla h^A\cdot\mathbf{D}>:\dot{\mathbf{E}}(\mathbf{x},t)+<\nabla h^A\cdot\mathbf{D}\cdot\nabla h^B>\cdot\dot{\mathbf{W}}^B(\mathbf{x},t),
\end{aligned}
$$

for every $\mathbf{x}\in\Omega_0$ and $t\in(t_0,t_f)$. The above equations will be referred to as the macro-constitutive equations for the linear visco-elastic composites.

Equations (4.4), (4.5) and (5.2), (5.7) represent the macro-internal variable model (MIV-model) of micro-periodic composites made of perfectly bonded visco-elastic constituents. For the linear elastic materials the above equations reduce to those of the refined macromechanics, which were obtained independently in [30] by means of certain heuristic hypotheses. For every micro-periodic composite solid (with constituents modelled as simple materials) the proposed model is uniquely determined by the choice of the micro-shape functions $h^A(\cdot)$, $A=1,\ldots,N$.

It has to be emphasized that for every class of motions specified by conditions (3.6), we obtain the pertinent MIV-model. In the analysis of special problems we have to take into account only these classes of motions which seem to be relevant from the viewpoint of the engineering applications of the theory.

Substituting the right-hand sides of Eqs. (5.7) into (4.4), (4.5), we obtain the system of three partial differential equations for the macrodisplacements $\mathbf{U}$ coupled with the system of $3N$ ordinary differential equations for the macro-internal variables $\mathbf{W}^A$. Hence, in formulations of the initial-boundary value problems, unknowns $\mathbf{W}^A(\cdot,t)$ do not enter the boundary conditions. That is why they were called the macro-internal variables (MIV). It can be shown that for homogeneous bodies and homogeneous initial conditions for MIV, we obtain the trivial solution $\mathbf{W}^A=\mathbf{0}$, $A=1,\ldots,N$, to every boundary value problem. Hence, the macro-internal variables play a crucial role in a description of the dynamic behaviour of solids with periodic microstructure, and that is why the models proposed were referred to as the macro-internal variable models. It has to be emphasized that solutions to special problems in the framework of MIV-models have the physical sense only if the fields $\mathbf{U}(\cdot,t)$, $\mathbf{W}^A(\cdot,t)$ as well as $\mathbf{S}(\cdot,t)$, $\mathbf{H}^A(\cdot,t)$, for every instant t, are the regular ε-macrofunctions. This requirement can be verified only *a posteriori*.

6. QIV-models

The MIV-model can be applied to composites which are periodic along every coordinate axis appearing in the problem under consideration. It means that the above models are applicable also to plane problems or to one-dimensional problems, provided that the corresponding basic cell of the composite is plane or one-dimensional, respectively. However, MIV-models cannot be applied if the dimension of the cell is smaller than the number of spatial coordinates in the problem under consideration. In this case the micro-shape functions are independent of some of the spatial coordinates, and hence the surface integral in the second Eqs. (4.11) cannot be neglected and the formula (5.3) does not hold. In this section we shall modify the previously obtained results for two important special types of composite materials.

6.1. Composites reinforced by a system of parallel fibres

Let $\Omega = \Pi \times (0, H)$, where Π is a regular region on the plane $0x_1x_2$. Assume that the composite has a material structure periodic only in the directions of the x_1-axis and x_2-axis. Let l_1, l_2 stand for the corresponding periods and define $A \equiv (-l_1/2, l_1/2) \times (-l_2/2, l_2/2)$. Such situation takes place, e.g., for composites reinforced by a system of periodically distributed fibres parallel to the x_3-axis. In these cases we shall deal with the A-periodic composites. Setting $\mathbf{X} := (x_1, x_2)$ and $A(\mathbf{X}) := \mathbf{X} + A$, we define the averaging operator on $A(\mathbf{X})$ given by

$$< f(\mathbf{z}, t) > (\mathbf{x}) := \frac{1}{l_1 l_2} \int\limits_{A(\mathbf{X})} f(z_1, z_2, x_3, t)\, dz_1\, dz_2\,, \tag{6.1}$$

which will be used throughout this subsection. Denoting $l = \sqrt{(l_1)^2 + (l_2)^2}$ we introduce the concept of ε-macrofunction (related to the region Π) as a function $\Phi(\cdot)$ defined on Π and such that

$$(\forall\ (\mathbf{X}, \mathbf{Y}) \in (\Pi)^2)[\|\mathbf{X} - \mathbf{Y}\| < l \quad \Rightarrow \quad |\Phi(\mathbf{X}) - \Phi(\mathbf{Y})| < \varepsilon_\Phi].$$

A function Φ can also depend on x_3 and/or t as parameters. Moreover, the micro-shape functions $h^A(\cdot)$ are now independent of the x_3-coordinate since the disturbances in displacements caused by the micro-periodic heterogeneity of the medium take place only in the x_α-axes directions, $\alpha = 1, 2$. The equations of motion for the class of composites under consideration can be obtained from Eqs. (4.11) by setting $V = A \times (-\delta, \delta)$, $\delta > 0$ and passing with δ to zero, $\delta \to 0$. Let $\mathbf{a}_3$ be the versor (unit vector) of the x_3-axis. Introducing kinematic fields averaged over $A(\mathbf{X})$

$$\mathbf{U}(\mathbf{x}, t) := <\mathbf{u}(\mathbf{z}, t)>(\mathbf{x}), \qquad \mathbf{W}^A(\mathbf{x}, t) := <\mathbf{w}_{\mathbf{x}}(\mathbf{z}, t) h^A(\mathbf{Z})>(\mathbf{x}) l^{-2}, \tag{6.1'}$$

and the stress fields averaged over $A(\mathbf{X})$

$$
\begin{aligned}
\mathbf{S}(\mathbf{x},t) &:= <\mathbf{s}(\mathbf{z},t)>(\mathbf{x}),\\
\mathbf{H}^A(\mathbf{x},t) &:= <\mathbf{s}(\mathbf{z},t)\cdot\nabla h^A(\mathbf{Z})>(\mathbf{x}),\\
\mathbf{R}^A{}_3(\mathbf{x},t) &:= <\mathbf{a}_3\cdot\mathbf{s}(\mathbf{z},t)h^A(\mathbf{Z})>(\mathbf{x}),
\end{aligned}
\tag{6.2}
$$

we shall assume that $\mathbf{U}(\cdot,x_3,t),\ldots,\mathbf{R}^A{}_3(\cdot,x_3,t)$ are regular ε-macrofunctions (related to Π) for every $x_3\in(0,H)$, $t\in(t_0,t_f)$. From (4.11), after some manipulations we obtain

$$
\begin{aligned}
&\operatorname{Div}\mathbf{S}(\mathbf{x},t)-<\varrho>(x_3)\,\ddot{\mathbf{U}}\,(\mathbf{x},t)+<\varrho>(x_3)\mathbf{b} = \mathbf{0},\\
&-\mathbf{R}^A{}_{3,3}(\mathbf{x},t)+<\varrho h^A h^B>(x_3)\,\ddot{\mathbf{W}}^{\mathbf{B}}\,(\mathbf{x},t)+\mathbf{H}^A(\mathbf{x},t) = \mathbf{0}.
\end{aligned}
\tag{6.3}
$$

For the sake of simplicity let us confine ourselves to the elastic materials, setting $\mathbf{s}=\mathbf{C}(\mathbf{X},x_3)\colon\mathbf{e}$, $\mathbf{e}=\operatorname{sym}\nabla\mathbf{u}$, where $\mathbf{C}(\cdot,x_3)$ is A-periodic for every $x_3\in(0,H)$. Substituting the right-hand side of the above constitutive relations into (6.2), by means of $\nabla\mathbf{u}=\mathbf{U}+h^A\nabla\mathbf{W}+\nabla h^A\otimes\mathbf{W}$, and using the procedure given in Sec. 5, we obtain

$$
\begin{bmatrix}\mathbf{S}\\ \mathbf{H}^{\mathbf{A}}\\ \mathbf{R}^A{}_3\end{bmatrix}
=
\begin{bmatrix}
<\mathbf{C}>, & <\mathbf{C}\cdot\nabla h^B>, & <\mathbf{C}\cdot\mathbf{a}_3h^B>\\
<\nabla h^A\cdot\mathbf{C}>, & <\nabla h^A\cdot\mathbf{C}\cdot\nabla h^B>, & <\nabla h^A\cdot\mathbf{C}\cdot\mathbf{a}_3h^B>\\
<h^A\mathbf{a}_3\cdot\mathbf{C}>, & <h^A\mathbf{a}_3\cdot\mathbf{C}\cdot\nabla h^B>, & <h^A\mathbf{a}_3\cdot\mathbf{C}\cdot\mathbf{a}_3h^B>
\end{bmatrix}
\times
\begin{bmatrix}\mathbf{E}\\ \mathbf{W}^{\mathbf{B}}\\ \mathbf{W}^{\mathbf{B}}{}_{,3}\end{bmatrix},
\tag{6.4}
$$

where the elements of the above matrix can be functions of an argument x_3. Equations (6.3), (6.4) together with (5.2) represent a model of the class of composites under consideration. In general we now deal with composites which can be non-periodic in the x_3-axis direction; if $\varrho(\cdot)$ and $\mathbf{C}(\cdot)$ are independent of x_3 then (6.3), (6.4) are equations with constant coefficients.

Substituting the right-hand sides of Eqs. (6.4) into Eqs. (6.3) we obtain the system of three partial differential equations of the second order for macro-displacements $\mathbf{U}$ coupled with the system of $3N$ equations for $\mathbf{W}^A$. The latter are the second order partial differential equations with respect to arguments t and x_3. Hence, in formulations of boundary-value problems, we have to prescribe on $\partial[\Pi\times(0,H)]$ three conditions for three components of $\mathbf{U}$. At the same time on $\Pi\times\{0\}$ and $\Pi\times\{H\}$ we have also to introduce $3N$ extra boundary conditions for $3N$ components of $\mathbf{W}^A$, $A=1,\ldots,N$. It follows that $\mathbf{W}^A$ can be treated as internal variables only as functions of the argument $\mathbf{X}=(x_1,x_2)$. That is why they will be called the quasi-internal variables. Hence, Eqs. (6.3), (6.4), (5.2) represent what will be called the quasi-internal variables model (QIV-model) of the fibre-reinforced composites.

6.2. Laminates

Following the line of approach performed in Subsec. 6.1 we also define $\Omega = \Pi \times (0, H)$ and assume that a laminate under consideration has the periodic material structure (with a period $l = l_3$) along the x_3-axis. The averaging operator used throughout this subsection will be given by

$$< f(\mathbf{z}) > (\mathbf{x}) := \frac{1}{l} \int\limits_{x_3 - l/2}^{x_3 + l/2} f(x_1, x_2, z_3)\, dz_3 .$$

Moreover, we introduce the concept of ε-macrofunction $\Phi(\cdot)$ (related to the line interval $(0, H)$) by means of

$$(\forall\ (x_3, y_3) \in (0, H)^2)[|x_3 - y_3| < l \quad \Rightarrow \quad |\Phi(x_3) - \Phi(y_3)| < \varepsilon_\Phi].$$

A function $\Phi(\cdot)$ can also depend on x_1, x_2 and/or t as parameters. The micro-shape functions $h^A(\cdot)$ depend now only on the argument x_3 since the oscillations of displacements caused by the periodic micro-heterogeneity of the medium take place only in the direction of the x_3-axis. Setting $V = (-\delta_1, \delta_1) \times (-\delta_2, \delta_2) \times (-l/2, l/2)$ and passing to the limit $\delta_1 \to 0$, $\delta_2 \to 0$, we obtain from Eqs. (4.11) the equations of motion for the composite under consideration. To this end we define by $\mathbf{a}_\alpha$ the versors of x_α-axes, $\alpha = 1, 2$, and introduce the fields

$$\begin{aligned} \mathbf{U}(\mathbf{x}, t) &:= \ < \mathbf{u}(\mathbf{z}, t) > (\mathbf{x}), \\ \mathbf{W}^A(\mathbf{x}, t) &:= \ < \mathbf{w}_{\mathbf{x}}(\mathbf{z}, t) h^A(z_3) > (\mathbf{x}) l^{-2}, \end{aligned} \tag{6.5}$$

and

$$\begin{aligned} \mathbf{S}(\mathbf{x}, t) &:= \ < \mathbf{s}(\mathbf{z}, t) > (\mathbf{x}), \\ \mathbf{H}^A(\mathbf{x}, t) &:= \ < \mathbf{s}(\mathbf{z}, t) \cdot \nabla h^A(z_3) > (\mathbf{x}), \\ \mathbf{R}^A{}_\alpha(\mathbf{x}, t) &:= \ < \mathbf{a}_\alpha \cdot \mathbf{s}(\mathbf{z}, t) h^A(z_3) > (\mathbf{x}). \end{aligned} \tag{6.6}$$

Moreover, let $\mathbf{U}(x_1, x_2, \cdot, t), \ldots, \mathbf{R}^A{}_\alpha(x_1, x_2, \cdot, t)$ be regular ε-macrofunctions (related to $(0, H)$) for every $\mathbf{X} = (x_1, x_2) \in \Pi$ and $t \in (t_0, t_f)$. In this case from (4.11) we obtain (summation over $\alpha = 1, 2$ holds!)

$$\begin{aligned} \operatorname{Div} \mathbf{S}(\mathbf{x}, t) - < \varrho > (\mathbf{X})\, \ddot{\mathbf{U}}\, (\mathbf{x}, t) + < \varrho > (\mathbf{X}) \mathbf{b} &= \mathbf{0}, \\ -\mathbf{R}^A{}_{\alpha,\alpha}(\mathbf{x}, t) + < \varrho h^A h^B > (\mathbf{X})\, \ddot{\mathbf{W}}^{\mathbf{B}}\, (\mathbf{x}, t) + \mathbf{H}^A(\mathbf{x}, t) &= \mathbf{0}. \end{aligned} \tag{6.7}$$

Using the procedure explained in Sec. 5, from (6.6), after some manipulations, we arrive at (α, β run over 1, 2; summation convention over β holds!)

$$
(6.8)\qquad \begin{bmatrix} \mathbf{S} \\ \mathbf{H}^A \\ \mathbf{R}^A{}_{\alpha} \end{bmatrix} = \begin{bmatrix} <\mathbf{C}>, & <\mathbf{C}\cdot\nabla h^A>, & <\mathbf{C}\cdot\mathbf{a}_{\beta} h^A> \\ <\nabla h^A\cdot\mathbf{C}>, & <\nabla h^A\cdot\mathbf{C}\cdot\nabla h^B>, & <\nabla h^A\cdot\mathbf{C}\cdot\mathbf{a}_{\beta} h^B> \\ <h^A\,\mathbf{a}_{\alpha}\cdot\mathbf{C}>, & <h^A\,\mathbf{a}_{\alpha}\cdot\mathbf{C}\cdot\nabla h^B>, & <h^A\,\mathbf{a}_{\alpha}\cdot\mathbf{C}\cdot\mathbf{a}_{\beta} h^B> \end{bmatrix} \times \begin{bmatrix} \mathbf{E} \\ \mathbf{W}^B \\ \mathbf{W}^B{}_{,\beta} \end{bmatrix},
$$

where the elements of the above matrix can depend on $\mathbf{X} \in \Pi$. Equations (6.7), (6.8) together with (5.2) constitute a model of the laminates which are periodic in x_3-axis direction. In the directions of the x_1 and x_2-axes, the material structure of those composites can be non-periodic. If $\varrho(\cdot)$ and $\mathbf{C}(\cdot)$ re independent of $\mathbf{X} = (x_1, x_2)$ then (6.7), (6.8) are equations with constant coefficients.

Substituting the right-hand sides of Eqs. (6.8) into Eqs. (6.7) we obtain the system of three second-order partial differential equations for $\mathbf{U}$ coupled with the system of $3N$ differential equations for $\mathbf{W}^A$. The latter are the second-order differential equations with respect to x_1, x_2 and t. It follows that in formulations of boundary-value problems we have to prescribe on $\partial\Omega$ three conditions for three components of $\mathbf{U}$. Moreover, on $\partial\Pi \times (0, H)$ we have also introduced $3N$ extra boundary conditions for $3N$ components of $\mathbf{W}^A$, $A = 1, ..., N$. Similarly to Subsec. 6.1, the unknowns $\mathbf{W}^A$ will be called the quasi-internal variables and hence the model given by Eqs. (6.7), (6.8) will be referred to as the quasi-internal variable model (QIV-model) of the laminated composites.

6.3. Final remarks

It can be seen that if the problem under consideration is independent of x_3-coordinate then the QIV-model for the fibre-reinforced composites reduces to the MIV-model. Similarly, if the problem is independent of x_1, x_2-coordinates then the QIV-model for laminates reduces to the MIV-model. The main feature of the QIV-models is that they can describe with a sufficient accuracy the conditions on these parts of boundary which are intersecting the periodic structure of a composite material. Using QIV-models we can also describe certain class of composites which are non-periodic in directions normal to the basic cell.

7. Conclusions

Let us summarize the advantages and drawbacks of both the MIV- and QIV-models of composites in the light of their possible applications to dynamics of composite solids. The main advantages can be listed as follows.

1. The MIV- and QIV-models describe the effect of the microstructure size on the dynamic behaviour of a composite body, contrary to models based on the concepts of the homogeneous equivalent body. Hence, using these models we

can investigate dispersion phenomena and determine higher wave propagation speeds and free vibration frequencies in composite materials. It can be observed that the MIV-models describe the length-scale effect on the composite body behaviour only in dynamic problems while QIV-models – also in the quasi-stationary problems.

2. The form of the governing equations of the MIV-models is relatively simple since all macro-internal variables as the extra unknowns are governed by the ordinary differential equations, involving only time derivatives of $\mathbf{W}^A$. Hence, the boundary conditions for the MIV-models have the form similar to that met in solid mechanics. Moreover, QIV-models make it possible to describe conditions on the boundary cross-sections of fibre composites and on lateral boundaries of laminates with the required accuracy. It has to be noticed that in the micromorphic models of composites, based on the concept of the extra local degrees of freedom (like the Cosserat-type continua), we deal with the large number of boundary conditions which may be not well motivated from the physical or engineering viewpoint. The same situation also holds for the asymptotic models involving higher-order approximations; this problem will be analyzed in a separate paper.

3. The governing equations of an arbitrary MIV-model have constant coefficients which can be easily determined by calculating the integrals over V and do not require any previous solution to the boundary value problem on the unit cell, contrary to models obtained via the asymptotic methods. Coefficients in the equations of QIV-models can also be easily obtained by the calculation of integrals over the basic area or line element.

4. The MIV- and QIV-models have a wide scope of applications since they can be postulated in the unified way for composites made of arbitrary simple materials. Moreover, the formal procedure presented in this contribution can be easily generalized to include the problem of finite deformations.

5. In some special problems, the MIV- and QIV-models have an adaptive character similar to that of the FEM. It means that they can be formulated on different levels of accuracy either by applying different truncations of the Fourier series or by changing the form of micro-shape functions. Moreover, the error of the obtained solutions to boundary value problems can be evaluated *a posteriori* by the evaluation of the residual fields $\mathbf{r}(\cdot, t)$ introduced in Sec. 4, provided that the stresses and displacements have been previously calculated in terms of macrodisplacements $\mathbf{U}$ and macro-internal variables $\mathbf{W}^A$ with a sufficient accuracy.

Among the drawbacks of the MIV- and QIV-models, the following ones seem to be the most relevant:

1. The analysis of the microdynamic effects is confined almost exclusively to the behaviour of a composite on a macro-level. The passage to microdynamics by using formulae (3.6) may require a very large number N of the micro-shape functions, which make the problem very difficult to solve.

2. The choice of the Fourier expansion of local oscillations and its truncation leading to the proper MIV- or QIV-models for the problem under consideration is not specified by the proposed approach. For some special problems (e.g. for laminated structures), the choice of the micro-shape functions can be based on the intuition of the researcher as a certain *a priori* postulated kinematic hypothesis not related to the aforementioned Fourier expansion.

3. Every MIV- and QIV-model is restricted only to the analysis of a special class of motions which from a qualitative viewpoint has to be postulated *a priori* by the choice of the micro-shape functions. Hence the above models can be applied mostly to problems in which we are interested in a dynamic body behaviour, under motions which can be assumed *a priori* as relevant for the problem under consideration.

Summarizing the above conclusions and taking into account the recently obtained results in this field (cf. the references mentioned in Introduction), one can suppose that the MIV- and QIV-models of composite material structures deserve a certain attention both from the theoretical and engineering point of view.

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