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Transport properties of finite and infinite composite materials and Rayleigh's sum

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The transport properties of a regular array of cylinders embedded in a homogeneous matrix material have been studied by the following method. Let us bound a part of the infinite material by a closed curve γ. Knowing the transport properties of this finite amount of material, we can evaluate the transport properties of the infinite material when γ tends to infinity. This method allows us to justify the method of Lord Rayleigh [1] for rectangular arrays of cylinders. Moreover, it is shown that in order to improve the Clausius–Mossotti approximation for a rectangular array, it is necessary to evaluate Rayleigh's sum.

1. Introduction

A regular array of cylinders is embedded in a homogeneous matrix material. The transport properties of this composite material can be studied by two approaches. The first approach is based on studying a boundary value problem in a cell representing a regular structure. A highly developed theory is used in this approach, from general investigations of homogenization to computation of the effective conductivity of the special composite materials. Results of this study are due to Lord Rayleigh [1], Bergman and Dunn [22] Kołodziej [9], Manteufel and Todreas [10], McPhedran *et al.* [4–7], Mityushev [8, 11, 23], Perrins *et al.* [2], Poladian *et al.* [3], Sangani and Acrivos [21] and many others. The previous results concern mainly isotropic homogenized materials: the square and hexagonal arrays of cylinders. Exceptions are [1, 11, 23], where general anisotropic homogenized materials are considered by analytical methods. Using the method of collocations, Kołodziej [9] computed also the effective conductivity in a fixed direction for the special arrays including anisotropic regular structures.

The present paper presents the direct approach which is based on the following idea. Let us bound a part of the infinite material body by a closed curve γ (Fig. 1). Suppose that we can study the transport properties of this finite composite material bounded by γ. Let the curve γ tend to infinity. We set up the hypothesis that the limit transport properties coincide with the transport properties of regular infinite material bodies. Anyway, it follows from the theory of homogenization. Therefore evaluating the limits, we can get the values in question for the infinite material.

We shall investigate the limit properties in the simplest case of circular cylinders packed in a rectangular array. However, following [23] it is easy to transfer the results to arbitrary arrays of parallelograms. The sides of the rectangle will be

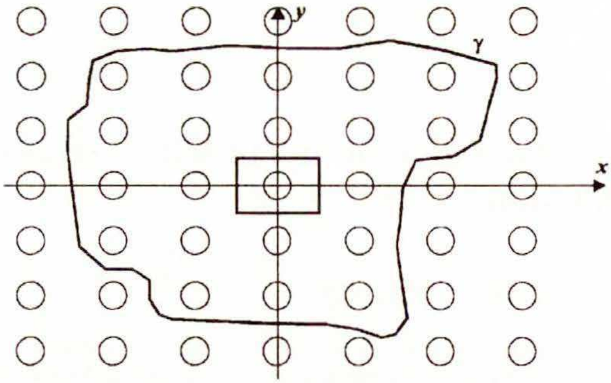

FIG. 1. Infinite rectangular array of circular cylinders and finite material bounded by γ.

denoted by α and β, and the radius of the cylinders by r. Assume that $\beta = \alpha^{-1}$, hence the area of the rectangle is equal to 1. We shall also assume that the state of the media is described by the two-dimensional Laplace equation. If the volume fraction of the cylinders is very small, then the effective conductivity can be evaluated by the Clausius–Mossotti formula (see Sec. 4, formula (4.5)). In order to improve this formula for the rectangular array of cylinders, Lord RAYLEIGH [1] introduced the absolutely divergent sum

$$S_2 := \sum{}' \frac{1}{(m_1\alpha + im_2\beta)^2},$$

where m_1 and m_2 run over all integers except $m_1 = m_2 = 0$ ($i^2 = -1$). The sum S_2 is conditionally convergent. Its value is dependent upon the shape of the exterior boundary of the pairs (m_1, m_2) which tends to infinity. The sum S_2 can be expressed by the integral

$$S_2 = \iint_{\mathbb{R}} \frac{1}{z^2} dx\, dy,$$

where $z = x + iy$. Lord RAYLEIGH [1] proposed to calculate S_2 by summation over a "needle-shaped" region, infinitely more extended along the x-axis than along the y-axis (Fig. 2). In this case

$$S_2 = S_2(\alpha^2) = \frac{2\pi^2}{\alpha^2}\left(\sum_{m=1}^{\infty} \sin^{-2}(im\pi\alpha^{-2}) + \frac{1}{6}\right). \tag{1.1}$$

Let us note that $S_2(1) = \pi$. Applying the theory of generalized functions MITYUSHEV [8] obtained the same result: $S_2(1) = \pi$. Since the sum S_2 is conditionally convergent, we can get any value for S_2 changing the shape of the exterior boundary. Using the effects of polarization, MCPHEDRAN *et al.* [7], PERRINS *et al.* [2]

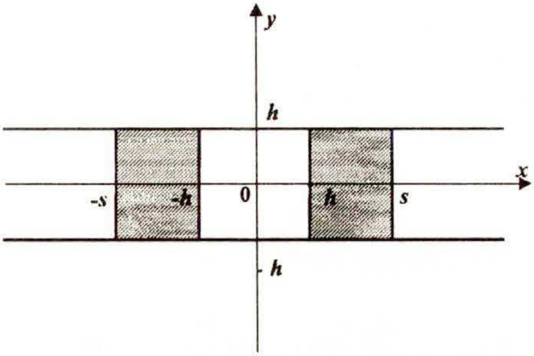

FIG. 2. Rayleigh's method of summation:
$$S_2 := \lim_{h\to\infty}\lim_{s\to\infty}\int_{-h}^{+h}\left(\int_{-s}^{-h}+\int_{h}^{s}\right)\frac{dx\,dy}{(x+iy)^2}.\quad \text{It is assumed that}\quad \int_{-h}^{+h}\int_{-h}^{+h}\frac{dx\,dy}{(x+iy)^2}=0.$$

proposed an explanation of this strange fact. In the present paper this fact has been explained completely.

At the beginning we consider two problems corresponding to infinite and finite material bodies separately. Then we shall compare the limiting effective conductivity of finite body and the effective conductivity of infinite body.

Let us assume the following independent variables. We shall write $z = x + iy$ if we consider a point inside the domain, and $t = x + iy$ if we consider a boundary point. Throughout the paper z and t are complex, x and y are real numbers.

2. Finite material body

Let G be a simply connected domain in the complex plane $\mathbb{C} := \{z = x + iy\}$ with the Lyapunov boundary γ. Let us introduce the points $a_1, a_2, ..., a_n \in G \cap E$ in the complex plane $\mathbb{C}$, where $E := \{m_1\alpha + im_2\alpha^{-1}$, m_1 and m_2 are integers$\}$. Consider mutually disjoint circles $D_k := \{z \in \mathbb{C},\ |z - a_k| < r\}$ $(k = 1, 2, ..., n)$ contained within the domain G. Suppose that $\bigcup\limits_{k=1}^{n} D_k$ and $D := G \backslash \bigcup_{k=1}^{n} D_k$ are occupied by two isotropic materials with conductivities λ_1 and λ, respectively. In order to determine the transport properties of G, we find the potentials $u(x, y)$, $u_1(x, y)$, $u_2(x, y)$, ..., $u_n(x, y)$ which are harmonic in the respective domains D, D_1, D_2, ..., D_n, continuously differentiable in the closures of these domains with the boundary conditions

$$\begin{aligned} &u = u_k, \qquad \lambda\frac{\partial u}{\partial n} = \lambda_1\frac{\partial u_k}{\partial n} \\ &\qquad\qquad \text{on}\ \ \partial D_k := \Big\{t \in \mathbb{C},\ \ |t - a_k| = r\Big\}, \qquad k = 1, 2, ..., n, \\ &u = f \qquad \text{on}\ \ \gamma, \end{aligned} \tag{2.1}$$

where $\partial/\partial n$ is a normal derivative, f is a given continuous function. We shall study the transport properties in the x-direction. Hence, we may take $f(t) = \operatorname{Re} t = x$. It is convenient to make the change

$$U_k(x,y) := \frac{\lambda_1 + \lambda}{2\lambda} u_k(x,y), \qquad k = 1, 2, ..., n.$$

Then the problem (2.1) takes the form

$$(2.2) \qquad u = (1-\varrho)U_k\,, \qquad \frac{\partial u}{\partial n} = (1+\varrho)\frac{\partial U_k}{\partial n} \qquad \text{on } \partial D_k\,, \quad u = f \quad \text{on } \gamma,$$

where $\varrho := (\lambda_1 - \lambda)/(\lambda_1 + \lambda)$.

General theory of the problem (2.2) is based on integral equations constructed by GAKHOV [13], MIKHAJLOV [14]. The problem (2.2) has been solved in an analytic form by MITYUSHEV [15, 16].

Let us consider certain auxiliary problems. The Dirichlet problem $V = f$ on ∂D for the function $V(x,y)$ harmonic in the domain D has the unique solution

$$V(z) = V(x,y) = \int_{\partial D} f \frac{\partial g}{\partial n} ds =: Sf(x,y) = Sf(z), \qquad z = x + iy \in D,$$

where g is Green's function of the domain D. The operator $S : f \to V$ transforms a continuously differentiable function into a function harmonic in D and continuously differentiable in $\overline{D}$ if ∂D is a smooth curve. Let us consider the domain $D_k^- := \{z \in \mathbb{C}, \quad |z - a_k| > r\}$ $(k = 1, 2, ..., n)$. We shall use the operators S_k corresponding to D_k^- and the operator S_γ corresponding to G. If $V(z)$ is harmonic in D_k, then

$$S_k V(z) = V(z_k^*), \qquad z \in D_k^-\,, \quad k = 1, 2, ..., n,$$

where points $z_k^* := r^2/(\overline{z - a_k}) + a_k$ and z are symmetric with respect to the circumference $|t - a_k| = r$. Let us consider the next auxiliary boundary value problem

$$(2.3) \qquad \frac{\partial U_0}{\partial n} + \frac{\partial S_\gamma U_0}{\partial n} = h \quad \text{on } \gamma,$$

for the function $U_0(z)$ harmonic in $\overline{\mathbb{C}}\backslash\overline{G}$ and vanishing at infinity.

LEMMA (MITYUSHEV [18]). Let h be a function continuously differentiable on γ. Then the boundary value problem (2.3) has a unique solution continuously differentiable in $\overline{\mathbb{C}}\backslash\gamma$.

If the function $u(x,y)$ from (2.2) is known, then using the above lemma, introduce the function $U_0(z)$ with $h = \partial u/\partial n - \partial x/\partial n$. Let us consider the function

$$\Phi(z) = \begin{cases} U_k(z) + \varrho \sum\limits_{\substack{m=1\\ m\neq k}}^{n} U_m(z_m^*) + S_\gamma U_0(z) - x, & |z-a_k| \le r, \quad k = 1,2,\dots,n, \\ U_0(z) + \varrho \sum\limits_{m=1}^{n} U_m(z_m^*), & z \in \overline{\mathbb{C}}\backslash G, \\ u(z) + \varrho \sum\limits_{m=1}^{n} U_m(z_m^*) + S_\gamma U_0(z) - x, & z \in D, \end{cases}$$

harmonic in $\mathbb{C}\backslash\partial D$. Using the boundary conditions (2.2) and (2.3), calculate the jumps of Φ on ∂D_k and γ

$$\Phi^+(t) - \Phi^-(t) = u(t) + \varrho U_k(t) - U_k(t) = 0, \qquad t \in \partial D_k, \quad k = 1,2,\dots,n,$$
$$\Phi^+(t) - \Phi^-(t) = u(t) + S_\gamma U_0(t) - x - U_0(t) = 0 \quad \text{on } \gamma.$$

Here $\Phi^+(t) := \lim_{\substack{z\to t\\ z\in D}} \Phi(z)$, $\Phi^-(t) := \lim_{\substack{z\to t\\ z\in D_k}} \Phi(z)$. Along the same lines

$$\begin{aligned} \frac{\partial\Phi^+}{\partial n}(t) - \frac{\partial\Phi^-}{\partial n}(t) &= \frac{\partial u}{\partial n}(t) + \varrho\frac{\partial}{\partial n}\left(U_k(t_k^*)\right) - \frac{\partial U_k}{\partial n}(t) \\ &= \frac{\partial u}{\partial n}(t) - (1+\varrho)\frac{\partial U_k}{\partial n}(t) = 0, \qquad t \in \partial D_k, \quad k = 1,2,\dots,n, \end{aligned}$$

since

$$\frac{\partial}{\partial n}\left(U_k(t_k^*)\right) = -\frac{\partial U_k}{\partial n}(t) \quad \text{on } \partial D_k.$$

Taking into account (2.3), we calculate

$$\frac{\partial\Phi^+}{\partial n}(t) - \frac{\partial\Phi^-}{\partial n}(t) = \frac{\partial u}{\partial n}(t) - \frac{\partial S_\gamma U_0}{\partial n}(t) - \frac{\partial x}{\partial n} - \frac{\partial U_0}{\partial n}(t) = 0 \qquad \text{on } \gamma.$$

The function $\Phi(z)$ is harmonic in $\overline{\mathbb{C}}\backslash\partial D$ and $\Phi^+ = \Phi^-$, $\partial\Phi^+/\partial n = \partial\Phi^-/\partial n$ on ∂D. According to the theorem of harmonic (analytic) continuation and Liouville's theorem we conclude that $\Phi(z) = c = \text{const}$. From the definition of $\Phi(z)$ we obtain the formulae

$$U_k(z) = -\varrho \sum_{\substack{m=1\\ m\neq k}}^{n} U_m(z_m^*) - S_\gamma U_0(z) + x + c, \quad |z-a_k| \le r, \quad k = 1,2,\dots,n,$$

$$U_0(z) = -\varrho \sum_{m=k}^{n} U_m(z_m^*) + c, \quad z \in \overline{\mathbb{C}}\backslash G.$$

From the last equality we determine $S_\gamma U_0(z)$ and substitute it in the previous equalities. As a result, we have the following system of functional equations

$$
(2.4) \qquad U_k(z) = -\varrho \left[\sum_{\substack{m=1 \\ m\neq k}}^{n} [U_m(z_m^*) - S_\gamma U_m(t_m^*)(z) \right] + \varrho S_\gamma U_k(t_k^*)(z) + x,
$$
$$
|z - a_k| \le r, \qquad k = 1, 2, ..., n,
$$

for $U_k(z)$ $(k = 1, 2, ..., n)$. Each harmonic function in a simply connected domain is the real part of an analytic function, which is uniquely determined with accuracy to an additive imaginary constant. Hence there exists such a function $\phi_k(z)$ analytic in $|z - a_k| \le r$ that $\operatorname{Re}\phi_k(z) = U_k(z)$. Let us introduce the operator T_γ^m which transforms a function $\phi_m(z)$ analytic in G in the following way. At the beginning calculate $\operatorname{Re}\phi_m(t_m^*) = U_m(t_m^*)$ on γ. Further on, by applying S_γ we obtain a harmonic function which is the real part of the analytic function $T_\gamma^m\phi_m(z)$. Actually in the last step we used the Schwarz operator of G studied by MIKHLIN [17]. We do not determine a pure imaginary constant in $T_\gamma^m\phi_m(z)$ because it does not affect the final result. So the system (2.4) is reduced to the following system of functional equations

$$
\phi_k(z) = -\varrho \sum_{\substack{m=1 \\ m\neq k}}^{n} \left[\overline{\phi_m(z_m^*)} - T_\gamma^m \phi_m(z) \right] + \varrho T_\gamma^k \phi_k(z) + z,
$$
$$
|z - a_k| \le r, \quad k = 1, 2, ..., n.
$$

Let us differentiate this system and obtain

$$
(2.5) \qquad \phi_k'(z) = \varrho \sum_{\substack{m=1 \\ m\neq k}}^{n} \left[\left(\frac{r}{z - a_m} \right)^2 \overline{\phi_m'(z_m^*)} + V_\gamma^m \phi_m'(z) \right] + \varrho V_\gamma^k \phi_k'(z) + 1,
$$
$$
|z - a_k| \le r, \quad k = 1, 2, ..., n,
$$

where $V_\gamma^m\phi_m'(z) := (T_\gamma^m\phi_m)'(z)$. The operator V_γ^m is correctly defined because MIKHLIN [17] has proved that T_γ^m is an integral operator.

THEOREM 1 (MITYUSHEV [15, 16, 18, 23]). *The system of functional equations* (2.5) *for the functions* $\phi_k'(z)$ *analytic in* $|z - a_k| < r$ *and continuous in* $|z - a_k| \le r$ $(k = 1, 2, ..., n)$ *has a unique solution. That solution can be found by the method of successive approximations converging uniformly in* $|z - a_k| \le r$ $(k = 1, 2, ..., n)$.

This theorem has the following important consequence.

THEOREM 2. *The function* $\dfrac{\partial U_k}{\partial x}(a_k) = \operatorname{Re}\phi_k'(a_k)$ *is analytic in the unit disc* $|\varrho| < 1$ *with respect to the variable* ϱ:

$$
\operatorname{Re}\phi_k'(a_k) = \sum_{p=1}^{\infty} A_p(k, n)\varrho^p,
$$

where

$$A_0(k,n) = 1, \qquad A_1(k,n) = \operatorname{Re}\left[\sum_{\substack{m=1\\ m\neq k}}^{n}\left[\left(\frac{r}{a_k - a_m}\right)^2 + V_\gamma^m\, 1(a_k)\right] + V_\gamma^k\, 1(a_k)\right].$$

3. Infinite material

A rectangular array of circular cylinders of conductivity λ_1 is embedded in a matrix of conductivity λ. Let us study the transport properties of the composite material in the x-direction. So we have the following problem in the cell $Q_0 := \{(x,y) \in \mathbb{R}^2,\ -\alpha/2 < x < \alpha/2,\ -1/(2\alpha) < y < 1/(2\alpha)\}$: find the potentials $w_1(x,y)$ and $w(x,y)$ harmonic in $Q_1 := \{(x,y) \in \mathbb{R}^2,\ x^2+y^2 < r^2\}$ and $Q := Q_0 \setminus \overline{Q}_1$ respectively, continuously differentiable in the closures of these domains with the boundary conditions

$$(3.1) \qquad w = w_1, \qquad \lambda\frac{\partial w}{\partial n} = \lambda_1\frac{\partial w_1}{\partial n} \quad \text{on the circumference} \quad x^2+y^2 = r^2,$$

$$(3.2) \qquad w(x+\alpha, y) = w(x,y) + \alpha, \qquad w(x, y+\alpha^{-1}) = w(x,y).$$

If $\lambda_1 = \lambda$ then $w = w_1 = x$, and the current $\mathbf{j} = -\operatorname{grad} x = (-1, 0)$.

References to papers with effective solutions of the problem (3.1), (3.2) are given in Sec. 1.

The problem (3.1), (3.2) is equivalent to the following boundary $\mathbb{R}$-value problem

$$(3.3) \qquad \phi(t) = \phi_1(t) - \varrho\overline{\phi_1(t)} - t, \qquad |t| = r,$$

where the unknown functions $\phi(z)$ and $\phi_1(z)$ are analytic in D and D_1, respectively, continuously differentiable in the closures of these domains. The function $\phi(z)$ is quasi-periodic:

$$\phi(z+\alpha) + i\gamma_1 = \phi(z) = \phi(z + i\alpha^{-1}) + i\gamma_2\,,$$

where γ_1 and γ_2 are real constants. The harmonic and analytic functions are related by the identities $w(x,y) = \operatorname{Re}(\phi(z)+z)$, $w_1(x,y)(\lambda+\lambda_1)/2\lambda = \operatorname{Re}\phi_1(z)$. The first condition (3.1) coincides with the real part of (3.3). The second condition (3.1) complies with the imaginary part of (3.3) differentiated along the tangential vector.

We assume that ϱ is a small parameter. A method of perturbation consists in finding a solution of the problem (3.3) in the form of the following expansions:

$$\phi(z) = \phi^0(z) + \varrho\phi^1(z) + \ldots, \qquad \phi_1(z) = \phi_1^0(z) + \varrho\phi_1^1(z) + \ldots .$$

By substituting these expansions in the boundary condition (3.3) and collecting terms with equal powers of ϱ^m, we obtain a cascade of the problems. The number zero problem is

$$\phi^0(t) = \phi_1^0(t) - t, \qquad |t| = r.$$

The first one is

$$\phi^1(t) = \phi_1^1(t) - \overline{\phi_1^0(t)}, \qquad |t| = r.$$

Since the solution of the zero problem has the form $\phi_1^0(z) = z$, the first problem becomes

$$\phi^1(t) = \phi_1^1(t) - \frac{r^2}{t}, \qquad |t| = r.$$

The last equality means that $\phi^1(z)$ is analytically continued into $1 < |z| < r$. Hence, the function $\phi^1(z)$ is analytic and quasi-periodic in $Q_0 \backslash \{z = 0\}$:

$$\phi^1(z + \alpha) + i\gamma_1^1 = \phi^1(z) = \phi^1(z + i\alpha^{-1}) + i\gamma_2^1. \tag{3.4}$$

It has a pole at the point $z = 0$. The residue of $\phi^1(z)$ at $z = 0$ is equal to $(-r^2)$. It follows from the theory of elliptic functions that

$$\phi^1(z) = r^2(Az - \zeta(z)), \tag{3.5}$$

where A is a constant, ζ is the Weierstrass function [19]. The relation (3.4) implies the equalities

$$\operatorname{Re}\left[\phi^1(z + \alpha) - \phi^1(z)\right] = \operatorname{Re}\left[\phi^1(z + i\alpha^{-1}) - \phi^1(z)\right] = 0.$$

Substituting (3.5) into the last relations we obtain that $\alpha \operatorname{Re} A = \eta_1$, $\operatorname{Im} A = 0$, where $\eta_1 := 2\zeta(\alpha/2)$ is a real number, hence $A = \alpha^{-1}\eta_1$. So we arrive at the following asymptotic representations

$$\phi_1(z) = z + \varrho r^2 \left[\alpha^{-1}\eta_1 z - ((\zeta(z) - 1/z)\right] + o(\varrho), \qquad \text{as } \varrho \to 0,$$

and

$$\phi_1'(0) = 1 + \varrho r^2 \alpha^{-1}\eta_1 + o(\varrho), \qquad \text{as } \varrho \to 0. \tag{3.6}$$

Let us consider the system (2.5). Let $R = R_0 h(\theta)$ be the equation of the curve γ in the polar coordinates (R, θ), R_0 is a positive constant. We shall say that γ tends to infinity ($\gamma \to \infty$) with a fixed shape if in the equation $R = R_0 h(\theta)$ the value R_0 tends to infinity ($R_0 \to \infty$). Let us fix $h(\theta)$. If we fix the shape of γ in such a way that the operators V_γ^m disappear in the limit $n \to \infty$, then the limiting system for infinite materials becomes

$$\psi(z) = \varrho \sum_m{}' \left(\frac{r}{z - a_m}\right)^2 \overline{\psi\left(\frac{r^2}{\overline{z - a_m}}\right)} + 1, \qquad |z| \le r. \tag{3.7}$$

The sum $\sum\limits_m{}'$ means that the term $a_0 = 0$ is missing. The unknown function $\psi(z) = \lim\limits_{\gamma\to\infty} \phi_k'(z)$. It is analytic in $|z| < r$, continuous in $|z| \le r$ and periodic: $\psi(z) = \psi(z + a_k)$ for each $a_k \in E = \{m_1\alpha + im_2\alpha^{-1}\}$, m_1 and m_2 are integers. The infinite sum in (3.7) is understood in the following sense

$$\text{(3.8)}\qquad \sum_m{}' \left(\frac{1}{z-a_m}\right)^2 \overline{\psi\left(\frac{r^2}{\overline{z-a_m}}\right)} := \sum_m{}' \left(\frac{1}{z-a_m}\right)^2 \left(\overline{\left(\psi\frac{r^2}{\overline{z-a_m}}\right)} - \overline{\psi(0)}\right) + \overline{\psi(0)}\left(\mathcal{P}(z) - \frac{1}{2} + S_2\right),$$

where S_2 is an undetermined quantity, and

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_m{}' \left[\left(\frac{1}{z-a_m}\right)^2 - \frac{1}{a_m^2}\right]$$

is the Weierstrass function [19]. If $\psi(z) \equiv 1$ and $z = 0$ in (3.8) then $\sum_m{}' \dfrac{1}{a_m^2} = S_2$. Using the method of successive approximations we conclude from (3.7) that

$$\text{(3.9)}\qquad \psi(0) = 1 + \varrho r^2 \sum_m{}' \frac{1}{a_m^2} + o(\varrho) = 1 + \varrho r^2 S_2 + o(\varrho), \quad \text{as } \varrho \to 0.$$

So we have the following quantities: $\phi_1'(0)$ from the problem for infinite material and $\psi(0)$ as the limit of $\phi_k'(a_k)$ with the special shape of γ. Since we assume that $\phi_1'(0) = \psi(0)$, then we conclude from (3.6) and (3.9) that $S_2 := \alpha^{-1}\eta_1$ for this special shape of γ. It follows from the theory of elliptic functions [19] that S_2 can be written in the form (1.1). This justifies the formula of Lord Rayleigh (1.1).

Let us show that the system of functional equations (3.7) is a continuous form of the infinite algebraic system of the method of Rayleigh. Introduce now the Taylor series for the function $\psi(z)$ inside the circle $z < r$

$$\psi(z) = \sum_{k=0}^{\infty} \psi_k z^k.$$

Then

$$\sum_m{}' \left(\frac{r}{z-a_m}\right)^2 \overline{\psi\left(\frac{r^2}{\overline{z-a_m}}\right)} = \sum_{k=0}^{\infty} \overline{\psi}_k\, r^{2(k+1)} \mathcal{S}_{k+2}(z,\alpha),$$

where

$$\mathcal{S}_{k+2}(z,\alpha) := \sum_m{}' \frac{1}{(z-a_m)^2} = \sum_m{}' \left(-\sum_{s=0}^{\infty} \frac{z^s}{a_m^{s+1}}\right)^{k+2}, \qquad k = 0, 1, 2, \dots .$$

The function $\mathcal{S}_2(z, \alpha)$ is understood in the following sense: $\mathcal{S}_2(z,\alpha) := \mathcal{P}(z) - \dfrac{1}{z^2} + S_2$. If $z = 0$ then $\mathcal{S}_2(0, \alpha) = S_2$. Substituting all series in (3.7) we arrive at an infinite system of linear algebraic equations. The real parts of this system coincide with the infinite system of Lord RAYLEIGH [1], MCPHEDRAN *et al.* [4–7], PERRINS *et al.* [2], POLADIAN *et al.* [3] for $\alpha = 1$. We will show it only for the number zero equation. Substituting $z = 0$ into (3.7) we obtain

$$\psi_0 = \varrho \sum_{k=0}^{\infty} \overline{\psi_k}\, r^{2(k+1)} \mathcal{S}_{k+2}(0,1) + 1,$$

where

$$\mathcal{S}_{k+2}(0,1) = \sum_m{}' \frac{1}{a_m^{k+2}}\,.$$

If we replace ψ_k by

$$\beta_{k+1} = (k+1)\varrho\, r^{2(k+1)} \operatorname{Re} \psi_k\,,$$

then we obtain the first equation of [2].

4. Effective conductivity and the sum S_2

Let us introduce the value

$$\lambda_e^x(n) = \frac{<j>}{<e>}\,,$$

where

$$<e>= J + \sum_{k=1}^{n} j_k\,, \qquad <j>= \lambda J + \sum_{k=1}^{n} \lambda_k j_k\,,$$

$$j_k := \iint\limits_{D_k} \frac{\partial u_k}{\partial x} dx\, dy = (1-\varrho) \iint\limits_{D_k} \frac{\partial U_k}{\partial x} dx\, dy = (1-\varrho) J_k\,,$$

$$J := \iint\limits_{D} \frac{\partial u}{\partial x} dx\, dy.$$

Using the Green's formula $\iint\limits_G \dfrac{\partial v}{\partial x} dx\, dy = \int\limits_\gamma v\, dy$ we obtain

$$\frac{\lambda_e^x(n)}{\lambda} = 1 + 2\varrho \frac{1}{|G|} \sum_{k=1}^{n} J_k\,,$$

where $|G|$ is the area of the domain G. Applying the mean value theorem to the integral J_k we have

$$\iint_{D_k} \frac{\partial U_k}{\partial x} dx\, dy = \pi r^2 \frac{\partial U_k}{\partial x}(a_k) = \pi r^2 \operatorname{Re} \phi_k'(a_k).$$

Therefore

$$\frac{\lambda_e^x(n)}{\lambda} = 1 + 2\varrho \frac{\pi r^2 n}{|G|} \frac{1}{n} \sum_{k=1}^{n} \operatorname{Re} \phi_k'(a_k). \tag{4.1}$$

It follows from the Theorem 2 of Sec. 2 that

$$\operatorname{Re} \phi_k'(a_k) = 1 + \operatorname{Re} \left[r^2 \sum_{\substack{m=1 \\ m \neq k}} \left(\frac{1}{a_k - a_m} \right)^2 + \sum_{m=1}^{n} V_\gamma^m \, 1(a_k) \right] \varrho + o(\varrho), \quad \text{as } \varrho \to 0.$$

Substituting this relation into (4.1) we have

$$\frac{\lambda_e^x(n)}{\lambda} = 1 + 2\varrho \frac{\pi r^2 n}{|G|} + 2\varrho^2 \frac{\pi r^2 n}{|G|} \kappa(n) + o(\varrho^2), \quad \text{as } \varrho \to 0,$$

where

$$\kappa(n) := r^2 \operatorname{Re} \left(S_2(n,\gamma) + \mu(n,\gamma) \right), \qquad S_2(n,\gamma) := \frac{1}{n} \sum_{k=1}^{n} \sum_{\substack{m=1 \\ m \neq k}} \left(\frac{1}{a_k - a_m} \right)^2,$$

$$\mu(n,\gamma) := \frac{1}{n} \sum_{k=1}^{n} \sum_{m=1}^{n} V_\gamma^m \, 1(a_k). \tag{4.2}$$

Calculating the limit $n \to \infty \Leftrightarrow \gamma$ tends to infinity, we arrive at the relation

$$\frac{\lambda_e^x}{\lambda} = 1 + 2\varrho\, v + 2\varrho^2 v \lim_{n \to \infty} \kappa(n) + o(\varrho^2), \quad \text{as } \varrho \to 0, \tag{4.3}$$

where v is the volume fraction of the inclusion. On the other hand, calculating λ_e^x in the cell Q_0 representing infinite material and using (3.6) we have

$$\frac{\lambda_e^x}{\lambda} = 1 + 2\varrho\, v \operatorname{Re} \phi_1'(0) = 1 + 2\varrho\, v + 2\varrho^2 v^2 \operatorname{Re} \frac{\eta_1}{\pi \alpha} + o(\varrho^2), \quad \text{as } \varrho \to 0. \tag{4.4}$$

Comparing (4.3) and (4.4) we conclude that $\lim_{n \to \infty} \kappa(n)$ must be equal to $\operatorname{Re}(v\eta_1/\pi\alpha)$. This conclusion allows us to explain strange properties of the sum S_2 (see Sec. 1). Let us fix the shape of γ and introduce the limits

$$S_2(\gamma) = \lim_{n \to \infty} S_2(n,\gamma), \qquad \mu(\gamma) = \lim_{n \to \infty} \mu(n,\gamma).$$

Thus we have the two equalities:

$$\kappa(n) = r^2 \operatorname{Re} S_2(n,\gamma) + \mu(n,\gamma) \qquad \text{and} \qquad \frac{v\eta_1}{\pi\alpha} = r^2 S_2(\gamma) + \mu(\gamma).$$

In the last equality the value $v\eta_1/\pi\alpha$ is independent of γ. It means that one may assume an arbitrary shape of γ and define $S_2(\gamma)$ at will. But it is necessary to account for the term $\mu(\gamma)$. For instance, if we take γ in such a way that $S_2(\gamma) = (\eta_1/\alpha)$, then $\mu(\gamma)$ must be equal to zero.

If $\alpha = 1$ then $\eta_1 = \pi$ and $\lim\limits_{n\to\infty} \kappa(n) = v$. In this case we arrive at the asymptotic formula

$$\frac{\lambda_e^x}{\lambda} = 1 + 2\varrho v + 2\varrho^2 v^2 + o(\varrho^2), \qquad \text{as } \varrho \to 0,$$

derived by Bergman and Milton. Thorough investigations of such representations involve the bounds on the effective tensor. The most important papers on those bounds are cited in the recent work by CLARK and MILTON [20].

The formula (4.4) is closely related to the famous Clausius–Mossotti approximation

$$\frac{\lambda_e}{\lambda} = \frac{1+\varrho v}{1-\varrho v} + o(v), \qquad \text{as } v \to 0. \tag{4.5}$$

Let us note that $\varrho \to 0$ in (4.4) and $v \to 0$ in (4.5). It follows from [23, p. 63–75] that

$$\frac{\lambda_e^x}{\lambda} = \frac{1 + \varrho v(2 - S_2/\pi) + \dfrac{6}{\pi^4} S_4^2 \varrho^3 v^5 \left[1 + \dfrac{S_2}{\pi}\varrho v - \varrho^2 v^2 \left(\dfrac{S_2}{\pi}\right)^2\right]}{1 - \varrho v S_2/\pi - \dfrac{6}{\pi^4} S_4^2 \varrho^3 v^5 \left[1 - \left(2 - \dfrac{S_2}{\pi}\right)\varrho v - \varrho^2 v^2 \left(2 - \dfrac{S_2}{\pi}\right)^2\right]} + o(v^7) \quad \text{as } v \to 0, \tag{4.6}$$

where $S_4 := \sum\limits_m{}' a_m^{-4}$ is the absolutely convergent Rayleigh sum of the fourth order [1]. Therefore, in order to improve (4.5) as $v \to 0$ or $\varrho \to 0$, we must use the value S_2. Calculation of the higher order Rayleigh sums S_{2n} $(n > 1)$ is only a computation problem, because they are absolutely convergent. Let us note that an exact formula for λ_e^x/λ has been derived in [8, 23], and formula (4.6) is an approximate consequence of this result. However, (4.6) is of a very simple form and can be easily used in technical calculations.

Applying the Dykhne–Keller identity [24] to (4.6) it is possible to get an analogous formula for λ_e^y/λ. Following [23] it is necessary to replace S_2 with $(2 - S_2/2)$.

5. Conclusions

In addition to the physical explanation of the equality $S_2 = \pi$ of MCPHEDRAN *et al.* [7] and PERRINS *et al.* [2] and the rigorous definition of S_2 of MITYUSHEV [8, 23], in the present paper we have given the rigorous mathematical explanation when and why the formula (1.1) is true. We have also analyzed the Rayleigh sum S_2 and its application to analytic formulae, determining the properties of the tensor. Moreover, we have justified the method of Lord Rayleigh.

Appendix

We shall prove that the value $S_2(\gamma)$ is correctly defined. Let us fix $h(\theta)$ in the equation $R = R_0 h(\theta)$ of the curve γ in the polar coordinates system (R, θ). According to [8]

$$J := v.p. \iint\limits_G \frac{1}{z^2} dx\, dy = \lim_{\varepsilon \to 0} \iint\limits_{G_\varepsilon} \frac{1}{z^2} dx\, dy,$$

$$G_\varepsilon := G \backslash \left\{ z \in \mathbb{C}, \quad |z| > \varepsilon \right\}.$$

Following to [8] we arrive at the formula

$$J = \int\limits_\gamma \mathrm{Re}\frac{1}{z} dy - \lim_{\varepsilon \to 0} \int\limits_{|z|=\varepsilon} \mathrm{Re}\frac{1}{z} dy = \frac{1}{2} \int\limits_0^{2\pi} \frac{h'(\theta)}{h(\theta)} (\sin 2\theta + i \cos 2\theta)\, d\theta.$$

Since $S_2(\gamma)$ can be considered as a limit of the Riemannian sum of the integral J, we have

$$S_2(\gamma) = \lim_{n \to \infty} S_2(n, \gamma) = \lim_{R_0 \to \infty} J.$$

One can see that J is independent of R_0. Hence $S_2(\gamma)$ is correctly defined by the integral

$$S_2(\gamma) = \frac{1}{2} \int_0^{2\pi} \frac{h'(\theta)}{h(\theta)} (\sin 2\theta + i \cos 2\theta)\, d\theta.$$

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