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Outlooks in Saint–Venant theory
III. Torsion and flexure in sections of variable thickness by formal expansions

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WE STUDY the Saint–Venant shear stress fields [1] arising in a family of sections we call Bredt-like [2, 3, 4], i.e. in a set of plane regions $\mathcal{D}_\varepsilon$ whose thickness we scale by a parameter ε. For each ε we build a coordinate mapping from a fixed plane domain $\mathcal{D}$ onto $\mathcal{D}_\varepsilon$. The shear stress field in $\mathcal{D}_\varepsilon$ can be represented by a Prandtl-like stress flow function [5, 6]. This is naturally done in torsion (*torsion*, [1]), while in flexure (*flexion inégale*, [1]) we face a gauge choice whose physical interpretation is uncertain [6]. We then consider the Helmholtz operator in a fixed system of coordinates in $\mathcal{D}$ and represent the shear stress field in a basis field which is not the covariant basis associated to any coordinate system. Formal ε-power series expansions for the shear stress field, the warping, the resultant force and torque and the shear shape factors tensor lead to hierarchies of perturbation problems for their coefficients. We obtain all the technical formulae at the lowest iteration steps and their generalization at higher steps – i.e., for thicker sections. No attempt is made to apply the methods proposed in [16] to estimate the distance between the generalized formulae we provide and the true solutions for the Saint–Venant shear stress problem.

1. Geometry of Bredt-like sections

WE CALL Bredt-like sections all the regions included in a plane $\mathcal{P}$ obtained by symmetrically thickening a curve $\mathcal{L} \in \mathcal{P}$ (*middle line*) along its Frenet normal with regularly varying thickness. The position vector field of the points of $\mathcal{L}$ with respect to any point $o \in \mathcal{P}$ is given as a function of its arc length s:

$$\mathcal{L} := \{q \in \mathcal{P} \,|\, q - o = \mathbf{r}_0(s), \qquad s \in [0, l]\}, \tag{1.1}$$

l is the length of $\mathcal{L}$. The Frenet orthonormal basis for the middle line is

$$\mathbf{l}(s) := \frac{\partial \mathbf{r}_0(s)}{\partial s} = \mathbf{r}_{0,s}(s), \qquad \mathbf{m}(s) := -*\frac{\partial \mathbf{r}_0(s)}{\partial s} = -*\mathbf{r}_{0,s}(s), \tag{1.2}$$

$*$ is Hodge operator in $\mathcal{P}$ ($\pi/2$ rotation in the positive orientation of $\mathcal{P}$); the comma denotes a derivative with respect to the indicated variable.

We define the ε-Bredt-like section as the collection of all the lines symmetrically shifted along $*\mathbf{l}(s)$ starting from $\mathcal{L}$, the total shift $\delta(s)$ being a regular function of s:

$$\mathcal{D}_\varepsilon := \{y \in \mathcal{P} \,|\, y - o = \mathbf{r}(s, z) = \mathbf{r}_0(s) + \varepsilon z \delta(s) * \mathbf{l}(s), \qquad s \in [0, l], \quad z \in [-1, 1]\}; \tag{1.3}$$

z is a coordinate along $\mathbf{l}(s)$ and ε is a thickness perturbation parameter. We regard s, z as rescaled coordinates over $\mathcal{D}_\varepsilon$, [7]; Eq. (1.3) implicitly defines a coordinate mapping from $\mathcal{D} := [0, l] \times [-1, 1]$ onto $\mathcal{D}_\varepsilon$.

The natural (covariant) basis associated with the coordinate system (s, z) is [8]

$$\begin{aligned} \mathbf{g}_1(s) &:= \frac{\partial \mathbf{r}}{\partial s} = [1 - \varepsilon z \kappa(s)\delta(s)]\,\mathbf{l}(s) + \varepsilon z \delta_{,s}(s) * \mathbf{l}(s), \\ \mathbf{g}_2(s) &:= \frac{\partial \mathbf{r}}{\partial z} = \varepsilon \delta(s) * \mathbf{l}(s); \end{aligned} \tag{1.4}$$

$\kappa(s)$ is the (suitably regular) curvature of the middle line. Henceforth, to lighten the notation we will drop the dependence of the indicated functions on the coordinates, when there is no risk of confusion.

The covariant components and the determinant of the metric tensor are

$$\begin{aligned} g_{11} &= (1 - \varepsilon z \kappa \delta)^2 + (\varepsilon z \delta_{,s})^2, \qquad g_{12} = g_{21} = z\varepsilon^2 \delta \delta_{,s}, \\ g_{22} &= \varepsilon \delta^2, \qquad g \doteq g_{11} g_{22} - (g_{11})^2 = (\varepsilon\delta)^2 (1 - \varepsilon z \kappa \delta)^2. \end{aligned} \tag{1.5}$$

The basis dual to the natural one is given by $\mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j$, where $\cdot$ stands for the usual scalar product in the vector space $\mathcal{V}$ of the translations of $\mathcal{P}$:

$$\mathbf{g}^1 = \frac{\mathbf{l}}{1 - \varepsilon z \kappa \delta}, \qquad \mathbf{g}^2 = -\frac{z\delta_{,s}}{\delta(1 - \varepsilon z \kappa \delta)}\mathbf{l} + \frac{1}{\varepsilon\delta} * \mathbf{l}. \tag{1.6}$$

As the section has variable thickness, the coordinate system (s, z) is not, in general, orthogonal (it is on the middle line, by construction). The Christoffel symbols (of second kind) associated with the coordinates s, z, according to $\left\{ {j \atop k\ l} \right\} := \mathbf{g}_{k,l} \cdot \mathbf{g}^j$, [3, 4, 8], are:

$$\begin{aligned} \left\{ {1 \atop 1\ 1} \right\} &= -\frac{\varepsilon z (2\kappa \delta_{,s} + \kappa_{,s}\delta)}{1 - \varepsilon z \kappa \delta}, \\ \left\{ {2 \atop 1\ 1} \right\} &= \frac{z^2 \varepsilon \delta_{,s}(2\kappa\delta_{,s} + \kappa_{,s}\delta)}{\delta(1 - \varepsilon z \kappa \delta)} + \frac{\kappa(1 - \varepsilon z \kappa \delta) + z\delta_{,ss}}{\varepsilon\delta}, \\ \left\{ {1 \atop 2\ 2} \right\} &= \left\{ {2 \atop 2\ 2} \right\} = 0, \\ \left\{ {1 \atop 2\ 1} \right\} &= \left\{ {1 \atop 1\ 2} \right\} = -\frac{\varepsilon\kappa\delta}{1 - \varepsilon z \kappa \delta}, \\ \left\{ {2 \atop 2\ 1} \right\} &= \left\{ {2 \atop 1\ 2} \right\} = \frac{\delta_{,s}}{\delta(1 - \varepsilon z \kappa \delta)}. \end{aligned} \tag{1.7}$$

The centroid of area b of the section is given, according to Eq. (1.3), by

$$b - o = \frac{\int_{\mathcal{D}_\varepsilon} (y - o)}{A_{\mathcal{D}_\varepsilon}} = \frac{\int_0^l \delta \mathbf{r}_0 - \frac{\varepsilon^2}{3}\int_0^l \delta^3 \kappa * \mathbf{l}}{\int_0^l \delta} =: \mathbf{b}_0 - \varepsilon^2 \mathbf{b}_2, \tag{1.8}$$

$A_{\mathcal{D}_\varepsilon}$ is the area of the section. In all the integrals the measure of integration is understood.

2. The elliptic problem for the shear stress field

The shear stress field $\mathbf{t}$ arising in the ε-section of a Saint-Venant cylinder is subjected to the elliptic problem [1, 6, 9]

$$\operatorname{div}\mathbf{t} = Y\mathbf{k}\cdot[*(y-b)] \qquad \text{in } \mathcal{D}^o_\varepsilon, \tag{2.1}$$

$$\operatorname{curl}\mathbf{t} = 2G[\tau + \nu\mathbf{k}\cdot(y-o)] \qquad \text{in } \mathcal{D}^o_\varepsilon, \tag{2.2}$$

$$\mathbf{t}\cdot\mathbf{n} = 0 \qquad \text{along } \partial\mathcal{D}_\varepsilon, \tag{2.3}$$

$$\oint_{\mathcal{C}} \mathbf{t}\cdot\mathbf{l}_{\mathcal{C}} = 2GA_{\mathcal{S}}[\tau + \nu\mathbf{k}\cdot(b_{\mathcal{S}}-o)] \qquad \forall\mathcal{C}\subset\mathcal{D}_\varepsilon. \tag{2.4}$$

In the former, Y, G, ν are the longitudinal (Young) elastic modulus, the tangential elastic modulus and Poisson's ratio, respectively; $\mathbf{n}$ is the outer unit normal to $\partial\mathcal{D}_\varepsilon$; $\mathcal{C}$ is a curve and $\mathbf{l}_{\mathcal{C}}$ is its unit tangent; $\mathcal{S}$ is the inner Jordan region of $\mathcal{C}$, $A_{\mathcal{S}}$ is its area and $b_{\mathcal{S}}$ its centroid; τ is the kinematic characteristic parameter of the torsion, representing the unit angle of twist (with respect to the point o); $\mathbf{k}$ is the kinematic characteristic parameter of the flexure, representing the (linear) variation of the curvature of the initially straight axis of the cylinder through o. Equation (2.1) describes local balance of contact force; Eq. (2.2) is a local compatibility condition, necessary and sufficient if the section is simply connected; Eq. (2.3) expresses the traction-free condition of the lateral surface of the cylinder; Eq. (2.4) is a global compatibility condition for sections with connection higher than 1.

It is well known that in general the problem (2.1)–(2.4) has no analytical solution in closed form, especially for sections with multiple connection. This makes it clear that for technical applications at least approximate solutions are to be found. So far as we know, in the literature there are no approximate formulae providing reliable results for the shear stress arising in thick sections. The aim of this paper is to look for such formulae, starting from a geometry of the section in which the thickening process depends on one parameter. No attempt will be done to estimate the error made in considering the formulae given here instead of the true solutions. This is a complicated mathematical issue to be faced with the methods proposed in [16].

As Eqs. (2.1)–(2.4) are linear in the two kinematical parameters, it is customary to divide the general problem of the shear stress field into two systems, each depending only on one parameter.

The system depending on τ describes the torsion:

$$\operatorname{div}\mathbf{t} = 0 \qquad \text{in } \mathcal{D}^o_\varepsilon, \tag{2.5}$$

$$\operatorname{curl} \mathbf{t} = 2G\tau \qquad \text{in} \quad \mathcal{D}^o_\varepsilon, \tag{2.6}$$

$$\mathbf{t}\cdot\mathbf{n} = 0 \qquad \text{along} \quad \partial\mathcal{D}_\varepsilon, \tag{2.7}$$

$$\oint_C \mathbf{t}\cdot\mathbf{l}_C = 2GA_\mathcal{S}\tau \qquad \forall C \subset \mathcal{D}_\varepsilon. \tag{2.8}$$

Equations (2.5) – (2.7) suggest that a flow function for **t** may be introduced, named after Prandtl [5]. Starting from such function, there are technical formulae providing approximate shear stress fields for thin sections with both simple and multiple connection [10, 11, 12]; in the latter case, the formula is due to BREDT [13].

In some previous works [2, 3, 4] the torsion problem, defined over a Bredt-like section, has been considered: the stress flow function turns out to depend on the thickness parameter. Once the *Ansatz* is made that the Prandtl function admits a formal ε-power series expansion, one obtains, starting from Eqs. (2.5) – (2.8), a hierarchy of perturbation problems. In [2, 3, 4] it has been shown that the solutions of this hierarchy provide all the known approximate formulae at the lowest steps (i.e., when the section is thin), plus their generalization at the following steps (when the section becomes thick).

The system depending on **k** describes the flexure:

$$\operatorname{div} \mathbf{t} = Y\mathbf{k}\cdot[*(y-b)] \qquad \text{in} \quad \mathcal{D}^o_\varepsilon, \tag{2.9}$$

$$\operatorname{curl} \mathbf{t} = 2G\nu\mathbf{k}\cdot(y-o) \qquad \text{in} \quad \mathcal{D}^o_\varepsilon, \tag{2.10}$$

$$\mathbf{t}\cdot\mathbf{n} = 0 \qquad \text{along} \quad \partial\mathcal{D}_\varepsilon, \tag{2.11}$$

$$\oint_C \mathbf{t}\cdot\mathbf{l}_C = 2GA_\mathcal{S}\nu\mathbf{k}\cdot(b_\mathcal{S}-o) \qquad \forall C \subset \mathcal{D}_\varepsilon. \tag{2.12}$$

In this case there is a technical formula, based on the integral counterpart of Eq. (2.9), due to Jouravski [10, 11, 12]; it provides a mean value for the shear stress component along $\mathcal{L}$, which is an accurate estimate of that component when the section is thin.

Let $\mathbf{t}_o$, $\mathbf{t}_{o'}$ be the field solutions of Eqs. (2.9) – (2.12) associated with any two arbitrarily chosen points o, $o' \in \mathcal{P}$. The field $\mathbf{t}_o - \mathbf{t}_{o'}$ satisfies the system

$$\operatorname{div}(\mathbf{t}_o - \mathbf{t}_{o'}) = 0 \qquad \text{in} \quad \mathcal{D}^o_\varepsilon, \tag{2.13}$$

$$\operatorname{curl}(\mathbf{t}_o - \mathbf{t}_{o'}) = 2G\nu\mathbf{k}\cdot(o'-o) \qquad \text{in} \quad \mathcal{D}^o_\varepsilon, \tag{2.14}$$

$$(\mathbf{t}_o - \mathbf{t}_{o'})\cdot\mathbf{n} = 0 \qquad \text{along} \quad \partial\mathcal{D}_\varepsilon, \tag{2.15}$$

$$\oint_C (\mathbf{t}_o - \mathbf{t}_{o'})\cdot\mathbf{l}_C = 2GA_\mathcal{S}\nu\mathbf{k}\cdot(o'-o) \qquad \forall C \subset \mathcal{D}_\varepsilon. \tag{2.16}$$

If we let $\bar{\tau} := \nu\mathbf{k}\cdot(o'-o)$ (both sides of this definition have the same physical dimensions), Eqs. (2.13) – (2.16) describe a torsion problem whose kinematic characteristic parameter is $\bar{\tau}$. That is to say, the choice of the origin o adds a torsional solution to the system of equations which describe the flexure. We are so free to choose a particular origin to simplify the search for a solution of Eqs. (2.9) – (2.12).

Let us choose $o = b$; it may be proven [6] that the field

$$\tilde{\mathbf{t}} = \frac{G}{4}\{(3 + 2\nu)[*(y - b) \otimes (y - b)] - (1 - 2\nu)[(y - b) \otimes *(y - b)]\}\mathbf{k}, \tag{2.17}$$

defined on the whole plane $\mathcal{P} \supset \mathcal{D}_\varepsilon$, is a particular solution of Eqs. (2.9) – (2.10). It turns out that

$$\int_{\mathcal{C}} *\tilde{\mathbf{t}}\cdot\mathbf{n}_{\mathcal{C}} \oint_{\mathcal{C}} \tilde{\mathbf{t}}\cdot\mathbf{l}_{\mathcal{C}} = 2GA_{\mathcal{S}}\nu\mathbf{k}\cdot(b_{\mathcal{S}} - b) \quad \forall\mathcal{C} \subset \mathcal{D}_\varepsilon \tag{2.18}$$

and the divergence and the curl of the field $\mathbf{t} - \tilde{\mathbf{t}}$ vanish. As a consequence, it is reasonable to look for a generalized stress flow function.

This can be easily done for simply connected sections and for sections whose symmetry group is that of the rectangle. In fact, (i) in simply connected sections the global compatibility condition, Eq. (2.12), is implied by the local condition Eq. (2.10), (ii) in sections with two axes of symmetry it is $b_{\mathcal{S}} = b$. In both cases, Eqs. (2.9) – (2.10), (2.12), (2.17) – (2.18) affirm the existence of a generalized stress flow function $\Psi : \mathcal{D}_\varepsilon \to \mathbb{R}$ [6]. Indeed, let $\mathcal{Q} \subset \mathcal{D}_\varepsilon$ be a region and $\mathcal{M} := \partial\mathcal{Q}\backslash\partial\mathcal{D}_\varepsilon$; we have

$$\begin{aligned}0 = \int_{\mathcal{Q}} \operatorname{div}(\mathbf{t} - \tilde{\mathbf{t}}) = \int_{\partial\mathcal{Q}} (\mathbf{t} - \tilde{\mathbf{t}})\cdot\mathbf{n}_{\partial\mathcal{Q}} = \oint_{\partial\mathcal{Q}} *(\mathbf{t} - \tilde{\mathbf{t}})\cdot\mathbf{l}_{\partial\mathcal{Q}} \\ = \oint_{\mathcal{M}} *(\mathbf{t} - \tilde{\mathbf{t}})\cdot\mathbf{l}_{\mathcal{M}} \Rightarrow \mathbf{t} = \tilde{\mathbf{t}} - *(\nabla\Psi),\end{aligned} \tag{2.19}$$

∇ being the spatial gradient operator in $\mathcal{P}$. For simply connected sections and for sections with double connection and two axes of symmetry it is then possible to build perturbation hierarchies similar to those obtained in [2, 3, 4] for torsion, and their iteration solution provides the known Jouravski formula at the lowest step and its generalization at higher steps [6].

Unfortunately, this procedure faces a serious *empasse* in the case of non-symmetrical sections with multiple connection. Indeed, Eq. (2.12) implies the existence of stress sources in the lacunae with given flow across all the closed lines $\mathcal{M}$. Now, there are infinitely many divergence-free stress fields which are gradients of Prandtl-like functions and whose flow coincides with that given by Eq. (2.18), so that we are to face a gauge choice whose physical meaning is rather uncertain [6].

It is then clear that, in order to provide a perturbation technique valid for all Bredt-like sections, we must change our point of view and abandon the method of the generalized stress flow function. The simpler idea is then to look for a perturbation method to be applied directly to the shear stress field $\mathbf{t}$.

3. The Helmholtz operator in the Bredt basis field

The ε parameter in (1.3) magnifies the thickness of sections with the same "shape". We look for a hierarchy of perturbation problems for $\mathbf{t}$ in ε, whose solution at each step should provide generalized approximate formulae for the shear stress field of thick sections. On this purpose, we will write the Helmholtz operator in Eqs. (2.1)–(2.4) in the coordinate system (s, z) in $\mathcal{D}$ as a function of the thickness parameter ε.

We define the *Bredt basis field* $\mathcal{B}$ in $\mathcal{D}_\varepsilon$ as:

$$\mathcal{B} : (s, z) \mapsto \mathbf{l}(s), \ *\mathbf{l}(s); \tag{3.1}$$

the components of its Lie bracket [14] in (s, z) are

$$[\mathbf{l}, *\mathbf{l}]_1 = \frac{\kappa}{(1-\varepsilon z\kappa\delta)^2}, \qquad [\mathbf{l}, *\mathbf{l}]_2 = -\frac{z\kappa\delta_{,s}}{\delta(1-\varepsilon z\kappa\delta)^2}. \tag{3.2}$$

As the Lie bracket of $\mathcal{B}$ is not, in general, the zero vector field, $\mathcal{B}$ is not, in general, the covariant natural basis associated with any coordinate system [15].

We do not represent the shear stress in the natural basis of the (s, z) coordinate system, but in the Bredt basis, as it is usually done in the applications:

$$\mathbf{t} = t^1\mathbf{g}_1 + t^2\mathbf{g}_2 = t_1\mathbf{g}^1 + t_2\mathbf{g}^2 = t_s\mathbf{l} + t_z * \mathbf{l}. \tag{3.3}$$

According to (3.3), we find the relations between contravariant and Bredt components of the shear stress:

$$t^1 = \frac{t_s}{1-\varepsilon z\kappa\delta}, \qquad t^2 = \frac{1}{\varepsilon\delta}t_z - \frac{z\delta_{,s}}{\delta(1-z\kappa\varepsilon\delta)}t_s. \tag{3.4}$$

Because of Eqs. (3.3)–(3.4), the divergence operator is written as [8]

$$\begin{aligned} \operatorname{div}\mathbf{t} = (\nabla\mathbf{t})^i_i &= t^1_{,s} + \left\{ {1 \atop j\ \ 1} \right\} t^j + t^2_{,z} + \left\{ {2 \atop j\ \ 2} \right\} t^j \\ &= \frac{t_{s,s}}{1-\varepsilon z\kappa\delta} - \frac{\kappa t_z}{1-\varepsilon z\kappa\delta} + \frac{t_{z,z}}{\varepsilon\delta}. \end{aligned} \tag{3.5}$$

The same expression is obtained for Bredt-like sections with constant thickness, i.e., when the coordinates s, z are orthogonal at each point [14]. That is to say, the divergence operator expressed in terms of t_s, t_z does not change its form also when the section has variable thickness and (s, z) becomes a non-orthogonal coordinate system. This is somehow physically reasonable, as the divergence operator is more linked to the variation of the field along the middle line than to that along the thickness. This also justifies the success of the approximate formulae for the shear stress of thin sections.

The covariant components of the shear stress may be found in terms of the Bredt components by $t_i = g_{ij}t^j$, once Eq. (3.3) is given:

$$t_1 = (1 - \varepsilon z \kappa \delta) t_s + \varepsilon z \delta_{,s} t_s\,, \qquad t_2 = \varepsilon \delta t_z\,. \tag{3.6}$$

We use Eq. (3.6) to calculate the curl operator and express it in terms of the Bredt components:

$$\begin{aligned} \operatorname{curl}\mathbf{t} &= \frac{1}{\sqrt{g}}(\nabla_1 t_2 - \nabla_2 t_1) = \frac{t_{2,s} - \left\{ {j \atop 1\ 2} \right\} t_j - t_{1,z} + \left\{ {j \atop 2\ 1} \right\} t_j}{\varepsilon\delta(1 - \varepsilon z\kappa\delta)} \\ &= \frac{t_{2,s} - t_{1,z}}{\varepsilon\delta(1-\varepsilon z\kappa\delta)} = \frac{t_{z,s}}{1 - \varepsilon z\kappa\delta} - \frac{t_{s,z}}{\varepsilon\delta} + \frac{\kappa t_s}{1-\varepsilon z\kappa\delta} - \frac{z\delta_{,s}t_{z,z}}{\delta(1-\varepsilon z\kappa\delta)}. \end{aligned} \tag{3.7}$$

Equation (3.7) differs only in the fourth term from the expression obtained for sections with constant thickness [14]. The additional term is proportional to the derivative of the z-component of the field with respect to z and to the variation of the thickness along $\mathcal{L}$. This is physically reasonable, as the curl operator is linked with the variation of the field along the thickness.

The gradient of a scalar function w in terms of the coordinates s, z is given by

$$\nabla w = \frac{\partial w}{\partial s}\mathbf{g}^1 + \frac{\partial w}{\partial z}\mathbf{g}^2 = \left[\frac{\delta w_{,s} - \varepsilon z \delta_{,s} w_{,z}}{\varepsilon\delta(1 - \varepsilon z\kappa\delta)}\right]\mathbf{l} + \frac{w_{,z}}{\varepsilon\delta} * \mathbf{l}; \tag{3.8}$$

Eq. (3.8) reduces to the expression of the gradient of a scalar function in an orthogonal system of coordinates [14] if we let $\delta_{,s} = 0$.

4. Formal expansions for the shear stress

As the thickness parameter describes a geometrical feature of the section, it is reasonable to let $\mathbf{t}$ be a function of ε as well as of s, z. We propose the following formal series expansion for $\mathbf{t}$ [7]:

$$\mathbf{t}(s, z; \varepsilon) = \sum_{n=0}^{N} \varepsilon^n \mathbf{t}_n(s, z) + o(\varepsilon^N), \tag{4.1}$$

where $o(\varepsilon^N)$ stands for terms of order higher than ε^N. Similar formal expansions hold for both components of the shear stress field with respect to the Bredt basis.

If we substitute Eqs. (4.1), (3.5) and (3.7) into Eqs. (2.5)–(2.8), (2.9)–(2.12), we now obtain two hierarchies of perturbation problems for the shear stress.

The first hierarchy describes the case of torsion:

$$\frac{\sum_{n=0}^{N}\{\varepsilon^{n+1}\delta[t_{sn,s}-\kappa(t_{zn}+zt_{zn,z})]+\varepsilon^{n}t_{zn,z}\}}{\varepsilon\delta(1-\varepsilon z\kappa\delta)}=0, \tag{4.2}$$

$$\frac{\sum_{n=0}^{N}\{\varepsilon^{n+1}[\delta t_{zn,s}+\kappa\delta(t_{sn}+zt_{sn,z})-z\delta_{,s}t_{zn,z}]-\varepsilon^{n}t_{sn,z}\}}{\varepsilon\delta(1-\varepsilon z\kappa\delta)}=2G\tau, \tag{4.3}$$

$$\sum_{n=0}^{N}\varepsilon^{n}\mathbf{t}\cdot\mathbf{n}=0, \tag{4.4}$$

$$\sum_{n=0}^{N}\oint_{z=0}\varepsilon^{n}t_{sn}=2G\tau A_{\mathcal{R}}(b_{\mathcal{R}}-o). \tag{4.5}$$

The other hierarchy describes the case of flexure:

$$\frac{\sum_{n=0}^{N}\{\varepsilon^{n+1}\delta[t_{sn,s}-\kappa(t_{zn}+zt_{zn,z})]+\varepsilon^{n}t_{zn,z}\}}{\varepsilon\delta(1-\varepsilon z\kappa\delta)} = Y\mathbf{k}\cdot(*\bar{\mathbf{r}}_0-\varepsilon z\delta\mathbf{l}+\varepsilon^2\mathbf{b}_2), \tag{4.6}$$

$$\frac{\sum_{n=0}^{N}\{\varepsilon^{n+1}[\delta t_{zn,s}+\kappa\delta(t_{sn}+zt_{sn,z})-z\delta_{,s}t_{zn,z}]-\varepsilon^{n}t_{sn,z}\}}{\varepsilon\delta(1-\varepsilon z\kappa\delta)} = 2G\nu\mathbf{k}\cdot(\mathbf{r}_0+\varepsilon z\delta*\mathbf{l}), \tag{4.7}$$

$$\sum_{n=0}^{N}\varepsilon^{n}\mathbf{t}\cdot\mathbf{n}=0, \tag{4.8}$$

$$\sum_{n=0}^{N}\oint_{z=0}\varepsilon^{n}t_{sn}=2G\nu\mathbf{k}\cdot A_{\mathcal{R}}(b_{\mathcal{R}}-o). \tag{4.9}$$

In Eqs. (4.2)–(4.9) the terms of order higher than ε^N have been dropped; in Eq. (4.6) $\bar{\mathbf{r}}_0:=\mathbf{r}_0-\mathbf{b}_0$; the outer normal vector $\mathbf{n}$ in Eqs. (4.4), (4.8) is evaluated in the following Eqs. (4.10), (4.13), (4.14); in Eqs. (4.5), (4.9) $\mathcal{R}$ is the inner Jordan region (if any) enclosed by $\mathcal{L}$ $(z=0)$ and $b_{\mathcal{R}}$ is its centroid.

If the section has double connection, we will call it *closed*, referring to its middle line, which is homotopic to a circumference. In this case, there are two different connected elements which compose the boundary of the section. The outer vector field normal to the boundary is given by

$$\mathbf{n}|_{z=-1}=-\varepsilon\delta_{,s}\mathbf{l}-(1+\varepsilon\kappa\delta)*\mathbf{l}, \qquad \mathbf{n}|_{z=1}=-\varepsilon\delta_{,s}\mathbf{l}+(1-\varepsilon\kappa\delta)*\mathbf{l}, \tag{4.10}$$

so that the boundary conditions for closed sections are, dropping the terms $o(\varepsilon^N)$,

$$\sum_{n=0}^{N} |\varepsilon^{n+1}(\delta_{,s} t_{sn} + \kappa\delta t_{zn}) + \varepsilon^n t_{zn}|_{z=-1} = 0, \tag{4.11}$$

$$\sum_{n=0}^{N} |-\varepsilon^{n+1}(\delta_{,s} t_{sn} + \kappa\delta t_{zn}) + \varepsilon^n t_{zn}|_{z=1} = 0. \tag{4.12}$$

If the section is simply connected, we will call it *open*, always referring to its middle line, now homotopic to a segment. The boundary is composed by a unique connected element, divided into four regular components. The outer vector field normal to the boundary is represented by

$$\mathbf{n}|_{s=0} = -\mathbf{l}, \qquad \mathbf{n}|_{z=-1} = -\varepsilon\delta_{,s}\mathbf{l} - (1 + \varepsilon\kappa\delta) * \mathbf{l}, \tag{4.13}$$

$$\mathbf{n}|_{s=l} = \mathbf{l}, \qquad \mathbf{n}|_{z=1} = -\varepsilon\delta_{,s}\mathbf{l} + (1 - \varepsilon\kappa\delta) * \mathbf{l}, \tag{4.14}$$

so that the boundary conditions for open sections are, always dropping the terms $o(\varepsilon^N)$,

$$\sum_{n=0}^{N} |\varepsilon^n t_{sn}|_{s=0} = 0, \tag{4.15}$$

$$\sum_{n=0}^{N} |\varepsilon^{n+1}(\delta_{,s} t_{sn} + \kappa\delta t_{zn}) + \varepsilon^n t_{zn}|_{z=-1} = 0, \tag{4.16}$$

$$\sum_{n=0}^{N} |\varepsilon^n t_{sn}|_{s=l} = 0, \tag{4.17}$$

$$\sum_{n=0}^{N} |-\varepsilon^{n+1}(\delta_{,s} t_{sn} + \kappa\delta t_{zn}) + \varepsilon^n t_{zn}|_{z=1} = 0. \tag{4.18}$$

In the next sections we will look for the solution of the hierarchies of perturbation problems (4.2)–(4.9), with the appropriate substitution of the boundary conditions (4.11)–(4.12), (4.15)–(4.18), both for closed and open sections.

As the structure of both hierarchies is the same, the only difference being brought in by the right-hand sides in Eqs. (4.2)–(4.3), (4.6)–(4.7), we obtain (see next section) the same recursive structure for the coefficients in Eq. (4.1), no matter if we study torsion or flexure.

In particular, there are common features of the two hierarchies which are worth remarking:

a. Each step can be solved within an unknown function of the coordinate s, which is determined only by solving the local balance equation, together with the boundary conditions, at the next step. This is a very well known possible feature of perturbation series [7].

b. Both for closed and open sections, the structure of the perturbation problem is the same but for two different boundary conditions. Anyway, in the case of open sections, we cannot force a system of *second* order to fulfill *four* independent boundary conditions. This is another well known phenomenon of perturbation procedures, when being in presence of a boundary layer (in this case, in a neighbourhood of the "short" sides of the section [2, 3, 4, 7]). We should provide two different expansions, called *outer* and *inner* [7], and then match them; but for the aim of this paper we will content ourselves with the outer expansion, valid outside the region of the boundary layer.

4.1. Shear stress coefficients in torsion

We will first give the solutions of the first steps of our perturbation series in the case of simply connected (open) sections. We remind that the following coefficients are those of the outer formal series expansion for the shear stress. It is

$$\mathbf{t}_0 = \mathbf{0}, \tag{4.19}$$

$$\mathbf{t}_1 = -(2G\tau z\delta)\mathbf{l}, \tag{4.20}$$

$$\mathbf{t}_2 = G\tau\left\{\left[(1-z^2)\kappa\delta^2 - \frac{2}{3}\int_0^s \delta(\kappa\delta^2)_{,s}\right]\mathbf{l} + \delta\delta_{,s}(z^2-3)*\mathbf{l}\right\}. \tag{4.21}$$

Equation (4.19) is a known result, though never explicitly affirmed in the literature: when the thickness vanishes, the only solution of the torsion problem is the zero field. Equation (4.20) is the result usually provided in the literature for thin sections, sometimes attributed to Kelvin [2, 3, 4, 10, 11, 12]; Eq. (4.21) is the generalization of this formula when the section becomes thicker.

Equation (4.19) verifies all the boundary conditions; Eq. (4.20) verifies the boundary conditions at the "short" sides of the section as a mean over the thickness; Eq. (4.21) verifies only the boundary conditions at the "long" sides of the sections. This implies that, even if at the lowest step of the hierarchy the zero field is a suitable solution also near the "short" sides, at the successive iteration steps a boundary layer in that region arises, named after Kelvin. We will provide in Sec. 6.1 a measure of this effect, the same as that presented in the literature.

In the case of sections with double connection (closed) we have

$$\mathbf{t}_0 = 2G\tau\frac{A_{\mathcal{R}}}{\delta\oint\limits_{\mathcal{L}}\dfrac{1}{\delta}}\mathbf{l} =: t_{s0}\mathbf{l}, \tag{4.22}$$

$$\mathbf{t}_1 = z[\delta(\kappa t_{s0} - 2G\tau)\mathbf{l} + \delta_{,s}*\mathbf{l}], \tag{4.23}$$

$$(4.24)\qquad \mathbf{t}_2 = \left[z^2\kappa\delta^2(\kappa t_{s0} - 2G\tau) - z^2\delta_{,s}t_{s0} + \frac{z^2}{2}\delta\delta_{,s}t_{s0} + \bar{t}_{s2}\right]\mathbf{l} + \left\{\delta\delta_{,s}\left[(z^2+1)\kappa t_{s0} + (z^2-3)G\tau\right] - \frac{z^2-1}{2}\delta^2\kappa_{,s}t_{s0}\right\} * \mathbf{l}.$$

Equation (4.22) is the well known Bredt formula [2, 3, 4, 10, 11, 12, 13]; Eqs. (4.23) – (4.24) provide its generalization when the section is thicker. In Eq. (4.24) $\bar{t}_{s2}$ is an unknown function of s alone, to be determined by solving the next step of the perturbation procedure.

We remark that for closed section we have a non-vanishing $\mathbf{t}_0$: this is physically grounded, because the stress flow runs along all the closed lines which fill the section; on the contrary, in open section the condition that the shear stress flow be zero necessarily implies that the shear stress must also be zero at the lowest step of the hierarchy.

If we let $\delta_{,s} = 0$ (δ = constant along $\mathcal{L}$) in the Eqs. (4.19) – (4.22), we recover all the results provided in [2] for sections with constant thickness.

4.2. Shear stress coefficients in flexure

As it has been done in the case of torsion, we will begin providing the solutions of the first steps of our perturbation procedure for open sections; these are also in this case the coefficients of an outer formal expansion. It is

$$(4.25)\qquad \mathbf{t}_0 = Y\mathbf{k}\cdot\frac{\int_0^s \delta * \bar{\mathbf{r}}_0}{\delta}\mathbf{l} =: t_{s0}\mathbf{l},$$

$$(4.26)\qquad \mathbf{t}_1 = z[\delta(\kappa t_{s0} - 2G\nu\mathbf{k}\cdot\mathbf{r}_0)\mathbf{l} + \delta_{,s} * \mathbf{l}],$$

$$(4.27)\qquad \mathbf{t}_2 = \left\{\left[z^2\delta^2\kappa(\kappa t_{s0} - 2G\nu\mathbf{k}\cdot\mathbf{r}_0) - G\nu\mathbf{k}\cdot * \mathbf{l}\right] + \frac{z^2}{2}\left[\delta(\delta_{,s}t_{s0})_{,s} - \delta_{,s}^2 t_{s0}\right]\right\}\mathbf{l} + \left\{\delta\delta_{,s}\left[(z^2+1)\kappa t_{s0} + (z^2-3)G\nu\mathbf{k}\cdot\mathbf{r}_0\right] - \frac{z^2-1}{2}\delta^2\left[\kappa_{,s}t_{s0} + \mathbf{k}\cdot(\mathbf{l} + Y\kappa * \bar{\mathbf{r}}_0)\right]\right\} * \mathbf{l}.$$

Equation (4.25) is the known Jouravski formula and it may be found in all the textbooks on strength of materials [10, 11, 12]: the integral in t_{s0} is the first moment of area of the part of the section enclosed by the values 0 and s of the arc length of the middle line. Jouravski formula in the literature is written in terms of the resultant shear stress; it is easy, though, to recognize the same expression once having remembered the linear relationship between the kinematic parameter $\mathbf{k}$ and the resultant shear stress $\mathbf{q}$ [1, 6, 9]. We will discuss this in Sec. 6.2, where we calculate the coefficients of a formal expansion for the resultant shear stress.

The component along $*\mathbf{l}$ of Eq. (4.26) is another known formula. In the literature it is sometimes found by means of heuristic and graphic deduction and it is said, also on heuristic grounds, that its magnitude is small if compared with that provided by Jouravski formula. From the point of view of our technique, this result is clearly interpreted as a higher order effect in terms of the perturbation parameter.

The components along $\mathbf{l}$ of Eq. (4.26) and Eq. (4.27) are not found in the literature, so far as we know. They are the first generalization of Jouravski formula for open sections which become thick.

Exactly as we have found in the case of torsion, Eq. (4.25) verifies all the boundary conditions; Eq. (4.26) verifies the boundary conditions at the "short" sides of the section as a mean over the thickness; Eq. (4.27) verifies only the boundary conditions at the "long" sides of the sections. That is, also in flexure a boundary layer near the 'short' sides arises, a measure of which we provide in Sec. 6.2.

In the case of sections with double connection (closed) we have

$$\mathbf{t}_0 = \left\{ \frac{Y\mathbf{k}}{\delta} \cdot \left[\int_0^s (\delta * \bar{\mathbf{r}}_0) - \frac{\oint_{\mathcal{L}} \frac{\int_0^s (\delta * \bar{\mathbf{r}}_0)}{\delta}}{\oint_{\mathcal{L}} \frac{1}{\delta}} \right] + \frac{2GA_{\mathcal{R}}\nu\mathbf{k}\cdot(b_{\mathcal{R}} - o)}{\delta \oint_{\mathcal{L}} \frac{1}{\delta}} \right\} \mathbf{l} =: t_{s0}\mathbf{l}, \tag{4.28}$$

$$\mathbf{t}_1 = z\{\delta[\kappa t_{s0} - 2G\nu\mathbf{k}\cdot\mathbf{r}_0]\mathbf{l} + (\delta_{,s}t_{s0}) * \mathbf{l}\}, \tag{4.29}$$

$$\begin{aligned} \mathbf{t}_2 = {} & \left\{ \frac{z^2}{2} \left[\delta(\delta_{,s}t_{s0})_{,s} - \delta_{,s}^2 t_{s0} \right] + z^2\delta^2 \left[\kappa^2 t_{s0} - G\nu\mathbf{k}\cdot(\kappa\mathbf{r}_0 + *\mathbf{l}) \right] + \bar{t}_{s2} \right\} \mathbf{l} \\ & + \left\{ \delta\delta_{,s} \left[(z^2+1)\kappa t_{s0} + (z^2-3)G\nu\mathbf{k}\cdot\mathbf{r}_0 \right] \right. \\ & \left. + (1-z^2)\delta^2 \left[\frac{\kappa_{,s}t_{s0}}{2} + \mathbf{k}\cdot(G\mathbf{l} + Y\kappa * \bar{\mathbf{r}}_0) \right] \right\} *\mathbf{l}. \end{aligned} \tag{4.30}$$

Equation (4.28) is the Jouravski formula for thin closed section, which in the literature is found by an application of Volterra distorsions and the principle of virtual power [11, 12]. The second addend in t_{s0} is sometimes called "torsion in the section as a whole", because it has the same form of a Bredt field in the torsion of a thin closed section [2, 3, 4, 10, 11, 12, 13]; on the basis of our procedure, this similarity is more strict, as both come from the integral condition of compatibility, Eq. (2.4) [2, 3, 4]. Equations (4.29) – (4.30) provide a generalization of Jouravski formula for closed sections when the section is thicker. In Eq. (4.30) $\bar{t}_{s2}$ is an unknown function of s alone, determined in an implicit form by solving the next step of the procedure.

5. Formal expansions for the warping

There is the following relationship between the shear stress field and the warping w of a section of a Saint-Venant cylinder [1, 6, 9]:

$$\mathbf{t} = G(\mathbf{v}' + \nabla w). \tag{5.1}$$

In Eq. (5.1) $\mathbf{v}'$ is a known function, given by

$$\mathbf{v}' = \tau * \mathbf{r} + \nu\{(\mathbf{r} \otimes *\mathbf{r}) + [*(y - b) \otimes \mathbf{r}\}\mathbf{k}; \tag{5.2}$$

it expresses the variation along the axis of the cylinder of the displacement of its substantial points in the plane of its sections. In Eq. (5.2) we dropped the addends which expressed a rigid contribution and used the definition $(\mathbf{a} \otimes \mathbf{b})\mathbf{c} := (\mathbf{a} \cdot \mathbf{c})\mathbf{b}$. The warping is then directly linked with the solution of the elliptic problem (2.1)–(2.4), and it is natural to try to integrate Eq. (5.1) to find its expression.

We assume that the following formal expansion holds:

$$w(s, z; \varepsilon) = \sum_{n=0}^{N} \varepsilon^n w_n(s, z) + o(\varepsilon^N). \tag{5.3}$$

Equation (5.3), substituted into Eqs. (3.8), (5.1)–(5.2), leads to a hierarchy of perturbation problems for the coefficients of the formal expansion of the warping. Due to the linearity of the problem in the two kinematical parameters, it is suggestive to split the hierarchy into two ones.

The first hierarchy describes the warping in torsion:

$$\begin{aligned} &\sum_{n=0}^{N} \left[\varepsilon^{n+1}(\delta w_{n,s} - z\delta_{,s} w_{n,z})\mathbf{l} + \varepsilon^n (1 - \varepsilon z \kappa \delta) w_{n,z} * \mathbf{l}\right] \\ &\quad = \frac{\sum_{n=0}^{N} \varepsilon^n (1 - \varepsilon z \kappa \delta)\delta \mathbf{t}_n}{G} - \tau \varepsilon \delta \left[*\mathbf{r}_0 - \varepsilon z \delta(\kappa * \mathbf{r}_0 + \mathbf{l}) + \varepsilon^2 z^2 \kappa \delta^2 \mathbf{l}\right]. \end{aligned} \tag{5.4}$$

The second hierarchy describes the warping in flexure:

$$\begin{aligned} &\sum_{n=0}^{N} \left[\varepsilon^{n+1}(\delta w_{n,s} - z\delta_{,s} w_{n,z})\mathbf{l} + \varepsilon^n (1 - \varepsilon z \kappa \delta) w_{n,z} * \mathbf{l}\right] \\ &\quad = \frac{\sum_{n=0}^{N} \varepsilon^n (1 - \varepsilon z \kappa \delta)\delta \mathbf{t}_n}{G} \\ &\quad -\nu\Big\{\mathrm{sym}(\mathbf{r}_0 \otimes *\mathbf{r}_0) - *\mathbf{b}_0 \otimes \mathbf{r}_0 + \varepsilon z \delta[\mathrm{sym}(*\mathbf{l} \otimes *\mathbf{r}_0 - \mathbf{r} \otimes \mathbf{l}) - (*\mathbf{b}_0 \otimes *\mathbf{l})] \\ &\quad + \varepsilon^2[z^2\delta^2 \mathrm{sym}(*\mathbf{l} \otimes \mathbf{l}) - (*\mathbf{b}_2 \otimes \mathbf{r}_0)] + \varepsilon^3 z \delta(*\mathbf{b}_2 \otimes *\mathbf{l})\Big\}\mathbf{k}, \end{aligned} \tag{5.5}$$

where sym stands for the symmetric part of the indicated tensor.

In both Eqs. (5.4) – (5.5) we have dropped the terms of order higher than ε^N. The two hierarchies have the same structure, and the only difference is brought in by the right-hand sides, just like in the problems of determining the coefficients of the formal expansion of the shear stress. It is important to remark that the structure of the hierarchies decouples the system of partial differential equations into a system of *ordinary* equations, which significantly simplifies all the calculation. In the following we will give the results of the first coefficients of the formal expansions for the warping, dropping the constant values of integration which will only imply an (inessential) rigid contribution.

5.1. Warping coefficients in torsion

We provide at first the results for open sections, for which we have

$$w_0 = -2\tau\Omega(s), \tag{5.6}$$

$$w_1 = -\tau z\delta\mathbf{r}_0\cdot\mathbf{l}. \tag{5.7}$$

In Eq. (5.6) $\Omega(s)$ is the area of the inner Jordan region of the curve composed by the position vectors $\mathbf{r}(0)$, $\mathbf{r}(s)$ and the arc $0,\ s$ of $\mathcal{L}$ (sectorial area). Equation (5.6) is a formula which is found in the literature in the framework of the so-called Vlasov theory of thin-walled beams [11]. Equation (5.7) coincides with a fundamental assumption of the aforementioned theory: it states that for thicker sections the warping will be a linear function of the coordinate along the thickness. Vlasov does not consider a Saint – Venant cylinder, but regards a thin-walled beam as a shell and postulates that the section of the shell (which we have interpreted as the section of a Saint – Venant cylinder) is not affected by any deformation in its own plane. In Vlasov theory Eq. (5.6) is a consequence of the introduction of Kirchhoff-type internal constraints given by Eq. (5.7). From the point of view of our perturbation approach, this result is naturally obtained solving the lowest step of the hierarchy, which corresponds to thin sections. Moreover, our procedure shows that Kirchhoff-type constraints seem to be naturally satisfied by Saint – Venant displacement fields.

For closed sections we have

$$w_0 = 2\tau\left[\frac{A_{\mathcal{R}}}{\oint\limits_{\mathcal{L}}\dfrac{1}{\delta}}\int_0^s\frac{1}{\delta} - \Omega(s)\right], \tag{5.8}$$

$$w_1 = -\tau z\delta\mathbf{r}_0\cdot\mathbf{l}. \tag{5.9}$$

Equation (5.8) gives again the warping according to the law of the sectorial area; this also is a known result for thin sections, based on Vlasov's theory. The additional term present in Eq. (5.8) with respect to Eq. (5.6) takes into account the necessity for the warping function to be periodical along the middle line.

Equation (5.9) affirms that the first higher order correction to the warping of thin sections is the same as that for open sections. This is physically reasonable because the behaviour of the warping along the thickness does not have to depend on the section being open or closed.

The same results presented in Eqs. (5.6)–(5.9) were obtained also in [2, 3, 4] by means of the Prandtl stress flow function. It is worth remarking that in [2] we obtained the same equations dealing with Bredt-like sections of constant thickness: that is, at the first two steps of the iteration procedure, the coefficients of the formal expansion for the warping are not affected by the variable thickness.

5.2. Warping coefficients in flexure

In the case of flexure we have

$$w_0 = \frac{1}{G}\int_0^s t_{s0} - \nu\mathbf{k}\cdot\int_0^s [\mathrm{sym}\,(\mathbf{r}_0 \otimes *\mathbf{r}_0) - \mathbf{r}_0 \otimes *\mathbf{b}_0]\mathbf{l}\,, \tag{5.10}$$

$$w_1 = -\nu z\delta\left\{[\mathrm{sym}\,(\mathbf{r}_0 \otimes *\mathbf{r}_0) - *\mathbf{b}_0 \otimes \mathbf{r}_0]\mathbf{k}\cdot(*\mathbf{l}) + 2\mathbf{k}\cdot\int_0^s \mathbf{r}_0\right\}. \tag{5.11}$$

Equation (5.10) is the warping for thin sections, according to the interpretation of the thickness perturbation parameter; Eq. (5.11) is its generalization. As in the case of torsion, the first order coefficient of the formal expansion of the warping is a linear function of the z-coordinate.

We remark that the structure of the solution is the same both for open and closed sections, the difference being brought in only by the different expression of t_{s0}. This behaviour could be seen also in the expressions of the warping coefficients of the torsion. So far as we know, such a general formulation for the warping cannot be found in the literature.

6. Formal expansions for the resultant force and torque and for the shear shape factors

In Saint–Venant cylinders the kinematic characteristic parameters on which the general solution of the problem depends are linearly related to the resultant actions (force and torque) which act on the basis of the cylinder. This is of great relevance from the point of view of the applications, because it makes it possible to project the results obtained in the three-dimensional Saint–Venant theory onto the one-dimensional beam theory. In this way, the stiffness of a beam is a global property which results from an integral defined over the section of a Saint–Venant cylinder. So, it is rather important to check if our perturbation procedure is able to provide good approximate results also for the resultant force and torque.

In the case of torsion, there is only one non-vanishing resultant action, which is the resultant torque due to the shear stress distribution:

$$T := \int_{\mathcal{D}_\varepsilon} [*(y - \bar{x})] \cdot \mathbf{t}, \tag{6.1}$$

where $\bar{x}$ is any point in $\mathcal{P}$, chosen as reduction pole for the torque. It is convenient (but unnecessary, of course) to put $\bar{x} = o$, so that the lever arm of the torque distribution is just the position vector of the places of the section, for which we have an ε-dependent expression, Eq. (1.3).

Also for the torque we obtain

$$T(\varepsilon) = \sum_{n=0}^{N} \varepsilon^n T_n = \int_{-1}^{1} \int_{0}^{l} \sqrt{g}(*\mathbf{r}_0 - \varepsilon z \delta \mathbf{l}) \cdot \sum_{n=0}^{N} \varepsilon^n \mathbf{t}_n\,, \tag{6.2}$$

where, as usual, we have dropped the terms $o(\varepsilon^N)$. On the basis of the results obtained in Sec. 4.1, we will calculate the coefficients of the formal expansion of the resultant torque and, as a consequence, the torsional rigidity (simply defined as the ratio of the torque and the unit angle of twist).

In the case of flexure, the peculiar resultant action is the shearing force:

$$\mathbf{q} := \int_{\mathcal{D}_\varepsilon} \mathbf{t}, \tag{6.3}$$

as a matter of fact, the resultant torque of the shear stress distribution is an effect of the choice of the origin of the plane $\mathcal{P}$, Eqs. (2.13) – (2.16), and vanishes when the so-called shear centre, or centre of flexure [9, 10, 11, 12] is chosen as origin. For $\mathbf{q}$ we obtain

$$\mathbf{q}(\varepsilon) = \sum_{n=0}^{N} \varepsilon^n \mathbf{q}_n = \int_{-1}^{1} \int_{0}^{l} \sqrt{g} \sum_{n=0}^{N} \varepsilon^n \mathbf{t}_n\,, \tag{6.4}$$

always dropping the terms $o(\varepsilon^N)$.

There is also another expression for $\mathbf{q}$, given by the theory of Saint-Venant [1, 6, 9]:

$$\mathbf{q} = Y * \mathbf{J}_b \mathbf{k}, \tag{6.5}$$

$$\mathbf{J}_b := \int_{\mathcal{D}_\varepsilon} [*(y - b) \otimes *(y - b)], \tag{6.6}$$

where $\mathbf{J}_b$ is a tensor of inertia of the section with respect to its centroid. Substituting the expression of the position vector of the places of the section, Eq. (1.3), into Eqs. (6.5), (6.6) it is found that $\mathbf{J}_b$ is a polynomial function of the thickness parameter:

$$(6.7)\qquad \mathbf{J}_b = 2\varepsilon \int_0^l \delta(*\bar{\mathbf{r}}_0 \otimes *\bar{\mathbf{r}}_0) + \frac{4}{3}\varepsilon^3 \int_0^l \delta^3[\mathbf{l}\otimes\mathbf{l} + \operatorname{sym}(\bar{\mathbf{r}}_0 \otimes \kappa\mathbf{l})] =: \varepsilon\mathbf{J}_{b1} + \varepsilon^3\mathbf{J}_{b3}.$$

According to Eqs. (6.5) – (6.7), then, the resultant shearing force may be written as

$$(6.8)\qquad \mathbf{q} = Y * (\varepsilon\mathbf{J}_{b1} + \varepsilon^3\mathbf{J}_{b3})\mathbf{k} =: \varepsilon\mathbf{q}_1 + \varepsilon^3\mathbf{q}_3\,.$$

In Sec. 6.2 we will compare the results obtained by substituting the coefficients of the shear stress given in Sec. 4.2 in Eq. (6.4) with the result given by Eqs. (6.5)–(6.8).

In the technical literature the symmetric tensor $\mathbf{K}$ of the shear shape factors is introduced, according to the following equivalence in power:

$$(6.9)\qquad \frac{1}{2}\mathbf{q}\cdot\mathbf{K}\frac{\mathbf{q}}{GA_{\mathcal{D}_\varepsilon}} = \frac{1}{2G}\int_{\mathcal{D}_\varepsilon}(\mathbf{t}\cdot\mathbf{t}) \;\Rightarrow\; \mathbf{q}\cdot\mathbf{K}\mathbf{q} = A_{\mathcal{D}_\varepsilon}\int_{\mathcal{D}_\varepsilon}(\mathbf{t}\cdot\mathbf{t}).$$

An ε-formal power series expansion holds also for $\mathbf{K}$, as, from Eq. (6.9), we easily obtain the following expression:

$$(6.10)\qquad \left(\sum_{n=0}^{N}\varepsilon^n\mathbf{q}_n\right)\cdot\left(\sum_{n=0}^{N}\varepsilon^n\mathbf{K}_n\sum_{n=0}^{N}\varepsilon^n\mathbf{q}_n\right) = 2\varepsilon\int_0^l \delta\int_{-1}^{1}\int_0^l \sqrt{g}\left(\sum_{n=0}^{N}\varepsilon^n\mathbf{t}_n\right)\cdot\left(\sum_{n=0}^{N}\varepsilon^n\mathbf{t}_n\right);$$

in the former, we have dropped the terms $o(\varepsilon^N)$.

6.1. Resultant torque coefficients in torsion

As previously done, we will first give the results for open sections, for which we have found only an outer expansion of the shear stress distribution:

$$(6.11)\qquad T_0 = T_1 = T_2 = 0,$$

$$(6.12)\qquad T_3 = \frac{4}{3}G\tau\int_0^l\left\{\left[\int_0^s \kappa\delta^2\delta_{,s}\right](*\mathbf{r}_0\cdot\mathbf{l}) - 4\delta^2\delta_{,s}(\mathbf{r}_0\cdot\mathbf{l}) + \delta^3(1 + *\mathbf{r}_0\cdot\kappa\mathbf{l})\right\}.$$

Equation (6.11) expresses a known result, which is not clearly affirmed in the literature, though: the resultant torque in the torsion of an open section of a

Saint-Venant cylinder appears only with the third power of the thickness. From the point of view of our technique, this is a natural result of the perturbation method.

Equation (6.12) is the generalization of another known result, that is to say, the so-called Kelvin formula for the resultant torque of thin sections. So far as we know, in the literature there is no general formula like Eq. (6.12). All the results given in the literature for each particular open section may be found starting from Eq. (6.12): this is particularly evident in the case of sections with constant thickness, for which we immediately recover the expressions given in [2, 3] and in the technical literature [10, 11, 12].

For instance, from Eq. (6.12) one obtains the exact resultant torque in the case of a semicircular section with constant thickness. In this case, the boundary layer effect, named after Kelvin, vanishes because of geometrical effects, and the outer expansion of the shear stress provides a very good approximation of the global effects of the actual distribution.

This does not happen in the case of rectangular sections, for which we obtain from Eq. (6.12)

$$T_3 = \frac{G\tau(2\delta)^3 l}{6}, \tag{6.13}$$

which is only one half of the actual resultant torque. The outer expansion of $\mathbf{t}$ loses information on the actual stress distribution near the short edges of the rectangle. The portion of resultant torque missing in Eq. (6.13) is a measure of the boundary layer (Kelvin) effect, and was already obtained in some previous works [2, 3, 4, 6].

We remark, though, that, if we start studying the torsion from a formal expansion of the Prandtl stress flow function, as was done in [2, 3, 4, 6], we obtain the exact result for the resultant torque. This is reasonable, because Prandtl function is in some sense a primitive for the shear stress (see Sec. 2), that is to say, it is richer in information. Thus, if we calculate the resultant torque as the integral of Prandtl function over the section [1, 6, 9, 12], we are able to recover also the global contribution of the boundary layer (but not its actual local behaviour).

In the case of closed sections we obtain

$$T_0 = 0, \tag{6.14}$$

$$T_1 = 4G\tau \frac{A_{\mathcal{R}}^2}{\oint\limits_{\mathcal{L}} \frac{1}{2\delta}}, \tag{6.15}$$

$$T_2 = 0. \tag{6.16}$$

Equation (6.14) is a known result, though never explicitly affirmed in the literature: the resultant torque must vanish when the thickness of the section fades.

Equation (6.15) is another well known result, that is to say, Bredt formula [10, 11, 12, 13]; the same expression was obtained also starting from Prandtl stress flow function in [2, 3, 4]. Equation (6.16) is a new result which affirms that the first generalization of Bredt formula has to be searched at orders higher than two in the thickness parameter: this is a justification of the validity of Bredt formula in all the applications.

6.2. Resultant force and shear shape factors coefficients in flexure

For both open and closed sections we obtain

$$\mathbf{q}_0 = \mathbf{0}, \tag{6.17}$$

$$\mathbf{q}_1 = Y * \mathbf{J}_{b1}\mathbf{k}, \tag{6.18}$$

$$\mathbf{q}_2 = \mathbf{0}. \tag{6.19}$$

As usual, Eq. (6.17) affirms that the resultant action must be zero for fading thickness. The most interesting result is represented by Eqs. (6.18) – (6.19): they show that, no matter if we use Eq. (6.4) or Eqs. (6.5) – (6.8), we obtain the same results. We remark also that the resultant shearing force, as a global result, does not take into account the section being open or closed, and the different forms of Jouravski formula in the two cases, Eqs. (4.25), (4.28). Equations (6.17) – (6.19) imply that Kelvin effect in flexure is at least a third order effect in ε. Such a phenomenon has been studied in [6] in the case of open Bredt-like sections with constant thickness. In [6] the flexure is studied starting from a generalized stress flow function; there it is shown that such a function verifies all boundary conditions up to the second order in an ε-formal expansion – i.e., a boundary effect arises only starting from the third order in ε.

Equations (6.18) – (6.19) confirm that, even if Jouravski formula was originally obtained in a heuristic way, its validity is really great for all the applications. Indeed it is simple to use and provides good global approximate results. We also think that, as we have been able to find and rationally justify these results, our generalization perturbation technique is meaningful.

As for the coefficients of the formal expansion of the shear shape factors tensor (6.10), we obtain

$$\mathbf{q}_1 \cdot \mathbf{K}_0 \mathbf{q}_1 = 4 \int_0^l \delta \int_0^l \delta (x t_0 \cdot \mathbf{t}_0), \tag{6.20}$$

$$\mathbf{q}_1 \cdot \mathbf{K}_1 \mathbf{q}_1 = 0 \quad \Rightarrow \quad \mathbf{K}_1 = \mathbf{0}, \tag{6.21}$$

$$\mathbf{q}_1 \cdot \mathbf{K}_2 \mathbf{q}_1 = 4 \int_0^l \delta \int_0^l \left[(2\mathbf{t}_0 \cdot \mathbf{t}_2 + \mathbf{t}_1 \cdot \mathbf{t}_1)\delta - 2z\kappa\delta^2 \mathbf{t}_1 \cdot \mathbf{t}_0 \right]. \tag{6.22}$$

It is easy and meaningful to calculate the coefficient given by Eq. (6.20) at least for simply connected (open) sections, using Eqs. (4.25) and (6.18). If we make use of an orthonormal basis whose elements are directed along the principal axes of inertia of the section, it is

$$\mathbf{K}_0 = \frac{A_{\mathcal{D}_\varepsilon}}{2}\begin{pmatrix} \dfrac{1}{J_{b1\,22}^2}\displaystyle\int_0^l \frac{S_2^2}{\delta} & \dfrac{1}{J_{b1\,11}J_{b1\,22}}\displaystyle\int_0^l \frac{S_1S_2}{\delta} \\ \dfrac{1}{J_{b1\,11}J_{b1\,22}}\displaystyle\int_0^l \frac{S_1S_2}{\delta} & \dfrac{1}{J_{b1\,11}^2}\displaystyle\int_0^l \frac{S_1^2}{\delta} \end{pmatrix}, \tag{6.23}$$

S_1, S_2 are the first moments of area with respect to the 1 and 2 directions as functions of the s coordinate.

So far as we know, Eq. (6.23) is not given in the literature; usually only one of the components of the main diagonal is calculated [11, 12], and coincides with that given in Eq. (6.23).

7. Applications

We will at first consider the torsion of an isosceles trapezium whose height is l and whose bases are $2h_1$, $2h_2$. We let the s coordinate run along the height; as the s coordinate line is a portion of a straight line, its curvature vanishes. An orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2)$ is given whose first element is parallel to the height of the trapezium,

$$\begin{aligned} \mathbf{r}_0 &= s\mathbf{e}_1, \quad 0 \le s \le l \ \Rightarrow \ \mathbf{l} = \mathbf{e}_1, \\ \delta(s) &= h_1 + (h_2 - h_1)\frac{s}{l} \ \Rightarrow \ \delta_{,s} = \frac{h_2 - h_1}{l}. \end{aligned} \tag{7.1}$$

Obviously, if $h_2 = 0$ the section reduces to an isosceles triangle and if $h_2 = h_1$, the trapezium degenerates into a rectangle.

Let us determine, as a meaningful example, the first nonvanishing term of the resultant torque coefficient given according to Eq. (6.12):

$$\frac{T_3}{G\tau} = \frac{4}{3}\int_0^l (\delta^3 - 4\delta^2\delta_{,s}\mathbf{r}_0 \cdot \mathbf{l}) = \frac{4}{3}l\left[\frac{7}{12}h_1(h_1^2 + h_1h_2 + h_2^2) - \frac{3}{4}h_2^3\right]. \tag{7.2}$$

If $h_1 \approx h_2$, we obtain once again the same result provided by Eq. (6.13), which was to be expected, as we have used only an outer expansion for $\mathbf{t}$. If $h_2 = 0$ (isosceles triangle), it is

$$\frac{T_3}{G\tau} = \frac{7}{72}(2h_1)^3 l. \tag{7.3}$$

In [11] we found that the value for $T_3/G\tau$ is $[(2h_1)^3 l]/6$; then Eq. (7.3) overestimates the actual torsional rigidity. The difference between the value given by Eq. (7.3) and that in [11] is a measure of Kelvin effect for isosceles triangles: it is less relevant than that for the rectangle, as the vertex of the triangle is a stagnation point for the shear stress flow and its contribution to the resultant torque is negligible. The result given in [3] was obtained by using Prandtl stress flow function. We remarked in the last section that Prandtl function provides more accurate results because, at least at the lowest steps of a perturbation hierarchy, it is not affected by a boundary layer effect as it happens for the formal expansion for **t** used in this paper.

As a second application of our method, let us consider the flexure of an isosceles triangle of height l and basis $2h$; as for the trapezium studied before, the s coordinate runs along the height and the curvature κ of the middle line (which coincides with the height) vanishes. Let us assume that the kinematical parameter **k** is orthogonal (the resultant shearing force is parallel, Eqs. (6.5), (6.6)) to the height, as it is usually done in the literature. An orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2)$ is given whose first element is parallel to the height of the triangle,

$$\begin{aligned} \mathbf{r}_0 &= s\mathbf{e}_1, \quad 0 \le s \le l \ \Rightarrow \ \mathbf{l} = \mathbf{e}_1, \\ \delta(s) &= \frac{h}{l}(l-s) \ \Rightarrow \ \delta_{,s} = -\frac{h}{l}. \end{aligned} \tag{7.4}$$

It is known from elementary geometry that

$$b - o = \mathbf{b}_0 = \frac{l}{3}\mathbf{e}_1 \ \Rightarrow \ \bar{\mathbf{r}}_0 = \left(s - \frac{l}{3}\right)\mathbf{e}_1. \tag{7.5}$$

Besides, the coordinates s, z run along the principal axes of inertia of the domain, so that (see Eq. (6.7))

$$\mathbf{J}_b = \begin{pmatrix} J_s & 0 \\ 0 & J_z \end{pmatrix} = \begin{pmatrix} J_{b3} & 0 \\ 0 & J_{b1} \end{pmatrix}, \qquad J_{b1} = \frac{hl^3}{18}. \tag{7.6}$$

The coefficients of the formal expansion of the shear stress fields are, with respect to $(\mathbf{l}, *\mathbf{l}) \equiv (\mathbf{e}_1, \mathbf{e}_2)$, (see Eqs. (4.25)–(4.27))

$$\mathbf{t}_0 = \frac{6s}{hl^3}(l-s)\mathbf{e}_1 =: t_{s0}\mathbf{e}_1, \tag{7.7}$$

$$\mathbf{t}_1 = z\delta_{,s}t_{s0}\mathbf{e}_2, \tag{7.8}$$

$$\mathbf{t}_2 = G\frac{\delta^2}{3}(2\nu - 1)\left(\frac{10}{3} - z^2\right)\mathbf{e}_1, \tag{7.9}$$

Eqs. (7.7)–(7.8) are known in literature [10, 11, 12]; the first one represents the Jouravski mean shear stress field, and the second is orthogonal to the direction of the first. In the literature it is said, on heuristic grounds, that the field given by Eq. (7.8) is of small magnitude if compared with the Jouravski field. Our procedure gives a rational justification of this result, since Eq. (7.8) is simply a coefficient of higher order in a formal expansion of **t**. Equation (7.9), so far as we know, is a new result, providing an estimate of the shear stress field in thick isosceles triangles.

8. Conclusions

In this paper we have presented a rational procedure of formal expansion of the shear stress field for a Saint - Venant cylinder, using techniques from differential geometry. The class of domains which we are able to describe is large enough to embrace many of the sections used in typical technological applications. We are able to obtain a general formulation of the elliptic problem both for torsion and flexure. The results we find cover all the known technical formulae given in the literature at the first steps of the formal expansion, that is to say, when the section is thin. We are also able to provide new approximate formulae which seem to be meaningful.

Further investigations should be addressed to the numerical testing of the new approximate expressions and to an attempt to regularize the procedure, which we know to supply – in the present form – diverging series [17]. Finally, it may be mathematically interesting, using the methods developed in [16], to estimate the distance between the generalized Jouravski formulae we provide and the true solutions to Saint - Venant shear stress problem.

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