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Symmetric boundary integral formulations of transient heat conduction: saddle-point theorems for BE analysis and BE-FE coupling

Dedicated to the memory of Professor Gaetano Fichera

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THE LINEAR PROBLEM of transient heat conduction over a bounded time interval in a homogeneous domain with boundary conditions for temperature and flux is formulated in terms of boundary integral equations with an integral operator which is shown to be symmetric with respect to a bilinear form (convolutive in time). This form generates a functional characterizing the solution by its stationarity. Making recourse to a suitable integral transform and to another special bilinear form, it is shown that the boundary solution over the unbounded time interval $0 \leq t < \infty$, is characterized by a saddle-point property with separation of variables. Separation means that the solution corresponds to a maximum with respect to the time history of temperatures on the Neumann boundary, and by a minimum with respect to the time history of fluxes on the Dirichlet boundary. Subsequently a domain decomposition is assumed in view of coupled BE-FE discretization and a variational basis to such heterogeneous multifield modelling is provided.

1. Introduction

IN THE LAST FEW YEARS a growing portion of the literature concerning boundary integral equation (BIE) approaches and boundary element methods (BEMs) is devoted to symmetric formulations and relevant solution procedures.

The traditional formulation rests on Somigliana's identity (generated by "static" sources) and on its space-discrete version achieved by field modelling and node-wise collocation. As for diffusion problems, representative contributions are those due to RIZZO and SHIPPY [1], SHAW [2], TANAKA and TANAKA [3], ROURES and ALARCON [4], PINA and FERNANDEZ [5], while a comprehensive survey can be found in SHARP and CROUCH [6]. In this now popular context, key operators turn out to be nonsymmetric (or non-selfadjoint). Symmetry can be conferred to these operators by suitably adopting as boundary sources both static (or *intensive* or *single layer*) and kinematic (or *extensive* or *double layer*) discontinuities and, after modelling, by enforcing two suitably chosen BIEs in a Galerkin weighted-residual sense, which implies double integrations. Thus, among various consequences, variational characterizations can be given to the solution of boundary-value problems and of their BE-discretized versions in elasticity and in potential problems such as steady-state heat conduction, Darcy filtration, Saint-Venant torsion analyses and their analogues (see e.g.: [7, 8, 9]).

Parallel results have been established in incremental plasticity for both the rate problem and the finite-step problem, by recourse to domain distributions of concentrated strain sources (and relevant additional terms in the two BIEs) and to a third integral equation for stresses over potentially yielding portions of the domain (see e.g. [10, 11, 12, 13, 14]).

No attempt is made here to survey the numerous contributions to the theoretical foundations and to related computational aspect (*in primis* double integrations of hypersingular integrands and computer implementations) of the Galerkin-symmetric BEM in linear and nonlinear boundary-value problems. Two recent books [15] and [16] provide fairly abundant information and references (updated to 1991 and 1995, respectively).

As for initial-value boundary-value problems, much less attention has been attracted so far by their symmetric Galerkin BIE formulations and consequent solution properties and BE techniques. These formulations and properties have been established making use of time-dependent discontinuity sources of two kinds, in a way basically similar to the one adopted for boundary-value problems. Thus the BIE analysis of transient heat conduction (diffusion) [17], elastodynamics [18], viscoelasticity [19] and elastic-plastic dynamics [20] have been conferred symmetry in space and time (with respect to a time-convolutive bilinear form) over the finite time-interval of interest. As a consequence, variational saddle-point characterizations of the time response of the system to a given history of external actions, have been established in all the mechanical contexts listed above.

The present paper is intended to provide a further contribution to the theoretical foundations of the symmetric, variational BIE-BE methods for time-dependent problems with reference to linear transient heat conduction.

First the diffusion problem with mixed boundary conditions is formulated in the context of the "direct" approaches by means of BIEs using boundary sources of two kinds, like in an earlier paper by the authors in the context of "indirect" approaches [17]. The integral operator arising from the set of the above BIEs is shown to be symmetric (self-adjoint) with respect to a suitably devised bilinear form. This is defined as usual in the space variables; as for the time variable, the bilinear form is generated by means of the Laplace transforms of the two functions involved and by integrating, with respect to the transform parameter s over the unbounded interval $0 \leq s \leq \infty$, the product of the two functions and a suitable weight function.

As a consequence of the symmetry achieved in the above sense, the time history of the unknown boundary temperatures and fluxes over the unbounded time interval turns out to be characterized by a saddle-point property with separation of the two kinds of variables; namely, by a minimum with respect to the temperature field (extensive, kinematic variables) and by a maximum with respect to the heat flux field (intensive, static variables).

In contrast to the present saddle-point theorem, the min-max property presented earlier by the authors did not exhibit the above separation of variables. However, it had been proved over any bounded time interval, instead of over $0 \le t \le \infty$ only. The path of reasoning leading to the present min-max theorem is inspired by the ones followed by GURTIN [21], TONTI [22], RAFALSKI [23, 24] and REISS and HAUG [25], in order to arrive at variational principles for initial/boundary value problems formulated by partial differential equations.

Domain decomposition for coupling of BEM and FEM (Finite Element Method) has been a topic of active research since years (see e.g. [26]). The purpose is to employ each method in the subdomain where its peculiarities can be exploited at best for the numerical solution of the problem. Galerkin symmetric BEM turns out to be especially suitable to BE-FE coupled solutions, as shown by HOLZER [27], POLIZZOTTO and ZITO [28]. For the present time-dependent (transient) heat transfer problem, a contribution to heterogeneous modelling in the above sense is provided by the variational approach developed herein in Sec. 5.

2. Governing equation, Green functions and their properties

2.1. The linear diffusion problem

The thermally isotropic material considered herein is characterized by the following constant parameters: thermal conductivity k (measured e.g. in the units: $\mathrm{J\,sec^{-1}m^{-1}K^{-1}}$); specific heat γ ($\mathrm{J\,K^{-1}kg^{-1}}$); density ϱ ($\mathrm{kg\,m^{-3}}$). The heat conduction in a homogeneous body obtained by filling with the above material the open bounded domain Ω of a space with d dimensions ($\mathcal{R}^d$, with $d = 1, 2$ or 3) is governed by Fourier's and energy conservation laws. These laws combined lead to the classical equation (see e.g. [29]):

$$\frac{\partial \theta(\mathbf{x}, t)}{\partial t} - \alpha \nabla^2 \theta(\mathbf{x}, t) = \frac{1}{\gamma \varrho} Q(\mathbf{x}, t) \qquad \text{in } \Omega \times T. \tag{2.1}$$

Here $\alpha = k\gamma^{-1}\varrho^{-1}$ is the diffusivity coefficient of the material (in $\mathrm{m^2\,sec^{-1}}$); ∇^2 means Laplace operator; θ denotes temperature (in Kelvin degrees K); $\mathbf{x}$ is the d-vector of space coordinates x_i in a Cartesian reference system; t denotes time and $T \equiv [0, \bar{t}]$ the time interval over which the phenomenon is to be studied; Q represents the (given) density of heat supply rate [i.e. the production of heat per unit volume and time ($\mathrm{J\,m^{-3}\,sec^{-1}}$)].

The initial and boundary conditions are:

$$\theta(\mathbf{x}, 0) = \bar{\theta}_0(\mathbf{x}) \qquad \text{in } \Omega, \tag{2.2}$$

$$\theta(\mathbf{x}, t) = \bar{\theta}(\mathbf{x}, t) \qquad \text{on } \Gamma_\theta \times T, \tag{2.3}$$

$$q(\mathbf{x}, t) = -k \frac{\partial \theta}{\partial n}(\mathbf{x}, t) = \bar{q}(\mathbf{x}, t) \qquad \text{on } \Gamma_q \times T, \tag{2.4}$$

$$q(\mathbf{x}, t) = [\theta(\mathbf{x}, t) - \theta_\infty]\, c \qquad \text{on } \Gamma_c \times T. \tag{2.5}$$

Equation $(2.4)_1$ defines the heat flux q ($\mathrm{J\,m^{-2}sec^{-1}}$) in direction $\mathbf{n}$; $\mathbf{n}$ is the outward unit normal to the boundary $\Gamma = \Gamma_\theta \cup \Gamma_q \cup \Gamma_c$ (Γ_θ, Γ_q and Γ_c being disjoint parts of Γ) and is supposed to be uniquely defined everywhere; $\bar{\theta}_0$, $\bar{\theta}$ and $\bar{q}$ are given functions; the convection coefficient c ($\mathrm{J\,sec^{-1}\,m^{-2}\,K^{-1}}$) and the far-field temperature θ_∞ (in K) are known parameters.

For the sake of formal simplicity, the present study will assume $\Gamma_c = 0$, but its results can easily be extended to Cauchy (convective) condition (2.5). The less easy extension to non-homogeneous, multidomain problems can be carried out according to the line of thought pointed out in [18]. Thermally anisotropic media are implicitly covered with recourse to the relevant fundamental solutions.

2.2. Green functions

Consider (and denote by Ω_∞) the space $\mathcal{R}^d$ embedding Ω and filled with the same material as Ω. The response of Ω_∞ to a *source* represented by a (pulse) unit heat supply, concentrated in $\boldsymbol{\xi}$ (*load* or *source* point) at the instant τ, is described by the classical formula (see e.g. [29]):

$$(2.6)\qquad G_{\theta\theta}(\mathbf{x},\boldsymbol{\xi};t-\tau) = \frac{1}{\gamma\varrho}\,\frac{e^{-\frac{r^2}{4\alpha(t-\tau)}}}{[4\pi\alpha(t-\tau)]^{d/2}} \qquad \text{for } t \geq \tau,$$

where $r \equiv \|\mathbf{x}-\boldsymbol{\xi}\| = [(x_i-\xi_i)(x_i-\xi_i)]^{1/2}$ with $i = 1,\ldots,d$, denoting by $\|\cdot\|$ the Euclidean norm. The Green function or kernel (2.6) is the *fundamental solution* to Eq. (2.1), in the sense that it solves Eq. (2.1), when one sets in it $Q(\mathbf{x},t) = Q\,\delta(\mathbf{x}-\boldsymbol{\xi})\,\delta(t-\tau)$ (δ being Dirac distribution and $Q = 1$) and assumes $\theta \to 0$ for $\|\mathbf{x}\| \to \infty$ as boundary condition (i.e. $\theta = 0$ on Γ_∞) in three-dimensional situations.

It is worth noting that the heat-impulse source which gives rise to the temperature field (2.6), can be interpreted as a unit flux discontinuity across Γ, concentrated in $\boldsymbol{\xi}$ and τ. In order to make this circumstance explicit, denote by $\boldsymbol{\xi}^+$ a point not belonging to $\Omega \cup \Gamma$ (i.e. internal to $\Omega_\infty - \Omega \cup \Gamma$) and infinitely close to $\boldsymbol{\xi} \in \Gamma$ and by Γ^+ the set of all $\boldsymbol{\xi}^+$. Similarly, denote by $\boldsymbol{\xi}^-$ and Γ^- the obvious counterparts defined for points belonging to Ω. The unit normal, indicated by $\boldsymbol{\nu}$ in $\boldsymbol{\xi}$ and $\mathbf{n}$ in $\mathbf{x}$, is assumed as outward with respect to Ω and common to Γ and Γ^-, but in $\boldsymbol{\xi}^+$ the normal is $\boldsymbol{\nu}^+ = -\boldsymbol{\nu}$ (outward with respect to $\Omega_\infty - \Omega$). By means of this notation, the above source can be described in the alternate form:

$$(2.7)\qquad \Delta q\,\delta(\mathbf{z}-\boldsymbol{\xi})\,\delta(t-\tau),$$

where

$$(2.7')\qquad \Delta q \equiv -q\left(\boldsymbol{\xi}^-,t\right) - q\left(\boldsymbol{\xi}^+,t\right) = 1, \qquad \mathbf{z},\boldsymbol{\xi} \in \Gamma, \qquad t,\ \tau \in T.$$

Here Δq denotes the jump of the heat flux across Γ in $\boldsymbol{\xi}$ and $\delta(\mathbf{z}-\boldsymbol{\xi})$ is the Dirac distribution defined over Γ (no longer over Ω).

Other kernels, all defined for $t \geq \tau$, are derived below for later use by taking derivatives of the two-point function $G_{\theta\theta}$, Eq. (2.6), in the direction $\mathbf{n}$ of the outward normal to Γ defined in the *field* or *receiver* point $\mathbf{x}$ or/and in the direction $\boldsymbol{\nu}$ of the outward normal in the load point $\boldsymbol{\xi}$. Whenever useful to remind of these normals in the expression of a kernel G, their symbols will show up in the argument or will be replaced by / to mark their absence.

$$(2.8) \qquad G_{q\theta}(\mathbf{x},\boldsymbol{\xi};\mathbf{n},/;t-\tau) \equiv -k\frac{\partial G_{\theta\theta}}{\partial x_i}n_i = \frac{k}{2\alpha(t-\tau)}(x_i-\xi_i)\,n_i G_{\theta\theta}(\mathbf{x},\boldsymbol{\xi};t-\tau),$$

$$(2.9) \qquad G_{\theta q}(\mathbf{x},\boldsymbol{\xi};/,\boldsymbol{\nu};t-\tau) \equiv -k\frac{\partial G_{\theta\theta}}{\partial \xi_i}\nu_i = \frac{-k}{2\alpha(t-\tau)}(x_i-\xi_i)\,\nu_i G_{\theta\theta}(\mathbf{x},\boldsymbol{\xi};t-\tau),$$

$$(2.10) \qquad G_{qq}(\mathbf{x},\boldsymbol{\xi};\mathbf{n},\boldsymbol{\nu};t-\tau) \equiv -k\frac{\partial G_{q\theta}}{\partial \xi_i}\nu_i = -k\frac{\partial G_{\theta q}}{\partial x_i}n_i$$
$$= -\frac{k^2}{2\alpha(t-\tau)}\left[\frac{1}{2\alpha(t-\tau)}(x_i-\xi_i)\,n_i\,(x_r-\xi_r)\,\nu_r - n_j\nu_j\right]G_{\theta\theta}(\mathbf{x},\boldsymbol{\xi};t-\tau).$$

Physically interpreted, Eq. (2.8) represents the flux response in the field point $\mathbf{x}$ and direction $\mathbf{n}$ at instant t to the heat impulse acting on Ω_∞ in load point $\boldsymbol{\xi}$ at instant τ. Kernels (2.9) and (2.10) represent the temperature at $\mathbf{x}$ and t and, respectively, the flux at $\mathbf{x}$ in direction $\mathbf{n}$ at time t, which are generated in Ω_∞ by a "double layer" *source* consisting of a unit discontinuity of temperature across a surface through $\boldsymbol{\xi}$ of normal $\boldsymbol{\nu}$, concentrated in space and time. This (concentrated) temperature discontinuity source can be formally described by a counterpart to Eq. (2.7), making recourse to the same Dirac distributions δ, $\Delta\theta$ denoting a jump of temperature across Γ:

$$(2.11) \qquad \Delta\theta\,\delta(\mathbf{z}-\boldsymbol{\xi})\,\delta(t-\tau),$$

where

$$(2.11') \qquad \Delta\theta \equiv -\theta\left(\boldsymbol{\xi}^+,t\right)+\theta\left(\boldsymbol{\xi}^-,t\right)=1, \qquad \mathbf{z},\boldsymbol{\xi}\in\Gamma, \qquad t,\tau\in T.$$

2.3. Properties of kernels

The Green functions (2.6), (2.8), (2.9) and (2.10), all defined over Ω_∞ and for $t \geq \tau$ (causality condition), for $\mathbf{x}\to\boldsymbol{\xi}$ and $t\to\tau$ exhibit singularities which depend on the ratio $r/(t-\tau)$ when both r and $(t-\tau)$ tend to zero. However, it can be shown (see Appendix A) that the usual singularities of the (time-independent)

Green functions for the stationary conduction are exhibited by the Laplace transforms of these functions; namely, in three-dimensional situations ($d = 3$):

$$\begin{aligned} \mathcal{L}(G_{\theta\theta}) &= O(r^{-1}), \qquad \mathcal{L}(G_{\theta q}) = O(r^{-2}), \\ \mathcal{L}(G_{q\theta}) &= O(r^{-2}), \qquad \mathcal{L}(G_{qq}) = O(r^{-3}). \end{aligned} \tag{2.12}$$

In Eq. (2.12) $\mathcal{L}$ means Laplace transform

$$\mathcal{L}(\phi(t), s) \equiv \int_0^\infty e^{-st}\phi(t)\,dt, \tag{2.13}$$

s being the transformation parameter and ϕ any $\mathcal{L}$-transformable function.

The following *reciprocity* relationships among the above kernels hold at any time in space for $\mathbf{x} \neq \boldsymbol{\xi}$ and can be readily justified by inspection of the relevant formulae, Eqs. (2.6), (2.8), (2.9) and (2.10):

$$G_{\theta\theta}(\mathbf{x}, \boldsymbol{\xi}; t-\tau) = G_{\theta\theta}(\boldsymbol{\xi}, \mathbf{x}; t-\tau), \tag{2.14}$$

$$G_{q\theta}(\mathbf{x}, \boldsymbol{\xi}; \mathbf{n}, /; t-\tau) = G_{\theta q}(\boldsymbol{\xi}, \mathbf{x}; /, \mathbf{n}; t-\tau), \tag{2.15}$$

$$G_{qq}(\mathbf{x}, \boldsymbol{\xi}; \mathbf{n}, \boldsymbol{\nu}; t-\tau) = G_{qq}(\boldsymbol{\xi}, \mathbf{x}; \boldsymbol{\nu}, \mathbf{n}; t-\tau). \tag{2.16}$$

The positive definiteness of $\mathcal{L}(G_{\theta\theta})$ and the negative (semi)definiteness of $\mathcal{L}(G_{qq})$ formally mean that:

$$\int_\Gamma\int_\Gamma \mathcal{L}(\Delta q(\mathbf{x}; t), s)\,\mathcal{L}(G_{\theta\theta}(\mathbf{x}, \boldsymbol{\xi}; t), s)\,\mathcal{L}(\Delta q(\boldsymbol{\xi}; t), s)\,d\Gamma\,d\Gamma > 0 \qquad \forall \Delta q \neq 0, \tag{2.17}$$

$$\int_\Gamma\int_\Gamma \mathcal{L}(\Delta\theta(\mathbf{x}; t), s)\,\mathcal{L}(G_{qq}(\mathbf{x}, \boldsymbol{\xi}; \mathbf{n}, \boldsymbol{\nu}; t), s)\,\mathcal{L}(\Delta\theta(\boldsymbol{\xi}; t), s)\,d\Gamma\,d\Gamma \leq 0 \qquad \forall \Delta\theta. \tag{2.18}$$

These properties are proved in Appendix B.

In view of the $O(r^{-3})$ singularity ("hypersingularity") of the Laplace transform of kernel G_{qq}, Eq. (2.10), the double integral (2.18) acquires a meaning only if special interpretations and computational provisions are adopted. These are extensively dealt with in the recent literature see e.g. [30, 31, 32 and 33]; therefore they will not be discussed here. An investigation and implementation of hypersingular integrals occurring in elastostatics are presented in [8].

It seems appropriate to mention here also the following features of the basic Green's function, Eq. (2.6), in three-dimensional situations, see e.g. [29]:

$$\lim_{t\to\infty} G_{\theta\theta} = 0, \qquad \lim_{r\to\infty} G_{\theta\theta} = 0, \qquad \lim_{t\to\tau} G_{\theta\theta} = \delta(\mathbf{x} - \boldsymbol{\xi}). \tag{2.19}$$

3. Symmetric boundary integral equations and first variational formulation

In order to simplify notation and formal developments, the following provisions will be adopted henceforth. (i) Only Dirichlet Eq. (2.3) and Neumann Eq. (2.4) boundary conditions will be considered (i.e. $\Gamma_c = 0$). (ii) The differentials $d\boldsymbol{\xi}$ (and $d\mathbf{x}$) henceforth will mean $d\Gamma$ or $d\Omega$ when the integration variables are the coordinates of the source point $\boldsymbol{\xi}$ (and of the receiver point $\mathbf{x}$, respectively), as the integration domain indicated near the integral symbol will remove any ambiguity. (iii) The convolutive integration with respect to time τ will be denoted by an asterisk, namely $\int_0^t \psi(t-\tau)\psi'(\tau)\,d\tau = \psi(t) * \psi'(t)$, where ψ and ψ' are any time functions.

3.1. Two governing boundary integral equations

Consider the time history of the temperature field $\theta(\mathbf{x}, t)$ in Ω (as a part of Ω_∞) due to the following *causes* acting on Ω_∞: discontinuities of flux $\Delta q(\boldsymbol{\xi}, \tau)$ and of temperature $\Delta\theta(\boldsymbol{\xi}, \tau)$, distributed along the boundary Γ; heat supply $Q(\boldsymbol{\xi}, t)$ in the domain Ω; temperature initial condition $\bar{\theta}_0(\boldsymbol{\xi}, 0)$ in the domain Ω and $\bar{\theta}_0(\boldsymbol{\xi}, 0) = 0$ outside Ω, i.e. in $\Omega_\infty - (\Omega \cup \Gamma)$. Obviously, the last two data define the initial temperature discontinuity $\Delta\theta(\boldsymbol{\xi})$ across Γ at $\tau = 0$.

Using the kernels $G_{\theta\theta}$ and $G_{\theta q}$ as influence functions of Ω_∞ and superposing effects, we can give $\theta(\mathbf{x}, t)$ the following representation:

$$\theta(\mathbf{x};t) = \int_\Gamma G_{\theta\theta}(\mathbf{x},\boldsymbol{\xi};t) * \Delta q(\boldsymbol{\xi};t)\,d\boldsymbol{\xi} + \int_\Gamma G_{\theta q}(\mathbf{x},\boldsymbol{\xi};/,\boldsymbol{\nu};t) * \Delta\theta(\boldsymbol{\xi};t)\,d\boldsymbol{\xi} + \int_\Omega G_{\theta\theta}(\mathbf{x},\boldsymbol{\xi};t) * Q(\boldsymbol{\xi},t)\,d\boldsymbol{\xi} + \gamma\varrho \int_\Omega G_{\theta\theta}(\mathbf{x},\boldsymbol{\xi};t)\bar{\theta}_0(\boldsymbol{\xi})\,d\boldsymbol{\xi}. \tag{3.1}$$

The last integral containing the initial temperature $\bar{\theta}_0$ can be justified by the path of reasoning expounded in [6].

A similar integral representation is given below to the flux $q(\mathbf{x};t)$ in Ω (as a part of Ω_∞), taking the derivatives of Eq. (3.1) with respect to $\mathbf{x}$ in direction $\mathbf{n}$ and multiplying this derivative by $-k$:

$$q(\mathbf{x};t) = \int_\Gamma G_{q\theta}(\mathbf{x},\boldsymbol{\xi};\mathbf{n},/;t) * \Delta q(\boldsymbol{\xi};t)\,d\boldsymbol{\xi} + \int_\Gamma G_{qq}(\mathbf{x},\boldsymbol{\xi};\mathbf{n},\boldsymbol{\nu};t) * \Delta\theta(\boldsymbol{\xi};t)\,d\boldsymbol{\xi} + \int_\Omega G_{q\theta}(\mathbf{x},\boldsymbol{\xi};\mathbf{n},/;t) * Q(\boldsymbol{\xi},t)\,d\boldsymbol{\xi} + \gamma\varrho \int_\Omega G_{q\theta}(\mathbf{x},\boldsymbol{\xi};\mathbf{n},/;t)\,\bar{\theta}_0(\boldsymbol{\xi})\,d\boldsymbol{\xi}. \tag{3.2}$$

Choosing a "direct" rather than an "indirect" approach, we identify now the discontinuity sources as jumps across Γ between actual quantities in the domain

Ω and their counterparts in an unheated constant-temperature exterior $\Omega_\infty - (\Omega \cup \Gamma)$. Namely, we set for any $t \in T$:

$$\begin{array}{llll} q(\boldsymbol{\xi}^+, t) = 0 & \text{on } \Gamma, & q(\boldsymbol{\xi}^-, t) = \bar{q}(\boldsymbol{\xi}, t) & \text{on } \Gamma_q, \\ q(\boldsymbol{\xi}^-, t) = q(\boldsymbol{\xi}, t) & \text{on } \Gamma_\theta, & \theta(\boldsymbol{\xi}^+, t) = 0 & \text{on } \Gamma, \\ \theta(\boldsymbol{\xi}^-, t) = \bar{\theta}(\boldsymbol{\xi}, t) & \text{on } \Gamma_\theta, & \theta(\boldsymbol{\xi}^-, t) = \theta(\boldsymbol{\xi}, t) & \text{on } \Gamma_q. \end{array} \tag{3.3}$$

Now, keeping in mind Eqs. $(2.7)_2$, $(2.11)_2$ and (3.3), let us enforce Eq. (3.1) in points $\mathbf{x}^- \in \Omega$ infinitely close to the Dirichlet boundary Γ_θ, and identify the temperature in these points with the boundary data $\bar{\theta}(\mathbf{x}; t)$ assigned there. Similarly, we write Eq. (3.2) in points $\mathbf{x}^- \in \Omega$ close to Neumann boundary Γ_q, identify the heat flux in these points with the assigned boundary data $\bar{q}(\mathbf{x}; t)$. Thus Eqs. (3.1) and (3.2) yield:

for $\mathbf{x} \in \Gamma_\theta^-$

$$\int_{\Gamma_\theta} G_{\theta\theta}(\mathbf{x}, \boldsymbol{\xi}; t) * q(\boldsymbol{\xi}; t)\, d\boldsymbol{\xi} - \int_{\Gamma_q} G_{\theta q}(\mathbf{x}, \boldsymbol{\xi}; \boldsymbol{\nu}; t) * \theta(\boldsymbol{\xi}; t)\, d\boldsymbol{\xi} = \bar{f}_\theta(\mathbf{x}; t), \tag{3.4}$$

for $\mathbf{x} \in \Gamma_q^-$

$$\int_{\Gamma_\theta} G_{q\theta}(\mathbf{x}, \boldsymbol{\xi}; \mathbf{n}; t) * q(\boldsymbol{\xi}; t)\, d\boldsymbol{\xi} - \int_{\Gamma_q} G_{qq}(\mathbf{x}, \boldsymbol{\xi}; \mathbf{n}, \boldsymbol{\nu}; t) * \theta(\boldsymbol{\xi}; t)\, d\boldsymbol{\xi} = -\bar{f}_q(\mathbf{x}; t), \tag{3.5}$$

having set:

$$\begin{aligned} \bar{f}_\theta(\mathbf{x}; t) \equiv -\bar{\theta}(\mathbf{x}; t) &+ \int_\Omega G_{\theta\theta}(\mathbf{x}, \boldsymbol{\xi}; t) * Q(\boldsymbol{\xi}, t)\, d\boldsymbol{\xi} \\ &+ \gamma\varrho \int_\Omega G_{\theta\theta}(\mathbf{x}, \boldsymbol{\xi}; t)\, \bar{\theta}_0(\boldsymbol{\xi})\, d\boldsymbol{\xi} - \int_{\Gamma_q} G_{\theta\theta}(\mathbf{x}, \boldsymbol{\xi}; t) * \bar{q}(\boldsymbol{\xi}, t)\, d\boldsymbol{\xi} \\ &+ \int_{\Gamma_\theta} G_{\theta q}(\mathbf{x}, \boldsymbol{\xi}; \boldsymbol{\nu}; t) \bar{\theta}(\boldsymbol{\xi}; t)\, d\boldsymbol{\xi}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \bar{f}_q(\mathbf{x}; t) \equiv \bar{q}(\mathbf{x}; t) &- \int_\Omega G_{q\theta}(\mathbf{x}, \boldsymbol{\xi}; \mathbf{n}; t) * Q(\boldsymbol{\xi}, t)\, d\boldsymbol{\xi} \\ &- \gamma\varrho \int_\Omega G_{q\theta}(\mathbf{x}, \boldsymbol{\xi}; \mathbf{n}; t)\, \bar{\theta}_0(\boldsymbol{\xi})\, d\boldsymbol{\xi} + \int_{\Gamma_q} G_{q\theta}(\mathbf{x}, \boldsymbol{\xi}; \mathbf{n}; t) * \bar{q}(\boldsymbol{\xi}, t)\, d\boldsymbol{\xi} \\ &- \int_{\Gamma_\theta} G_{qq}(\mathbf{x}, \boldsymbol{\xi}; \mathbf{n}, \boldsymbol{\nu}; t)\, \bar{\theta}(\boldsymbol{\xi}; t)\, d\boldsymbol{\xi}. \end{aligned} \tag{3.7}$$

It is worth stressing that the singular integrals which intervene in Eqs. (3.4) and (3.5) must be interpreted in a suitable sense (in Cauchy and Hadamard sense for kernels $G_{\theta q}$ and $G_{q\theta}$ and for the "hypersingular" one G_{qq}, respectively). Also the existence and computability of integrals involving G_{qq} set special continuity requirements on functions θ and on interpolations to employ in its modelling. The analytical and numerical integrations in the presence of singularity and hypersingularity will not be discussed here. They are the object of the recent literature cited in Sec. 2 (and mostly concerning basically similar elastostatic and elastodynamic problems).

The boundary integral equations (3.4) and (3.5) govern the time histories, over the time interval T, of the unknown boundary fields $q(\boldsymbol{\xi};\tau)$ on Γ_θ, $\theta(\boldsymbol{\xi},\tau)$ on Γ_q. Thermal quantities which will actually occur elsewhere in the body considered will be recovered by quadratures from the boundary solution through the representation formulae (3.1) and (3.2) collocated at any point $\mathbf{x}$ and instant t of interest, account being taken of Eqs. $(2.7)_2$, $(2.11)_2$ and (3.3).

Therefore, the integral boundary equations (3.4) and (3.5) can be regarded as an alternative formulation of the original initial/boundary value problem, Eqs. (2.1) – (2.4). What follows is intended to point out some peculiar and hopefully computationally useful consequences of the above nonconventional "direct" BIE formulation (3.4) – (3.7) of linear transient heat conduction.

3.2. Symmetry and boundary variational theorem

It is convenient for subsequent developments to re-write the boundary integral equations (3.4) and (3.5) using a compact (operatorial) notation:

$$\mathbf{L}\,\mathbf{y} = \mathbf{f}. \tag{3.8}$$

In Eq. (3.8) $\mathbf{y}$ and $\mathbf{f}$ are vectors which gather boundary unknown functions and data, respectively:

$$\mathbf{y} \equiv \begin{Bmatrix} q(\boldsymbol{\xi};t) \\ \theta(\boldsymbol{\xi};t) \end{Bmatrix} \begin{matrix} \text{on} & \Gamma_\theta^- \times T, \\ \text{on} & \Gamma_q^- \times T, \end{matrix} \qquad \mathbf{f} \equiv \begin{Bmatrix} \bar{f}_\theta(\mathbf{x};t) \\ \bar{f}_q(\mathbf{x};t) \end{Bmatrix} \begin{matrix} \text{on} & \Gamma_\theta^- \times T, \\ \text{on} & \Gamma_q^- \times T, \end{matrix} \tag{3.9}$$

and $\mathbf{L}$ represents the linear integral operator:

$$\mathbf{L} \equiv \begin{bmatrix} \displaystyle\int_{\Gamma_\theta} G_{\theta\theta} * [\cdot]\, d\boldsymbol{\xi} & \displaystyle -\int_{\Gamma_q} G_{\theta q} * [\cdot]\, d\boldsymbol{\xi} \\ \displaystyle -\int_{\Gamma_\theta} G_{q\theta} * [\cdot]\, d\boldsymbol{\xi} & \displaystyle\int_{\Gamma_q} G_{qq} * [\cdot]\, d\boldsymbol{\xi} \end{bmatrix} \begin{matrix} \text{on} & \Gamma_\theta^- \times T, \\ \\ \text{on} & \Gamma_q^- \times T. \end{matrix} \tag{3.10}$$

A bilinear form over $\Gamma \times T$, *convolutive in time*, is defined as follows, $\mathbf{y}$ and $\mathbf{y}^*$ being two functions over $\Gamma \times T$ (superscript T denoting transpose):

$$(3.11) \qquad < \mathbf{y}, \mathbf{y}^* > \equiv \int_{\Gamma} \int_0^{\bar{t}} \mathbf{y}^T(\mathbf{x}; \bar{t} - t)\, \mathbf{y}^*(\mathbf{x}; t)\, d\mathbf{x}\, dt.$$

With reference to this notion, the two theorems stated below have been established in [16].

PROPOSITION 1. The integral operator $\mathbf{L}$, Eq. (3.10), of the governing boundary equations (3.8) is symmetric with respect to the bilinear form (3.11) convolutive in time; namely, the following equality holds for any vector of functions defined on $\Gamma \times T$ according to Eq. $(3.9)_1$:

$$(3.12) \qquad < \mathbf{L}\mathbf{y}, \mathbf{y}^* > = < \mathbf{L}\mathbf{y}^*, \mathbf{y} > \qquad \forall \quad \mathbf{y}, \mathbf{y}^*.$$

PROPOSITION 2. The time histories of boundary fields [flux $q(\mathbf{x}; t)$ on $\Gamma_\theta \times T$ and temperature $\theta(\mathbf{x}; t)$ on $\Gamma_q \times T$] which solve the diffusion problem in the direct boundary formulation (3.4) – (3.5), are characterized (as a sufficient and necessary condition) by the stationarity of the quadratic functional:

$$\begin{aligned}
(3.13) \qquad F\left(q(\mathbf{x}; t), \theta(\mathbf{x}; t)\right) \equiv & \frac{1}{2} < \mathbf{L}\mathbf{y}, \mathbf{y} > - < \mathbf{f}, \mathbf{y} > \\
= & \frac{1}{2} \int_0^{\bar{t}} \int_{\Gamma_\theta} q(\mathbf{x}; \bar{t} - t) \int_0^{t} \int_{\Gamma_\theta} G_{\theta\theta}\left(\mathbf{x}, \boldsymbol{\xi}; t - \tau\right) q\left(\boldsymbol{\xi}; \tau\right)\, d\boldsymbol{\xi}\, d\tau\, d\mathbf{x}\, dt \\
& - \frac{1}{2} \int_0^{\bar{t}} \int_{\Gamma_\theta} q(\mathbf{x}; \bar{t} - t) \int_0^{t} \int_{\Gamma_q} G_{\theta q}\left(\mathbf{x}, \boldsymbol{\xi}; t - \tau\right) \theta\left(\boldsymbol{\xi}; \tau\right)\, d\boldsymbol{\xi}\, d\tau\, d\mathbf{x}\, dt \\
& - \frac{1}{2} \int_0^{\bar{t}} \int_{\Gamma_q} \theta(\mathbf{x}; \bar{t} - t) \int_0^{t} \int_{\Gamma_\theta} G_{q\theta}\left(\mathbf{x}, \boldsymbol{\xi}; t - \tau\right) q\left(\boldsymbol{\xi}; \tau\right)\, d\boldsymbol{\xi}\, d\tau\, d\mathbf{x}\, dt \\
& + \frac{1}{2} \int_0^{\bar{t}} \int_{\Gamma_q} \theta(\mathbf{x}; \bar{t} - t) \int_0^{t} \int_{\Gamma_q} G_{qq}\left(\mathbf{x}, \boldsymbol{\xi}; t - \tau\right) \theta\left(\boldsymbol{\xi}; \tau\right)\, d\boldsymbol{\xi}\, d\tau\, d\mathbf{x}\, dt \\
& - \int_0^{\bar{t}} \int_{\Gamma_\theta} q(\mathbf{x}; \bar{t} - t) \bar{f}_\theta(\mathbf{x}; t)\, d\mathbf{x}\, dt - \int_0^{\bar{t}} \int_{\Gamma_q} \theta\left(\mathbf{x}; \bar{t} - t\right) \bar{f}_q(\mathbf{x}; t)\, d\mathbf{x}\, dt.
\end{aligned}$$

REMARKS

A. The above boundary statements have been established in [17], starting from earlier work by GURTIN [21] and TONTI [22] on variational principles for linear non-self-adjoint operators. The proof of Proposition 1 is based on the reciprocity properties (2.14)–(2.16) and the time-convolutive nature of the bilinear form (3.11). This proof is relegated to Appendix C. Proposition 2 follows from Proposition 1 through a customery path of reasoning which is outlined below. Through differentiation, and using the above symmetry (3.12), we may write:

$$\delta F = < \mathbf{L}\mathbf{y}, \delta\mathbf{y} > - < \mathbf{f}, \delta\mathbf{y} > + \frac{1}{2} < \mathbf{L}\delta\mathbf{y}, \delta\mathbf{y} > . \tag{3.14}$$

The first variation in (3.14) can be rewritten as $\delta^{(1)}F = < \mathbf{L}\mathbf{y} - \mathbf{f}, \delta\mathbf{y} >$ and this shows that indeed, the circumstance $\delta^{(1)}F = 0$ for any $\delta\mathbf{y}$ is a sufficient and necessary condition for $\mathbf{L}\,\mathbf{y} = \mathbf{f}$, i.e. for solving problem (3.4) and (3.5) in the boundary unknowns θ, q over the time T.

B. The second variation in Eq. (3.14), i.e. $\delta^{(2)}F = \frac{1}{2} < \mathbf{L}\delta\mathbf{y}, \delta\mathbf{y} >$, is not in general a sign-definite quadratic form. Therefore, the variational property stated by Proposition 2 corresponds to a saddle-point, not to an extremum of functional F. However, the saddle-point for F cannot be proved to represent an extremum point of F with respect to q and θ, separately. This remark motivates our search for stronger statements which led to the results expounded in the next Section.

C. The discretization in space and time, resting on the variational basis provided in what precedes, has been preliminarily discussed in [17] and implemented in [34] (with numerical integrations in space and analytical in time). Possible correlation between time interval and typical element length might be required in order to ensure the desired computational futures (primarily algorithmic stability). Issues of this kind, however, are beyond our present purposes.

4. Symmetry with respect to a bilinear form and a saddle-point theorem with variable separation

4.1. A further bilinear form and relevant variational theorem

Let $W(s)$ indicate an assigned function of the Laplace transform parameter s (interpreted as time). This "weight function" will be suitably chosen later within a broad class of alternatives, under the condition expressed below in (4.4), that it should be nonnegative everywhere, and not identically zero.

Taking over a concept put forward and used by RAFALSKI [23, 24] and REISS and HAUG [25] in linear initial-value problems, we introduce the following new bilinear form, denoted by the symbol $\ll \cdot, \cdot \gg$ involving the Laplace transforms of boundary field histories ($\mathbf{y}$ and $\mathbf{y}^*$) and defined over $\Gamma \times T_\infty$, denoting by T_∞ an unbounded from above time interval, namely $0 \le t < \infty$:

$$\ll \mathbf{y}, \mathbf{y}^* \gg \equiv \int_\Gamma \int_0^\infty W(s)\,[\mathcal{L}(\mathbf{y}(\mathbf{x};t), s)\mathcal{L}(\mathbf{y}^*(\mathbf{x};\tau), s)]\; ds\, d\mathbf{x} \tag{4.1}$$

or, more concisely:

$$\ll \mathbf{y}, \mathbf{y}^* \gg \equiv \int_{\Gamma} \int_0^{\infty} \int_0^{\infty} g(t+\tau)\mathbf{y}(\mathbf{x};t)\mathbf{y}^*(\mathbf{x};\tau)\,dt\,d\tau\,d\mathbf{x} \tag{4.2}$$

having set:

$$g(t+\tau) \equiv \int_0^{\infty} W(s)e^{-(t+\tau)s}\,ds \tag{4.3}$$

under the conditions:

$$W(s) \geq 0, \qquad W(s) \not\equiv 0. \tag{4.4}$$

On the basis of the new bilinear form (4.1) or (4.2), two further properties are stated below, as Propositions 3 and 4, which parallel Propositions 1 and 2, respectively.

PROPOSITION 3. The linear boundary operator **L**, Eq. (3.10), is symmetric with respect to the bilinear form (4.2); namely, the following equality holds for any pair of functions $\mathbf{y}$ and $\mathbf{y}^*$ defined by Eq. $(3.9)_1$ over the time-unbounded set $\Gamma \times T_\infty$:

$$\ll \mathbf{L}\mathbf{y}, \mathbf{y}^* \gg = \ll \mathbf{L}\mathbf{y}^*, \mathbf{y} \gg, \qquad \forall\, \mathbf{y}, \mathbf{y}^*. \tag{4.5}$$

PROPOSITION 4. A time-history of flux $q(\mathbf{x};t)$ on Γ_θ and temperature $\theta(\mathbf{x};t)$ on Γ_q, both defined over the unbounded time interval T_∞, represent the actual boundary response of the body to the external input, if and only if they make the following quadratic functional stationary:

$$\begin{aligned}
F^*(q(\mathbf{x};t), \theta(\mathbf{x};t)) &\equiv \frac{1}{2} \ll \mathbf{L}\mathbf{y}\,,\,\mathbf{y} \gg \;-\; \ll \mathbf{f}\,,\,\mathbf{y} \gg \\
&= \frac{1}{2}\int_{\Gamma_\theta}\int_0^{\infty}\int_0^{\infty} g(t+\eta)q(\mathbf{x};\eta)\int_{\Gamma_\theta} G_{\theta\theta}\,(\mathbf{x},\boldsymbol{\xi};t) * q\,(\boldsymbol{\xi};t)\;dt\,d\eta\,d\boldsymbol{\xi}\,d\mathbf{x} \\
&\quad - \frac{1}{2}\int_{\Gamma_\theta}\int_0^{\infty}\int_0^{\infty} g(t+\eta)q(\mathbf{x};\eta)\int_{\Gamma_q} G_{\theta q}\,(\mathbf{x},\boldsymbol{\xi};t) * \theta\,(\boldsymbol{\xi};t)\;dt\,d\eta\,d\boldsymbol{\xi}\,d\mathbf{x} \\
&\quad - \frac{1}{2}\int_{\Gamma_q}\int_0^{\infty}\int_0^{\infty} g(t+\eta)\theta(\mathbf{x};\eta)\int_{\Gamma_\theta} G_{q\theta}\,(\mathbf{x},\boldsymbol{\xi};t) * q\,(\boldsymbol{\xi};t)\;dt\,d\eta\,d\boldsymbol{\xi}\,d\mathbf{x} \\
&\quad + \frac{1}{2}\int_{\Gamma_q}\int_0^{\infty}\int_0^{\infty} g(t+\eta)\theta(\mathbf{x};\eta)\int_{\Gamma_q} G_{qq}\,(\mathbf{x},\boldsymbol{\xi};t) * \theta\,(\boldsymbol{\xi};t)\;dt\,d\eta\,d\boldsymbol{\xi}\,d\mathbf{x}
\end{aligned} \tag{4.6}$$

(4.6) [cont.]
$$
\begin{aligned}
&- \int_{\Gamma_\theta} \int_0^\infty \int_0^\infty g(t+\eta) \bar{f}_\theta(\mathbf{x};t) q(\mathbf{x};\eta)\, dt\, d\eta\, d\mathbf{x} \\
&- \int_{\Gamma_q} \int_0^\infty \int_0^\infty g(t+\eta) \bar{f}_q(\mathbf{x};t) \theta(\mathbf{x};\eta)\, dt\, d\eta\, d\mathbf{x}.
\end{aligned}
$$

Our formal proof of Proposition 3, still resting on the reciprocity relationships (2.14)–(2.16), implies rather lengthy manipulations and, hence, is confined to Appendix D. Proposition 4 is a straightforward consequence of Proposition 3, through the same familiar argument which led from Propositions 1 to 2 and, hence, its proof will not be duplicated here. The present task, pursued below, is to strengthen Proposition 4 into a stronger statement, a purpose which was not possible to achieve for Proposition 2.

4.2. A saddle-point theorem, extremum for flux and temperature, separately

The two quadratic forms, one in flux q and the other in temperature θ, contained in functional F, Eq. (3.13), turn out to be not defined in sign in general. On the contrary, the two quadratic forms in functional F^*, Eq. (4.6), which represent the counterparts to those in Eq. (3.13), do exhibit sign-definiteness as shown below.

PROPOSITION 5. The following sign-definiteness properties hold for the quadratic forms associated with Green functions $G_{\theta\theta}$ and G_{qq}, respectively, in functional F^*:

$$
\int_{\Gamma_\theta} \int_0^\infty \int_0^\infty g(t+\eta) q(\mathbf{x};\eta) \int_{\Gamma_\theta} \int_0^t G_{\theta\theta}(\mathbf{x},\boldsymbol{\xi};t-\tau)\, q(\boldsymbol{\xi};\tau)\, d\tau\, d\boldsymbol{\xi}\, d\eta\, dt\, d\mathbf{x} > 0, \qquad \forall q \not\equiv 0, \tag{4.7}
$$

$$
\int_{\Gamma_q} \int_0^\infty \int_0^\infty g(t+\eta) \theta(\mathbf{x};\eta) \int_{\Gamma_q} \int_0^t G_{qq}(\mathbf{x},\boldsymbol{\xi};t-\tau)\, \theta(\boldsymbol{\xi};\tau)\, d\tau\, d\boldsymbol{\xi}\, d\eta\, dt\, d\mathbf{x} \le 0, \qquad \forall \theta. \tag{4.8}
$$

The latter inequality can be strengthened into a strict inequality ($< 0, \forall\, \theta \not\equiv 0$, i.e. negative definiteness instead of semi-definiteness), if Γ_q splits Ω_∞ into disjoint parts.

P r o o f. Using Eq. (4.3) and the Laplace transform

$$
\mathcal{L}(\mathbf{y}(t), s) \equiv \int_0^\infty e^{-st} \mathbf{y}(t)\, dt, \tag{4.9}
$$

the quadratic form (4.7) can be given the following alternative expression:

$$\int_{\Gamma_\theta}\int_{\Gamma_\theta}\int_0^\infty W(s)\mathcal{L}(q(\mathbf{x};t),s)\mathcal{L}(G_{\theta\theta}(\mathbf{x},\boldsymbol{\xi};t),s)\mathcal{L}(q(\boldsymbol{\xi};t),s)\,ds\,d\mathbf{x}\,d\boldsymbol{\xi}. \tag{4.10}$$

Let us combine the expression in (4.10), with the positive definiteness property (2.17) of kernel $G_{\theta\theta}$, and with the assumed nonnegativeness of $W(s)$. This straightforwardly leads to the desired conclusion (4.7).

A proof of property (4.8) follows the same path of reasoning starting from kernel property (2.18) and, hence, is omitted here for brevity.

PROPOSITION 6. (Saddle-point theorem) Let $\hat{q}(\mathbf{x};t)$ over $\Gamma_\theta \times T_\infty$ and $\hat{\theta}(\mathbf{x};t)$ over $\Gamma_q \times T_\infty$ represent the boundary solution of the diffusion problem in its integral formulation (3.4)–(3.5), and let the uncapped symbols denote any pair of fields (flux q and temperature θ) defined there. Then the following inequalities hold:

$$F^*(\hat{q},\theta) \le F^*\left(\hat{q},\hat{\theta}\right) \le F^*\left(q,\hat{\theta}\right), \tag{4.11}$$

where the equality signs hold if and only if $q = \hat{q}$ and $\theta = \hat{\theta}$.

P r o o f. Compute the varied functional F^*, Eq. (4.6), after a perturbation $\delta\hat{\theta}$, $\delta\hat{q}$ around the solution and gather the first-order and second-order terms in $\delta^{(1)}F^*$ and $\delta^{(2)}F^*$, respectively. The addend $\delta^{(1)}F^*$ which contains the first-order terms vanishes because of the variational property Proposition 4. As for $\delta^{(2)}F^*$ which collects the second-order terms we notice that setting $\delta\hat{q} = 0$, this addend is negative for any $\delta\hat{\theta} \neq 0$ by virtue of Eq. (4.8). This justifies the former of inequalities (4.11) for infinitesimal perturbations around the solution (i.e. *in the small*). However, in view of the quadratic nature of the functional, the inequality must be fulfilled also *in the large*. The latter inequality (4.11) is achieved by similar argumentation setting $\delta\hat{\theta} = 0$ and making use of Eq. (4.7).

5. Coupling

Let the domain Ω be subdivided into two disjoint complementary open subdomains Ω^F and Ω^B, separated by interface Γ^C (so that $\Gamma^C = \bar{\Omega}^F \cap \bar{\Omega}^B$, denoting by bars that boundaries are included). The present purpose is to establish a unified variational basis for approximate solutions of the initial-boundary-value problem in point by means of two discretization procedures simultaneously, namely by a finite element method in Ω^F and a symmetric Galerkin boundary element method in Ω^B. To this coupling (or "multifield modelling") purpose, we rewrite below, suitably adjusted, two "strong" formulations of the transient heat conduction problem over Ω^F and Ω^B.

For the former subdomain Ω^F let the problem be formulated in terms of partial derivative equations and boundary conditions. Denoting by Γ_q^F and Γ_θ^F the Neumann and Dirichlet boundary, respectively, and being understood that all equations hold over the unbounded time interval T_∞, the initial-boundary value problem, after the domain decomposition in point, can be formulated as follows.

$$-\frac{\partial q_i^F}{\partial x_i} + Q^F = \gamma \varrho \frac{\partial \theta^F}{\partial t} \qquad \text{in } \Omega^F, \tag{5.1}$$

$$q_i^F = k p_i^F \qquad \text{in } \Omega^F, \tag{5.2}$$

$$p_i^F = -\frac{\partial \theta^F}{\partial x_i} \qquad \text{in } \Omega^F, \tag{5.3}$$

$$n_i q_i^F = \bar{q}^F \qquad \text{on } \Gamma_q^F, \tag{5.4}$$

$$\theta^F = \bar{\theta}^F \qquad \text{on } \Gamma_\theta^F, \tag{5.5}$$

$$\theta^F(\mathbf{x}, 0) = \bar{\theta}_0^F \qquad \text{in } \bar{\Omega}^F. \tag{5.6}$$

For the latter subdomain Ω^B let the same problem be governed by the boundary integral equations of the symmetric kind developed in Secs. 2 and 3. These are rewritten below by referring to points $\mathbf{x} \in \Gamma$ (no longer Γ^-) and, therefore, by making explicit certain consequences of the singularities in the integrands. In fact, integrals concerning the strongly singular kernels $G_{\theta q}$ and $G_{q\theta}$, give rise to Cauchy principal parts (marked by $\fint$ in what follows) and "free terms". Similarly, integrals involving the hypersingular kernel G_{qq} lead to Hadamard finite parts (marked by $\neq\!\!\!\!\int$) and "free terms". In the above free terms the coefficient, say β, depends on the geometry of surface Γ in a neighbourhood of field point $\mathbf{x}$ with $\beta = 1/2$ in smooth points as assumed herein.

For $\mathbf{x} \in \Gamma_\theta^B$

$$\int_{\Gamma_\theta^B} G_{\theta\theta}(\mathbf{x}, \boldsymbol{\xi}; t) * q^B(\boldsymbol{\xi}; t)\, d\boldsymbol{\xi} + \int_{\Gamma^C} G_{\theta\theta}(\mathbf{x}, \boldsymbol{\xi}; t) * q^C(\boldsymbol{\xi}; t)\, d\boldsymbol{\xi}$$
$$- \fint_{\Gamma_q^B} G_{\theta q}(\mathbf{x}, \boldsymbol{\xi}; \boldsymbol{\nu}; t) * \theta^B(\boldsymbol{\xi}; t)\, d\boldsymbol{\xi} - \fint_{\Gamma^C} G_{\theta q}(\mathbf{x}, \boldsymbol{\xi}; \boldsymbol{\nu}; t) * \theta^C(\boldsymbol{\xi}; t)\, d\boldsymbol{\xi} = \bar{f}_\theta^B, \tag{5.7}$$

for $\mathbf{x} \in \Gamma_q^B$

$$- \fint_{\Gamma_\theta^B} G_{q\theta}(\mathbf{x}, \boldsymbol{\xi}; \mathbf{n}; t) * q^B(\boldsymbol{\xi}; t)\, d\boldsymbol{\xi} - \fint_{\Gamma^C} G_{q\theta}(\mathbf{x}, \boldsymbol{\xi}; \mathbf{n}; t) * q^C(\boldsymbol{\xi}; t)\, d\boldsymbol{\xi}$$
$$+ \neq\!\!\!\!\!\!\int_{\Gamma_q^B} G_{qq}(\mathbf{x}, \boldsymbol{\xi}; \mathbf{n}, \boldsymbol{\nu}; t) * \theta^B(\boldsymbol{\xi}; t)\, d\boldsymbol{\xi} + \neq\!\!\!\!\!\!\int_{\Gamma^C} G_{qq}(\mathbf{x}, \boldsymbol{\xi}; \mathbf{n}, \boldsymbol{\nu}; t) * \theta^C(\boldsymbol{\xi}; t)\, d\boldsymbol{\xi} = \bar{f}_q^B, \tag{5.8}$$

for $\mathbf{x} \in \Gamma^C$

$$(5.9)\qquad \int_{\Gamma_\theta^B} G_{\theta\theta}(\mathbf{x},\boldsymbol{\xi};t) * q^B(\boldsymbol{\xi};t)\,d\boldsymbol{\xi} + \int_{\Gamma^C} G_{\theta\theta}(\mathbf{x},\boldsymbol{\xi};t) * q^C(\boldsymbol{\xi};t)\,d\boldsymbol{\xi}$$

$$- \fint_{\Gamma_q^B} G_{\theta q}(\mathbf{x},\boldsymbol{\xi};\boldsymbol{\nu};t) * \theta^B(\boldsymbol{\xi};t)\,d\boldsymbol{\xi} - \fint_{\Gamma^C} G_{\theta q}(\mathbf{x},\boldsymbol{\xi};\boldsymbol{\nu};t) * \theta^C(\boldsymbol{\xi};t)\,d\boldsymbol{\xi} + \frac{1}{2}\theta^C = \bar{f}_\theta^C,$$

for $\mathbf{x} \in \Gamma^C$

$$(5.10)\qquad -\fint_{\Gamma_\theta^B} G_{q\theta}(\mathbf{x},\boldsymbol{\xi};\mathbf{n};t) * q^B(\boldsymbol{\xi};t)\,d\boldsymbol{\xi} - \fint_{\Gamma^C} G_{q\theta}(\mathbf{x},\boldsymbol{\xi};\mathbf{n};t) * q^C(\boldsymbol{\xi};t)\,d\boldsymbol{\xi}$$

$$+ \neq\!\!\!\!\!\int_{\Gamma_q^B} G_{qq}(\mathbf{x},\boldsymbol{\xi};\mathbf{n},\boldsymbol{\nu};t) * \theta^B(\boldsymbol{\xi};t)\,d\boldsymbol{\xi} + \neq\!\!\!\!\!\int_{\Gamma^C} G_{qq}(\mathbf{x},\boldsymbol{\xi};\mathbf{n},\boldsymbol{\nu};t) * \theta^C(\boldsymbol{\xi};t)\,d\boldsymbol{\xi} - \frac{1}{2}q^C = \bar{f}_q^C.$$

On the interface between the two subdomains, the continuity conditions concern flux and temperature, namely:

$$(5.11)\qquad n_i q_i^F + q^C = 0 \qquad \text{on } \Gamma^C,$$

$$(5.12)\qquad \theta^F - \theta^C = 0 \qquad \text{on } \Gamma^C.$$

In the BIEs, Eqs. (5.7), (5.8), (5.9) and (5.10), the terms containing data only are, respectively:

$$(5.13)\qquad \bar{f}_\theta^B = -\frac{1}{2}\bar{\theta}^B(\mathbf{x};t) - \int_{\Gamma_q^B} G_{\theta\theta}(\mathbf{x},\boldsymbol{\xi};t) * \bar{q}^B(\boldsymbol{\xi};t)\,d\boldsymbol{\xi}$$

$$+ \fint_{\Gamma_\theta^B} G_{\theta q}(\mathbf{x},\boldsymbol{\xi};\boldsymbol{\nu};t) * \bar{\theta}^B(\boldsymbol{\xi};t)\,d\boldsymbol{\xi} + \int_{\Omega^B} G_{\theta\theta}(\mathbf{x},\boldsymbol{\xi};t) * Q^B(\boldsymbol{\xi};t)\,d\boldsymbol{\xi}$$

$$+ \gamma\varrho \int_{\Omega^B} G_{\theta\theta}(\mathbf{x},\boldsymbol{\xi};t)\,\bar{\theta}_0^B(\boldsymbol{\xi})\,d\boldsymbol{\xi},$$

$$(5.14)\qquad \bar{f}_q^B = \frac{1}{2}\bar{q}^B(\mathbf{x};t) + \fint_{\Gamma_q^B} G_{q\theta}(\mathbf{x},\boldsymbol{\xi};\mathbf{n};t) * \bar{q}^B(\boldsymbol{\xi};t)\,d\boldsymbol{\xi}$$

$$- \neq\!\!\!\!\!\int_{\Gamma_\theta^B} G_{qq}(\mathbf{x},\boldsymbol{\xi};\mathbf{n},\boldsymbol{\nu};t) * \bar{\theta}^B(\boldsymbol{\xi};t)\,d\boldsymbol{\xi} - \fint_{\Omega^B} G_{q\theta}(\mathbf{x},\boldsymbol{\xi};\mathbf{n};t) * Q^B(\boldsymbol{\xi};t)\,d\boldsymbol{\xi}$$

$$- \gamma\varrho \fint_{\Omega^B} G_{q\theta}(\mathbf{x},\boldsymbol{\xi};\mathbf{n};t)\,\bar{\theta}_0^B(\boldsymbol{\xi})\,d\boldsymbol{\xi},$$

$$(5.15)\qquad \bar f_\theta^C = -\int_{\Gamma_q^B} G_{\theta\theta}(\mathbf{x},\boldsymbol{\xi};t) * \bar q^B(\boldsymbol{\xi};t)\,d\boldsymbol{\xi} + -\!\!\!\!\!\!\int_{\Gamma_\theta^B} G_{\theta q}(\mathbf{x},\boldsymbol{\xi};\boldsymbol{\nu};t) * \bar\theta^B(\boldsymbol{\xi};t)\,d\boldsymbol{\xi}$$
$$+ \int_{\Omega^B} G_{\theta\theta}(\mathbf{x},\boldsymbol{\xi};t) * Q^B(\boldsymbol{\xi};t)\,d\boldsymbol{\xi} + \gamma\varrho\int_{\Omega^B} G_{\theta\theta}(\mathbf{x},\boldsymbol{\xi};t)\,\bar\theta_0^B(\boldsymbol{\xi})\,d\boldsymbol{\xi},$$

$$(5.16)\qquad \bar f_q^C = -\!\!\!\!\!\!\int_{\Gamma_q^B} G_{q\theta}(\mathbf{x},\boldsymbol{\xi};\mathbf{n};t) * \bar q^B(\boldsymbol{\xi};t)\,d\boldsymbol{\xi} - =\!\!\!\!\!\!\int_{\Gamma_\theta^B} G_{qq}(\mathbf{x},\boldsymbol{\xi};\mathbf{n},\boldsymbol{\nu};t) * \bar\theta^B(\boldsymbol{\xi};t)\,d\boldsymbol{\xi}$$
$$- -\!\!\!\!\!\!\int_{\Omega^B} G_{q\theta}(\mathbf{x},\boldsymbol{\xi};\mathbf{n};t) * Q^B(\boldsymbol{\xi};t)\,d\boldsymbol{\xi} - \gamma\varrho -\!\!\!\!\!\!\int_{\Omega^B} G_{q\theta}(\mathbf{x},\boldsymbol{\xi};\mathbf{n};t)\,\bar\theta_0^B(\boldsymbol{\xi})\,d\boldsymbol{\xi}.$$

The equations (5.1) – (5.3) concerning Ω^F, Eqs. (5.4), (5.5) concerning its boundary not in common with Ω^B and the initial conditions (5.6) can be given, respectively, the following compact operatorial formulations (in matrix notation):

$$(5.17)\qquad \begin{bmatrix} -k & 1 & 0 \\ 1 & 0 & \partial(\cdot)/\partial x_i \\ 0 & -\partial(\cdot)/\partial x_i & -\gamma\varrho(\partial(\cdot)/\partial t) \end{bmatrix} \begin{bmatrix} p_i^F \\ q_i^F \\ \theta^F \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -Q^F \end{bmatrix} \to \mathbf{N}_1\mathbf{y}_1 = \mathbf{h}_1\,,$$

$$(5.18)\qquad \begin{bmatrix} 0 & -n_i \\ n_i & 0 \end{bmatrix} \begin{bmatrix} q_i^F \\ \theta^F \end{bmatrix} = \begin{bmatrix} -n_i\bar\theta \\ \bar q \end{bmatrix} \quad \begin{matrix} \text{on } \Gamma_\theta^F \\ \text{on } \Gamma_q^F \end{matrix} \to \mathbf{N}_2\mathbf{y}_2 = \mathbf{h}_2\,,$$

$$(5.19)\qquad -\gamma\varrho\,\theta^F(\mathbf{x},0) = -\gamma\varrho\,\bar\theta_0^F \qquad \text{on } \bar\Omega^F \text{ for } t=0 \to N_3 y_3 = h_3\,.$$

Similarly the BIE's (5.7) – (5.10) which concern Ω^B and its boundary will be expressed in the more compact forms:

$$(5.20)\qquad \begin{bmatrix} \int_{\Gamma^C} G_{\theta\theta} * [\cdot]\,d\boldsymbol{\xi} & -\!\!\!\!\!\!\int_{\Gamma^C} G_{\theta q} * [\cdot]\,d\boldsymbol{\xi} + 1/2 & \int_{\Gamma_\theta^B} G_{\theta\theta} * [\cdot]\,d\boldsymbol{\xi} & - -\!\!\!\!\!\!\int_{\Gamma_q^B} G_{\theta q} * [\cdot]\,d\boldsymbol{\xi} \\ - -\!\!\!\!\!\!\int_{\Gamma^C} G_{q\theta} * [\cdot]\,d\boldsymbol{\xi} - 1/2 & =\!\!\!\!\!\!\int_{\Gamma^C} G_{qq} * [\cdot]\,d\boldsymbol{\xi} & - -\!\!\!\!\!\!\int_{\Gamma_\theta^B} G_{q\theta} * [\cdot]\,d\boldsymbol{\xi} & + =\!\!\!\!\!\!\int_{\Gamma_q^B} G_{qq} * [\cdot]\,d\boldsymbol{\xi} \\ \int_{\Gamma^C} G_{\theta\theta} * [\cdot]\,d\boldsymbol{\xi} & - -\!\!\!\!\!\!\int_{\Gamma^C} G_{\theta q} * [\cdot]\,d\boldsymbol{\xi} & \int_{\Gamma_\theta^B} G_{\theta\theta} * [\cdot]\,d\boldsymbol{\xi} & - -\!\!\!\!\!\!\int_{\Gamma_q^B} G_{\theta q} * [\cdot]\,d\boldsymbol{\xi} \\ - -\!\!\!\!\!\!\int_{\Gamma^C} G_{q\theta} * [\cdot]\,d\boldsymbol{\xi} & =\!\!\!\!\!\!\int_{\Gamma^C} G_{qq} * [\cdot]\,d\boldsymbol{\xi} & - -\!\!\!\!\!\!\int_{\Gamma_\theta^B} G_{q\theta} * [\cdot]\,d\boldsymbol{\xi} & + =\!\!\!\!\!\!\int_{\Gamma_q^B} G_{qq} * [\cdot]\,d\boldsymbol{\xi} \end{bmatrix} \begin{bmatrix} q^C \\ \theta^C \\ q^B \\ \theta^B \end{bmatrix} = \begin{bmatrix} \bar f_\theta^C \\ \bar f_q^C \\ \bar f_\theta^B \\ \bar f_q^B \end{bmatrix} \quad \begin{matrix} \text{on } \Gamma^C \\ \text{on } \Gamma^C \\ \text{on } \Gamma_\theta^B \\ \text{on } \Gamma_q^B \end{matrix} \to \mathbf{N}_4\mathbf{y}_4 = \mathbf{h}_4\,.$$

Finally, the interface conditions on Γ^C Eq. (5.11) and (5.12) are rewritten (two times) for convenience as follows.

$$(5.21)\qquad \frac{1}{2}\begin{bmatrix} 0 & -n_i & 0 & n_i \\ n_i & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ n_i & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} q_i^F \\ \theta^F \\ q^C \\ \theta^C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \to \mathbf{N}_5\mathbf{y}_5 = \mathbf{h}_5 .$$

Consistently with the above adopted compact notation, let us gather the (scalar) variables in the vector $\mathbf{y}^T = \left[p_i^F, q_i^F, \theta^F, q^C, \theta^C, q^B, \theta^B\right]$ (superscript T denoting transpose), the data in vector

$$\mathbf{h}^T = \left[\mathbf{h}_1^T, \mathbf{h}_2^T, h_3, \mathbf{h}_4^T, \mathbf{h}_5^T\right] = \left[0, 0, -Q^F, -n_i\bar{\theta}, \bar{q}, -\gamma\varrho\bar{\theta}_0^F, \bar{f}_\theta^C, \bar{f}_q^C, \bar{f}_\theta^B, \bar{f}_q^B, 0, 0, 0, 0\right]$$

and the operators into the matrix

$$(5.22)\qquad \mathbf{N} \equiv \begin{bmatrix} \overline{\mathbf{N}}_1 \\ \overline{\mathbf{N}}_2 \\ \overline{\mathbf{N}}_3 \\ \overline{\mathbf{N}}_4 \\ \overline{\mathbf{N}}_5 \end{bmatrix},$$

where the barred symbols have the following meaning: $\overline{\mathbf{N}}_1 = [\mathbf{N}_1, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}]$, $\overline{\mathbf{N}}_2 = [\mathbf{0}, \mathbf{N}_2, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}]$, $\overline{\mathbf{N}}_3 = [0, 0, N_3, 0, 0, 0, 0]$, $\overline{\mathbf{N}}_4 = [\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{N}_4]$, $\overline{\mathbf{N}}_5 = [\mathbf{0}, \mathbf{N}_5, \mathbf{0}, \mathbf{0}]$. Now, denoting by $\mathbf{y}$ and $\mathbf{y}'$ two vectors of fields which belong to the domain of the above defined operator $\mathbf{N}$ ($\mathbf{y}, \mathbf{y}' \in \mathcal{D}(\mathbf{N})$), a bilinear form associated to this operator $\mathbf{N}$ can be generated as follows, according to the pattern adopted in Sec. 4, Eqs. (4.1)–(4.4), on the basis of operator $\mathbf{L}$:

$$(5.23)\qquad \ll \mathbf{N}\mathbf{y}, \mathbf{y}' \gg \\ = \int_0^\infty\int_0^\infty g(t+\tau)\Bigg\{\int_{\Omega^F} [\mathbf{N}_1\mathbf{y}_1(t)]^T\,\mathbf{y}_1'(\tau)\,d\Omega + \int_{\Gamma^F} [\mathbf{N}_2\mathbf{y}_2(t)]^T\,\mathbf{y}_2'(\tau)\,d\tau \\ + \int_{\Gamma^B+\Gamma^C} [\mathbf{N}_4\mathbf{y}_4(t)]^T\,\mathbf{y}_4'(\tau)\,d\Gamma + \int_{\Gamma^C} [\mathbf{N}_5\mathbf{y}_5(t)]^T\,\mathbf{y}_5'(\tau)\,d\Gamma\Bigg\}\,dt\,d\tau \\ + \int_0^\infty\int_{\Omega^F} g(\tau)\,[N_3y_3(t)]\,y_3'(\tau)\,d\Omega\,d\tau.$$

In full analogy with Eq. (4.6), we construct the quadratic functional:

$$
\begin{aligned}
(5.24)\quad & \mathcal{F}\left(p_i^F, q_i^F, \theta^F, q^C, \theta^C, q^B, \theta^B\right) = \frac{1}{2} \ll \mathbf{N}\mathbf{y}, \mathbf{y} \gg - \ll \mathbf{h}, \mathbf{y} \gg \\
&= \int_0^\infty \int_0^\infty g(t+\tau) \Bigg[\frac{1}{2} \int_{\Omega^F} -\gamma \varrho \frac{\partial \theta^F(t)}{\partial t} \theta^F(\tau)\, d\Omega - \frac{1}{2} \int_{\Omega^F} \frac{\partial q_i^F}{\partial x_i}(t)\, \theta^F(\tau)\, d\Omega \\
&- \frac{1}{2} \int_{\Omega^F} k\, p_i^F(t)\, p_i^F(\tau)\, d\Omega + \int_{\Omega^F} q_i^F(t)\, p_i^F(\tau)\, d\Omega + \frac{1}{2} \int_{\Omega^F} \frac{\partial \theta^F}{\partial x_i}(t)\, q_i^F(\tau)\, d\Omega \\
&- \frac{1}{2} \int_{\Gamma_\theta^F} n_i \theta^F(t)\, q_i^F(\tau)\, d\Gamma + \frac{1}{2} \int_{\Gamma_q^F} n_i q_i^F(t)\, \theta^F(\tau)\, d\Gamma + \int_{\Omega^F} Q^F(t)\, \theta^F(\tau)\, d\Omega \\
&+ \int_{\Gamma_\theta^F} n_i \bar{\theta}^F(t)\, q_i^F(\tau)\, d\Gamma - \int_{\Gamma_q^F} \bar{q}(t)\, \theta^F(\tau)\, d\Gamma \Bigg] dt\, d\tau \\
&- \frac{1}{2} \int_0^\infty g(t) \int_{\Omega^F} \gamma \varrho\, \theta^F(t) \left(\theta^F(0) - 2\bar{\theta}_0^F\right) d\Omega\, dt \\
&+ \int_0^\infty \int_0^\infty g(t+\tau) \Bigg[\frac{1}{2} \int_{\Gamma^C} \int_{\Gamma^C} G_{\theta\theta} * q^C(t)\, q^C(\tau)\, d\boldsymbol{\xi}\, d\mathbf{x} \\
&- \frac{1}{2} \int_{\Gamma^C} \fint_{\Gamma^C} G_{\theta q} * \theta^C(t)\, q^C(\tau)\, d\boldsymbol{\xi}\, d\mathbf{x} + \frac{1}{2} \int_{\Gamma^C} \int_{\Gamma_\theta^B} G_{\theta\theta} * q^B(t)\, q^C(\tau)\, d\boldsymbol{\xi}\, d\mathbf{x} \\
&- \frac{1}{2} \int_{\Gamma^C} \fint_{\Gamma_q^B} G_{\theta q} * \theta^B(t)\, q^C(\tau)\, d\boldsymbol{\xi}\, d\mathbf{x} - \frac{1}{2} \int_{\Gamma^C} \fint_{\Gamma^C} G_{q\theta} * q^C(t)\, \theta^C(\tau)\, d\boldsymbol{\xi}\, d\mathbf{x} \\
&+ \frac{1}{2} \fint_{\Gamma^C} \fint\!\!\!\!=_{\Gamma^C} G_{qq} * \theta^C(t)\, \theta^C(\tau)\, d\boldsymbol{\xi}\, d\mathbf{x} - \frac{1}{2} \int_{\Gamma^C} \fint_{\Gamma_\theta^B} G_{q\theta} * q^B(t)\, \theta^C(\tau)\, d\boldsymbol{\xi}\, d\mathbf{x} \\
&+ \frac{1}{2} \fint_{\Gamma^C} \fint\!\!\!\!=_{\Gamma_q^B} G_{qq} * \theta^B(t)\, \theta^C(\tau)\, d\boldsymbol{\xi}\, d\mathbf{x} + \frac{1}{2} \int_{\Gamma_\theta^B} \int_{\Gamma^C} G_{\theta\theta} * q^C(t)\, q^B(\tau)\, d\boldsymbol{\xi}\, d\mathbf{x} \\
&- \frac{1}{2} \int_{\Gamma_\theta^B} \fint_{\Gamma^C} G_{\theta q} * \theta^C(t)\, q^B(\tau)\, d\boldsymbol{\xi}\, d\mathbf{x} + \frac{1}{2} \int_{\Gamma_\theta^B} \int_{\Gamma_\theta^B} G_{\theta\theta} * q^B(t)\, q^B(\tau)\, d\boldsymbol{\xi}\, d\mathbf{x} \\
&- \frac{1}{2} \int_{\Gamma_\theta^B} \fint_{\Gamma_q^B} G_{\theta q} * \theta^B(t)\, q^B(\tau)\, d\boldsymbol{\xi}\, d\mathbf{x} - \frac{1}{2} \int_{\Gamma_q^B} \fint_{\Gamma^C} G_{q\theta} * q^C(t)\, \theta^B(\tau)\, d\boldsymbol{\xi}\, d\mathbf{x}
\end{aligned}
$$

(5.24) [cont.]
$$+\frac{1}{2}\int_{\Gamma_q^B}\fint_{\Gamma^C} G_{qq} * \theta^C(t)\,\theta^B(\tau)\,d\boldsymbol{\xi}\,d\mathbf{x} - \frac{1}{2}\int_{\Gamma_q^B}\fint_{\Gamma_\theta^B} G_{q\theta} * q^B(t)\,\theta^B(\tau)\,d\boldsymbol{\xi}\,d\mathbf{x}$$
$$+\frac{1}{2}\int_{\Gamma_q^B}\fint_{\Gamma_q^B} G_{qq} * \theta^B(t)\,\theta^B(\tau)\,d\boldsymbol{\xi}\,d\mathbf{x} - \int_{\Gamma^C}\bar{f}_\theta^C(t)\,q^C(\tau)\,d\mathbf{x}$$
$$\left.-\int_{\Gamma^C}\bar{f}_q^C(t)\,\theta^C(\tau)\,d\mathbf{x} - \int_{\Gamma_\theta^B}\bar{f}_\theta^B(t)\,q^B(\tau)\,d\mathbf{x} - \int_{\Gamma_q^B}\bar{f}_q^B(t)\,\theta^B(\tau)\,d\mathbf{x}\right]dt\,d\tau$$
$$+\int_0^\infty\int_0^\infty g(t+\tau)\left[\frac{1}{2}\int_{\Gamma^C} n_i q_i^F(t)\,\theta^C(\tau)\,d\mathbf{x} + \frac{1}{2}\int_{\Gamma^C}\theta^F(t)\,q^C(\tau)\,d\mathbf{x}\right]dt\,d\tau.$$

At this stage, the following two statements can be formulated.

PROPOSITION 7. The operator **N**, Eq. (5.22), (which is both differential and boundary-integral), is symmetric with respect to the bilinear form (5.23).

PROPOSITION 8. In the body $\bar{\Omega}$ subdivided into subdomains $\bar{\Omega}^B$ and $\bar{\Omega}^F$, and over the unbounded time $0 \le t < \infty$, flux $q^B(\mathbf{x},t)$ on Γ_θ^B and Γ^C, temperature $\theta^B(\mathbf{x},t)$ on Γ_q^B and Γ^C, temperature $\theta^B(\mathbf{x},t)$ in $\bar{\Omega}^F$, flux $q_i^F(\mathbf{x},t)$ in $\bar{\Omega}^F$ and temperature gradient $p_i^F(\mathbf{x},t)$ also in $\bar{\Omega}^F$ represent the actual response of the body to a given time history of external actions, if and only if they make stationary the above functional $\mathcal{F}$, Eq. (5.24).

The proof of statement 7 for the present coupled formulation can be given following a rather lengthy path of reasoning similar to that adopted in Sec. 4 and Appendix D and, hence, will not be expounded here for brevity. It is worth noting that the operator symmetry disrupted by the coefficient $1/2$ and $-1/2$ of the free terms in Eq. (5.20) is recovered in the operator of the coupled problem (if there is no interface Γ^C decomposing the domain Ω, those coefficients show up only in terms of data as seen in the preceding Sections).

REMARKS

A. In the expression (5.24) of the functional $\mathcal{F}$ the first addend, denoted henceforth by symbol A_1, contains the product of the temperature field θ^F and its time derivative, over the finite element subdomain $\bar{\Omega}^F$. It is easy and computationally useful to transform A_1 into the sum of two quadratic terms in θ^F alone. In fact we may write a sequence of alternative formulations for A_1:

$$A_1 = \frac{1}{2}\int_{\Omega^F}\int_0^\infty\int_0^\infty\int_0^\infty \gamma\varrho\,W(s)\,e^{-(t+\tau)s}\,\frac{\partial\theta^F(t)}{\partial t}\,\theta^F(\tau)\,ds\,dt\,d\tau\,d\Omega \tag{5.25}$$

(5.25) [cont.]
$$= \frac{1}{2}\int_{\Omega^F}\gamma\varrho\int_0^\infty W(s)\int_0^\infty e^{-st}\,\frac{\partial\theta^F(t)}{\partial t}\,dt\int_0^\infty e^{-s\tau}\,\theta^F(\tau)\,d\tau\,ds\,d\Omega$$
$$= \frac{1}{2}\int_{\Omega^F}\gamma\varrho\int_0^\infty W(s)\left[-\theta^F(0)+s\int_0^\infty e^{-st}\,\theta^F(t)\,dt\int_0^\infty e^{-s\tau}\,\theta^F(\tau)\,d\tau\right]ds\,d\Omega$$
$$= \frac{1}{2}\int_0^\infty\int_0^\infty g^*(t+\tau)\int_{\Omega^F}\gamma\varrho\,\theta^F(t)\,\theta^F(\tau)\,d\Omega\,d\tau\,dt$$
$$+ \frac{1}{2}\int_0^\infty g(t)\int_{\Omega^F}\gamma\varrho\,\theta^F(t)\,\theta^F(0)\,d\Omega\,dt\,.$$

The first of the above expressions of A_1 has been achieved through Eq. (4.3), the second by rearranging the integrations, the third through an integration by parts over $0 \le t \le \infty$, the fourth by setting:

$$g^*(t+\tau) = \int_0^\infty s\,W(s)\,e^{-(t+\tau)s}\,ds. \tag{5.26}$$

B. The functional $\mathcal{F}$ defined by Eq. (5.24) is a multifield functional in the sense that over the subdomain $\bar{\Omega}^F$ it depends on temperature θ^F, its gradient $\partial\theta^F/\partial x_i$ and heat flux q_i^F, which represent independent unknowns on the subdomain Ω^F and, as such, can independently be modelled over Ω^F as a finite element discretization. Alternatively, Eqs. (5.2) and (5.3) can be *a priori* enforced, so that $\mathcal{F}$ reduces to a functional of temperature only over $\Omega^F \times T_\infty$, besides of temperature and flux on $\Gamma_q^B \times T_\infty$ and $\Gamma_\theta^B \times T_\infty$, respectively. The former case, (multifield functional $\mathcal{F}$) might be desirable in order to construct parametric variational principles with possible computational benefits, as pointed out by FELIPPA [35] in the context of the finite element methods.

C. Operator $\mathbf{N}$ for the coupled problem turns out to be symmetric also with respect to the bilinear form (3.11) convolutive in time i.e. $<\mathbf{N}\,\mathbf{y},\mathbf{y}'> = <\mathbf{N}\,\mathbf{y}',\mathbf{y}>$, besides with respect to the new bilinear form (4.2)–(4.4), i.e. $\ll \mathbf{N}\,\mathbf{y},\mathbf{y}' \gg = \ll \mathbf{N}\,\mathbf{y}',\mathbf{y} \gg$, as stated by Proposition 7.

6. Closing remarks

With reference to the transient heat conduction in a homogeneous body as a typical linear initial-boundary-value problem, what precedes presented the results outlined and commented below as conclusions.

(**a**) A formulation in terms of boundary integral equations, constructed by means of single and double layer sources, in such way that the boundary integral operator is symmetric with respect to a bilinear form convolutive in time over the boundary and a finite time interval. The reciprocity properties of the time-dependent Green functions for the thermally homogeneous space were crucial to establish this circumstance.

(b) A variational characterization, corresponding to a saddle point, for the solution of the boundary problem formulated over a bounded time T_∞ as outlined at (a).

(c) A further, different saddle point characterization of the boundary solution over the unbounded time interval T_∞, which is shown to represent a minimum with respect to the temperature and a maximum with respect to the flux, separately. This variational property and the variable separation in it have been achieved by generating another special bilinear form in the Laplace transforms of the boundary variable fields, and by using the sign-definiteness (proved in Appendix B) of the Laplace transforms of the two Green functions for temperature and heat flux due to discontinuities (concentrated in space and time) of flux and temperature, respectively.

(d) An extension of the variational theorem (c), preserving the variable separation, in order to cover cases where transient heat conduction is governed in a subdomain by the symmetric system of boundary (and now interface too) integral equations, and in the complementary subdomain by the original partial differential equations (Fourier and conservation laws) and the relevant mixed (Dirichlet and Neumann) boundary conditions. The computational potentialities and applications of the results expounded in this paper and summarized above, are regarded to be beyond the present purposes and will be discussed elsewhere. However, the following remarks may envisage possible developments towards the use of these results in numerical solution methods.

(e) The variational approach mentioned at (b) and developed in Sec. 3, by modelling in space and time (either simultaneously or separately) the boundary fields over a time interval T, leads to a boundary element algebraic linear equation system endowed with symmetric coefficient matrix (the same is attainable from result (a) by means of a Galerkin weighted-residual approximate enforcement of the integral equations). Both the computational benefits of such symmetry and the difficulties of the hypersingular integrals are fairly well understood in the recent BEM literature, though with reference to physically different problems, see e.g. [8, 9, 14, 15, 16, 30, 31, 32, 33].

(f) The boundary element discretization based on the saddle-point theorem derived in Sec. 4 and above mentioned at (c) preserves symmetry in the resulting algebraic equations and appears to be computationally promising for short-duration transient analysis, in view of the use of field modelling by means of shape functions with exponential, asymptotical decay in time, as pointed out in a forthcoming paper.

(g) The uselfulness, in terms of computing cost-effectiveness of large-size analyses, of multifields (or heterogeneous) models when approximating initial-boundary-value problems over complex domain, has been demonstrated by a growing literature in various contexts, see e.g. [36]. This fact motivates the variational approach (d) developed in Sec. 5 in view of a domain decomposition for BE-FE coupling motivated like in other contexts [26, 27, 28]. However, further work is required to assess the expected computational merits of result (d) from this standpoint (account taken of the present more stringent continuity requirements on the temperature field, compared to those in traditional BEMs). Another issue worth being pursued elsewhere concerns parametric variational principles in the sense of FELIPPA [35], which might be generated in the symmetric BE context, with possible computational advantages.

Appendix A

With reference to Sec. 2.3 on the properties of the time-dependent Green's functions for heat conduction in isotropic space Ω_∞, the statement given there on their singularities is corroborated here below by formal developments concerning kernel $G_{\theta\theta}$ alone for brevity. In two-dimensional situations ($d = 2$), Eq. (2.6) specializes to:

$$G_{\theta\theta} = \frac{1}{4\pi\alpha t} e^{-\frac{r^2}{4\alpha t}}. \tag{A.1}$$

The Laplace transform of kernel (A.1) reads:

$$\mathcal{L}(G_{\theta\theta}) = \frac{1}{2\pi\alpha} K_0\left(\frac{r\sqrt{s}}{\sqrt{\alpha}}\right). \tag{A.2}$$

Here s is the transformation paremeter, γ denotes Euler constant ($\gamma = 0.577\ldots$) and K_0 represents the modified zero-order Bessel function, namely, z being its argument (see [37])

$$K_0(z) = -\left[\log\left(\frac{z}{2}\right) + \gamma\right] I_0(z) + \frac{\frac{1}{4}z^2}{(1!)^2} + \left(1 + \frac{1}{2}\right)\frac{\left(\frac{1}{4}z^2\right)^2}{(2!)^2} + \left(1 + \frac{1}{2} + \frac{1}{3}\right)\frac{\left(\frac{1}{4}z^2\right)^3}{(3!)^2} + \ldots, \tag{A.3}$$

where

$$I_0(z) = 1 + \frac{\frac{1}{4}z^2}{(1!)^2} + \frac{\left(\frac{1}{4}z^2\right)^2}{(2!)^2} + \frac{\left(\frac{1}{4}z^2\right)^3}{(3!)^2} + \ldots. \tag{A.4}$$

From Eqs. (A.2) – (A.4) it turns out that the only singular term in $\mathcal{L}(G_{\theta\theta})$ is $\log r$:

$$\text{(A.5)} \qquad \log\left(\frac{z}{2}\right) = \log\left(\sqrt{\frac{s}{4\alpha}}\right) + \log r.$$

In three-dimensional problems ($d = 3$), Eq. (2.6) becomes:

$$\text{(A.6)} \qquad G_{\theta\theta} = \frac{1}{[4\pi\alpha t]^{3/2}} e^{-\frac{r^2}{4\alpha t}}.$$

The Laplace transform of kernel (A.6) reads:

$$\text{(A.7)} \qquad \frac{2\sqrt{\alpha}}{(4\alpha)^{3/2}\pi}\left(\frac{1}{r}\right) e^{-\frac{r\sqrt{s}}{\sqrt{\alpha}}},$$

where it can be noticed that the only singular factor is r^{-1}.

Appendix B. A proof of sign semi-definiteness for the Laplace transforms of kernels $G_{\theta\theta}$ and G_{qq}

If Laplace integral transform (4.9) is applied to both its sides, the diffusion equation (2.1) becomes (s denoting the transform parameter):

$$\text{(B.1)} \qquad \alpha\nabla^2\mathcal{L}(\theta) = s\mathcal{L}(\theta) - \theta(\mathbf{x}, 0) - \frac{1}{\gamma\varrho}\mathcal{L}(Q).$$

On the boundary Γ, interpreted as a surface in the space Ω_∞, consider a distribution of temperature discontinuities $\Delta\theta = -\theta^+ + \theta^-$ and another one of heat flux jumps $\Delta q = -q^+ - q^-$, according to Eqs. $(2.11)_2$ and $(2.7)_2$, respectively. By virtue of the Gauss lemma and of equation (B.1) in the transform space, keeping in mind that the above sources are now the only external actions on Ω_∞, we write:

$$\text{(B.2)} \qquad \int\limits_{\Gamma-} \mathcal{L}(\theta^-)\frac{\partial\mathcal{L}(\theta^-)}{\partial n}\, d\Gamma = \int\limits_{\Gamma-} \mathcal{L}(\theta^-)\frac{\partial\mathcal{L}(\theta^-)}{\partial x_i} n_i^-\, d\Gamma$$

$$= \int\limits_{\Omega} \frac{\partial\left(\mathcal{L}(\theta)\dfrac{\partial\mathcal{L}(\theta)}{\partial x_i}\right)}{\partial x_i}\, d\Omega = \int\limits_{\Omega} \frac{\partial\mathcal{L}(\theta)}{\partial x_i}\frac{\partial\mathcal{L}(\theta)}{\partial x_i}\, d\Omega + \int\limits_{\Omega} \mathcal{L}(\theta)\frac{\partial^2\mathcal{L}(\theta)}{\partial x_i \partial x_i}\, d\Omega$$

$$= \int\limits_{\Omega} \frac{\partial\mathcal{L}(\theta)}{\partial x_i}\frac{\partial\mathcal{L}(\theta)}{\partial x_i}\, d\Omega + \frac{s}{\alpha}\int\limits_{\Omega} \mathcal{L}(\theta)\mathcal{L}(\theta)\, d\Omega \geq 0.$$

Similarly, denoting by Ω^* the exterior domain, $\Omega^* = \Omega_\infty - (\Omega \cup \Gamma)$, we obtain an inequality over Γ^+:

$$\text{(B.3)} \qquad \int_{\Gamma^+} \mathcal{L}(\theta^+)\frac{\partial\mathcal{L}(\theta^+)}{\partial n}\,d\Gamma = -\int_{\Gamma^+} \mathcal{L}(\theta^+)\frac{\partial\mathcal{L}(\theta^+)}{\partial x_i}n_i^+\,d\Gamma$$
$$= -\int_{\Omega^*} \frac{\partial\left(\mathcal{L}(\theta)\dfrac{\partial\mathcal{L}(\theta)}{\partial x_i}\right)}{\partial x_i}\,d\Omega$$
$$= -\int_{\Omega^*} \frac{\partial\mathcal{L}(\theta)}{\partial x_i}\frac{\partial\mathcal{L}(\theta)}{\partial x_i}\,d\Omega - \frac{s}{\alpha}\int_{\Omega^*}\mathcal{L}(\theta)\mathcal{L}(\theta)\,d\Omega \le 0.$$

Fourier's law, $q_i = -k\dfrac{\partial\theta}{\partial x_i}$, formulated at $\mathbf{x}^+$ and $\mathbf{x}^-$ in the Laplace transform space with $k = 1$ and $\mathbf{n}^+ = -\mathbf{n} = -\mathbf{n}^-$, yields:

$$\text{(B.4)} \qquad \frac{\partial\mathcal{L}(\theta^-)}{\partial n} = -\mathcal{L}(q^-), \qquad \frac{\partial\mathcal{L}(\theta^+)}{\partial n} = \mathcal{L}(q^+).$$

By adding inequality (B.2) to inequality (B.3) reversed in sign and, subsequently, by using Eqs. (B.4), we obtain:

$$\text{(B.5)} \qquad -\int_{\Gamma^-}\mathcal{L}(\theta^-)\frac{\partial\mathcal{L}(\theta^-)}{\partial n}\,d\Gamma + \int_{\Gamma^+}\mathcal{L}(\theta^+)\frac{\partial\mathcal{L}(\theta^+)}{\partial n}\,d\Gamma$$
$$= \int_{\Gamma^-}\mathcal{L}(\theta^-)\mathcal{L}(q^-)\,d\Gamma + \int_{\Gamma^+}\mathcal{L}(\theta^+)\mathcal{L}(q^+)\,d\Gamma \le 0.$$

Now, like in Sec. 3 for the symmetric BIE formulation, let the sources Δq be confined to the portion Γ_θ of Γ, the sources $\Delta\theta$ to Γ_q. In terms of their Laplace transforms this means that:

$$\text{(B.6)} \qquad \left.\begin{aligned}\mathcal{L}(\Delta q) &= -\mathcal{L}(q^+) - \mathcal{L}(q^-)\\ \mathcal{L}(\theta^+) &= \mathcal{L}(\theta^-) = \mathcal{L}(\theta)\end{aligned}\right\} \quad \text{on } \Gamma_\theta,$$

$$\text{(B.7)} \qquad \left.\begin{aligned}\mathcal{L}(\Delta\theta) &= -\mathcal{L}(\theta^+) + \mathcal{L}(\theta^-)\\ -\mathcal{L}(q^+) &= \mathcal{L}(q^-) = \mathcal{L}(q)\end{aligned}\right\} \quad \text{on } \Gamma_q.$$

As a consequence of Eqs. (B.6) and (B.7), the inequality (B.5) becomes:

$$\text{(B.8)} \qquad \int_{\Gamma_\theta^-}\mathcal{L}(\theta)\mathcal{L}(q^-)\,d\Gamma + \int_{\Gamma_\theta^+}\mathcal{L}(\theta)\mathcal{L}(q^+)\,d\Gamma + \int_{\Gamma_q^-}\mathcal{L}(\theta^-)\mathcal{L}(q)\,d\Gamma$$
$$- \int_{\Gamma_q^+}\mathcal{L}(\theta^+)\mathcal{L}(q)\,d\Gamma = -\int_{\Gamma_\theta}\mathcal{L}(\theta)\mathcal{L}(\Delta q)\,d\Gamma + \int_{\Gamma_q}\mathcal{L}(\Delta\theta)\mathcal{L}(q)\,d\Gamma \le 0.$$

In view of the definitions and mechanical interpretations of the Green functions $G_{\theta\theta}$ and G_{qq} in Sec. 2, Eq. (B.8) can be re-written in the form:

$$\text{(B.9)} \qquad -\int_{\Gamma_\theta}\int_{\Gamma_\theta} \mathcal{L}(\Delta q)\mathcal{L}\left(G_{\theta\theta}\right)\mathcal{L}(\Delta q)\, d\Gamma\, d\Gamma + \int_{\Gamma_q}\int_{\Gamma_q} \mathcal{L}(\Delta\theta)\mathcal{L}\left(G_{qq}\right)\mathcal{L}(\Delta\theta)\, d\Gamma\, d\Gamma \leq 0.$$

Since the source fields are arbitrary (so that $\mathcal{L}(\Delta q) \equiv 0$ and $\mathcal{L}(\Delta\theta) \equiv 0$ are feasible choices), Eq. (B.9) yields the two inequalities which embody the sign-semidefiniteness of the two kernels in Laplace transform space:

$$\text{(B.10)} \qquad \begin{aligned} &\int_{\Gamma_\theta}\int_{\Gamma_\theta} \mathcal{L}(\Delta q)\mathcal{L}\left(G_{\theta\theta}\right)\mathcal{L}(\Delta q)\, d\Gamma\, d\Gamma \geq 0 \qquad \forall\, \Delta q,\\ &\int_{\Gamma_q}\int_{\Gamma_q} \mathcal{L}(\Delta\theta)\mathcal{L}\left(G_{qq}\right)\mathcal{L}(\Delta\theta)\, d\Gamma\, d\Gamma \leq 0 \qquad \forall\, \Delta\theta \qquad \text{q. e. d.}\end{aligned}$$

Appendix C. A proof of the symmetry of the boundary integral operator with respect to a time-convolutive bilinear form over $\Gamma \times T$ (Proposition 1)

In order to prove Eq. (3.12), let us write the bilinear form (3.11), in terms of $\mathbf{y}$ according to Eq. $(3.9)_1$ and of $\mathbf{y}^*$ interpreted as the integral transform of another field $\mathbf{y}'$ through operator (3.10):

$$\text{(C.1)} \qquad \begin{aligned} &< \mathbf{Ly}, \mathbf{y}' > \\ &= \int_0^{\bar t}\int_{\Gamma_\theta}\left[\int_0^{t}\int_{\Gamma_\theta} G_{\theta\theta}\left(\mathbf{x},\boldsymbol{\xi};t-\tau\right) q\left(\boldsymbol{\xi};\tau\right)\, d\boldsymbol{\xi}\, d\tau\right] q'\left(\mathbf{x};\bar t - t\right)\, d\mathbf{x}\, dt \\ &- \int_0^{\bar t}\int_{\Gamma_\theta}\left[\int_0^{t}\int_{\Gamma_q} G_{\theta q}\left(\mathbf{x},\boldsymbol{\xi};t-\tau\right) \theta\left(\boldsymbol{\xi};\tau\right)\, d\boldsymbol{\xi}\, d\tau\right] q'\left(\mathbf{x};\bar t - t\right)\, d\mathbf{x}\, dt \\ &- \int_0^{\bar t}\int_{\Gamma_q}\left[\int_0^{t}\int_{\Gamma_\theta} G_{q\theta}\left(\mathbf{x},\boldsymbol{\xi};t-\tau\right) q\left(\boldsymbol{\xi};\tau\right)\, d\boldsymbol{\xi}\, d\tau\right] \theta'\left(\mathbf{x};\bar t - t\right)\, d\mathbf{x}\, dt \\ &+ \int_0^{\bar t}\int_{\Gamma_q}\left[\int_0^{t}\int_{\Gamma_q} G_{qq}\left(\mathbf{x},\boldsymbol{\xi};t-\tau\right) \theta\left(\boldsymbol{\xi};\tau\right)\, d\boldsymbol{\xi}\, d\tau\right] \theta'\left(\mathbf{x};\bar t - t\right)\, d\mathbf{x}\, dt. \end{aligned}$$

Let us change the integration order and use the Heaviside function $H(t-\tau)$ ($=0$ for $\tau > t$; $=1$ for $\tau < t$):

$$\text{(C.2)}\quad \int_0^{\bar t}\int_0^{\bar t}\left[\int_{\Gamma_\theta}\int_{\Gamma_\theta} q(\boldsymbol{\xi};\tau)\,G_{\theta\theta}(\mathbf{x},\boldsymbol{\xi};t-\tau)\,q'(\mathbf{x};\bar t-t)\;d\boldsymbol{\xi}\,d\mathbf{x}\right.$$
$$-\int_{\Gamma_\theta}\int_{\Gamma_q}\theta(\boldsymbol{\xi};\tau)\,G_{\theta q}(\mathbf{x},\boldsymbol{\xi};t-\tau)\,q'(\mathbf{x};\bar t-t)\;d\boldsymbol{\xi}\,d\mathbf{x}$$
$$-\int_{\Gamma_q}\int_{\Gamma_\theta} q(\boldsymbol{\xi};\tau)\,G_{q\theta}(\mathbf{x},\boldsymbol{\xi};t-\tau)\,\theta'(\mathbf{x};\bar t-t)\;d\boldsymbol{\xi}\,d\mathbf{x}$$
$$\left.+\int_{\Gamma_q}\int_{\Gamma_q}\theta(\boldsymbol{\xi};\tau)\,G_{qq}(\mathbf{x},\boldsymbol{\xi};t-\tau)\,\theta'(\mathbf{x};\bar t-t)\;d\boldsymbol{\xi}\,d\mathbf{x}\right]H(t-\tau)\,d\tau\,dt.$$

Now take into account the kernel reciprocity properties, Eqs. (2.14)–(2.16), and adopt for convenience new time variables $\sigma\equiv\bar t-\tau$, $s\equiv\bar t-t$:

$$\text{(C.3)}\quad \int_0^{\bar t}\int_0^{\bar t}\left[\int_{\Gamma_\theta}\int_{\Gamma_\theta} q(\boldsymbol{\xi};\bar t-\sigma)\,G_{\theta\theta}(\mathbf{x},\boldsymbol{\xi};\sigma-s)\,q'(\mathbf{x};s)\;d\boldsymbol{\xi}\,d\mathbf{x}\right.$$
$$-\int_{\Gamma_\theta}\int_{\Gamma_q}\theta(\boldsymbol{\xi};\bar t-\sigma)\,G_{\theta q}(\mathbf{x},\boldsymbol{\xi};\sigma-s)\,q'(\mathbf{x};s)\;d\boldsymbol{\xi}\,d\mathbf{x}$$
$$-\int_{\Gamma_q}\int_{\Gamma_\theta} q(\boldsymbol{\xi};\bar t-\sigma)\,G_{q\theta}(\mathbf{x},\boldsymbol{\xi};\sigma-s)\,\theta'(\mathbf{x};s)\;d\boldsymbol{\xi}\,d\mathbf{x}$$
$$\left.+\int_{\Gamma_q}\int_{\Gamma_q}\theta(\boldsymbol{\xi};\bar t-\sigma)\,G_{qq}(\mathbf{x},\boldsymbol{\xi};\sigma-s)\,\theta'(\mathbf{x};s)\;d\boldsymbol{\xi}\,d\mathbf{x}\right]H(\sigma-s)\,d\sigma\,ds.$$

By re-arranging Eq. (C.3), using again symbol t instead of σ and τ instead of s, and, finally, by interpreting the role of the Heaviside function in terms of integration intervals, an expression is achieved from which the symmetry property to prove clearly emerges:

$$\text{(C.4)}\quad \int_0^{\bar t}\int_{\Gamma_\theta}\left[\int_0^{t}\int_{\Gamma_\theta} G_{\theta\theta}(\mathbf{x},\boldsymbol{\xi};t-\tau)\,q'(\mathbf{x};\tau)\;d\boldsymbol{\xi}\,d\tau\right]q(\boldsymbol{\xi};\bar t-t)\;d\boldsymbol{\xi}\,dt$$
$$-\int_0^{\bar t}\int_{\Gamma_\theta}\left[\int_0^{t}\int_{\Gamma_q} G_{\theta q}(\mathbf{x},\boldsymbol{\xi};t-\tau)\,q'(\mathbf{x};\tau)\;d\mathbf{x}\,d\tau\right]\theta(\boldsymbol{\xi};\bar t-t)\;d\boldsymbol{\xi}\,dt$$

(C.4) [cont.]

$$- \int_0^{\bar t} \int_{\Gamma_q} \left[\int_0^{t} \int_{\Gamma_\theta} G_{q\theta}(\mathbf{x}, \boldsymbol{\xi}; t-\tau)\, \theta'(\mathbf{x};\tau)\, d\mathbf{x}\, d\tau \right] q(\boldsymbol{\xi}; \bar t - t)\, d\boldsymbol{\xi}\, dt$$

$$+ \int_0^{\bar t} \int_{\Gamma_q} \left[\int_0^{t} \int_{\Gamma_q} G_{qq}(\mathbf{x}, \boldsymbol{\xi}; t-\tau)\, \theta'(\mathbf{x};\tau)\, d\mathbf{x}\, d\tau \right] \theta(\boldsymbol{\xi}; \bar t - t)\, d\boldsymbol{\xi}\, dt$$

$$= < \mathbf{L}\mathbf{y}', \mathbf{y} > \qquad \text{q. e. d.}$$

Appendix D. A proof of the symmetry of the boundary integral operator with respect to a new bilinear form over $\Gamma \times T_\infty$ (Proposition 3)

A path of reasoning similar to that in Appendix C can be followed here again with reference to the bilinear form defined by Eq. (4.2) in order to prove the symmetry property expressed by Eq. (4.5), with the same interpretation of $\mathbf{y}$, $\mathbf{y}'$ and $\mathbf{L}$ as in Appendix C.

(D.1)

$$\ll \mathbf{L}\mathbf{y}, \mathbf{y}' \gg$$

$$= \int_{\Gamma_\theta} \int_0^\infty \int_0^\infty g(t+\eta)\, q'(\mathbf{x};\eta) \int_{\Gamma_\theta} \int_0^t G_{\theta\theta}(\mathbf{x}, \boldsymbol{\xi}; t-\tau)\, q(\boldsymbol{\xi};\tau)\, d\tau\, d\boldsymbol{\xi}\, d\eta\, dt\, d\mathbf{x}$$

$$- \int_{\Gamma_\theta} \int_0^\infty \int_0^\infty g(t+\eta)\, q'(\mathbf{x};\eta) \int_{\Gamma_q} \int_0^t G_{\theta q}(\mathbf{x}, \boldsymbol{\xi}; t-\tau)\, \theta(\boldsymbol{\xi};\tau)\, d\tau\, d\boldsymbol{\xi}\, d\eta\, dt\, d\mathbf{x}$$

$$- \int_{\Gamma_q} \int_0^\infty \int_0^\infty g(t+\eta)\, \theta'(\mathbf{x};\eta) \int_{\Gamma_\theta} \int_0^t G_{q\theta}(\mathbf{x}, \boldsymbol{\xi}; t-\tau)\, q(\boldsymbol{\xi};\tau)\, d\tau\, d\boldsymbol{\xi}\, d\eta\, dt\, d\mathbf{x}$$

$$+ \int_{\Gamma_q} \int_0^\infty \int_0^\infty g(t+\eta)\, \theta'(\mathbf{x};\eta) \int_{\Gamma_q} \int_0^t G_{qq}(\mathbf{x}, \boldsymbol{\xi}; t-\tau)\, \theta(\boldsymbol{\xi};\tau)\, d\tau\, d\boldsymbol{\xi}\, d\eta\, dt\, d\mathbf{x}.$$

Let us now take into account again the definition (4.2) and invert the integration sequence, to obtain:

(D.2)

$$\int_{\Gamma_\theta} \int_{\Gamma_\theta} \int_0^\infty W(s) \int_0^\infty e^{-s\eta} q'(\mathbf{x};\eta)\, d\eta \int_0^\infty e^{-st} G_{\theta\theta}(\mathbf{x}, \boldsymbol{\xi}; t)\, dt$$

$$\times \int_0^\infty e^{-s\tau} q(\boldsymbol{\xi};\tau)\, d\tau\, ds\, d\boldsymbol{\xi}\, d\mathbf{x}$$

(D.2) [cont.]

$$-\int_{\Gamma_\theta}\int_{\Gamma_q}\int_0^\infty W(s)\int_0^\infty e^{-s\eta} q'(\mathbf{x};\eta)\,d\eta \int_0^\infty e^{-st} G_{\theta q}(\mathbf{x},\boldsymbol{\xi};t)\,dt \times \int_0^\infty e^{-s\tau}\theta(\boldsymbol{\xi};\tau)\,d\tau\,ds\,d\boldsymbol{\xi}\,d\mathbf{x}$$

$$-\int_{\Gamma_q}\int_{\Gamma_\theta}\int_0^\infty W(s)\int_0^\infty e^{-s\eta} \theta'(\mathbf{x};\eta)\,d\eta \int_0^\infty e^{-st} G_{q\theta}(\mathbf{x},\boldsymbol{\xi};t)\,dt \times \int_0^\infty e^{-s\tau} q(\boldsymbol{\xi};\tau)\,d\tau\,ds\,d\boldsymbol{\xi}\,d\mathbf{x}$$

$$+\int_{\Gamma_q}\int_{\Gamma_q}\int_0^\infty W(s)\int_0^\infty e^{-s\eta} \theta'(\mathbf{x};\eta)\,d\eta \int_0^\infty e^{-st} G_{qq}(\mathbf{x},\boldsymbol{\xi};t)\,dt \times \int_0^\infty e^{-s\tau}\theta(\boldsymbol{\xi};\tau)\,d\tau\,ds\,d\boldsymbol{\xi}\,d\mathbf{x}.$$

The reciprocity properties (2.14)-(2.16) of the Green functions in point lead to a final expression which evidences the new symmetry of operator **L**, as stated by Proposition 3:

(D.3)

$$\int_{\Gamma_\theta}\int_0^\infty\int_0^\infty g(t+\eta)\,q(\mathbf{x};\eta)\int_{\Gamma_\theta}\int_0^t G_{\theta\theta}(\mathbf{x},\boldsymbol{\xi};t-\tau)\,q'(\boldsymbol{\xi};\tau)\,d\tau\,d\boldsymbol{\xi}\,d\eta\,dt\,d\mathbf{x}$$

$$-\int_{\Gamma_\theta}\int_0^\infty\int_0^\infty g(t+\eta)\,\theta(\mathbf{x};\eta)\int_{\Gamma_q}\int_0^t G_{\theta q}(\mathbf{x},\boldsymbol{\xi};t-\tau)\,q'(\boldsymbol{\xi};\tau)\,d\tau\,d\boldsymbol{\xi}\,d\eta\,dt\,d\mathbf{x}$$

$$-\int_{\Gamma_q}\int_0^\infty\int_0^\infty g(t+\eta)\,q(\mathbf{x};\eta)\int_{\Gamma_\theta}\int_0^t G_{q\theta}(\mathbf{x},\boldsymbol{\xi};t-\tau)\,\theta'(\boldsymbol{\xi};\tau)\,d\tau\,d\boldsymbol{\xi}\,d\eta\,dt\,d\mathbf{x}$$

$$+\int_{\Gamma_q}\int_0^\infty\int_0^\infty g(t+\eta)\,\theta(\mathbf{x};\eta)\int_{\Gamma_q}\int_0^t G_{qq}(\mathbf{x},\boldsymbol{\xi};t-\tau)\,\theta'(\boldsymbol{\xi};\tau)\,d\tau\,d\boldsymbol{\xi}\,d\eta\,dt\,d\mathbf{x}$$

$$= \ll \mathbf{L}\mathbf{y}', \mathbf{y} \gg \qquad \text{q. e. d.}$$

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