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Crack in an anisotropic medium

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A NUMERICAL method for the 3D problem of cracks in anisotropic media is developed, based on the variational approach to the crack opening problem. Properties of the pseudo-differential operator of the crack equilibrium problem are considered. Numerical examples are presented.

1. Introduction

SOLUTION of the 3D problem of a plane crack in anisotropic medium is not simple in view of the absence of effective algorithms for determination of fundamental solutions of the equilibrium equations.

Presumably the first integro-differential equation for the plane crack in anisotropic medium was constructed in [1, 2] by means of the Fourier and Radon transforms. The main difficulty in that approach lies in the necessity of constructing several auxiliary solutions to the problem of determination of the root of elliptic polynomials in three variables. In fact, in the case of arbitrary anisotropy, the latter problem can be solved only numerically. That does not allow us to obtain qualitative and quantitative results for cracks, which are known for isotropic case [3, 4].

The method developed for solution of the 3D problem for a plane crack of arbitrary shape in anisotropic medium is based on the construction of the elliptic pseudo-differential operator (p.d.o.) and application of the Goldstein – Klein – Eskin variational method [5] for a numerical solution.

2. Basic operators

Anisotropic elastic medium is considered, for which Lamé's equations of equilibrium can be written in the form

$$\mathbf{A}(\partial_x)\mathbf{u}(\mathbf{x}) \equiv -\operatorname{div}_x \mathbf{C} \cdots \nabla_x \mathbf{u}(\mathbf{x}) = 0, \tag{2.1}$$

where $\mathbf{A}$ is the matrix differential operator of the equilibrium equations, $\mathbf{C}$ is a fourth-order elasticity tensor, assumed to be strongly elliptic, while the medium itself is assumed to be hyperelastic, and $\mathbf{u}$ is the displacement vector field.

Application of the integral Fourier transform

$$g^{\wedge}(\xi) = \int_{R^3} g(\mathbf{x}) \exp(-2\pi i \mathbf{x}\cdot\xi)\, dx$$

to (2.1) yields the matrix operator **A**

$$\mathbf{A}^\wedge(\xi) = (2\pi)^2 \xi \cdot \mathbf{C} \cdot \xi \,. \tag{2.2}$$

As it follows from (2.2), operator $\mathbf{A}^\wedge$ is strongly elliptic, positive definite of degree 2 and analytic in R^3.

Now, formal identity following from the definition of the fundamental solution

$$\mathbf{A}^\wedge(\xi) \cdot \mathbf{E}^\wedge(\xi) = \mathbf{I}, \tag{2.3}$$

where $\mathbf{E}^\wedge$ is the fundamental solution, and **I** is a unit (diagonal) matrix, enables us to write $\mathbf{E}^\wedge$ in the form

$$\mathbf{E}^\wedge(\xi) = \mathbf{A}_0^\wedge(\xi) / \det \mathbf{A}^\wedge(\xi), \tag{2.4}$$

where $\mathbf{A}_0^\wedge$ is the cofactor of $\mathbf{A}^\wedge$. This formula shows that operator $\mathbf{E}^\wedge$ is also strongly elliptic, positive definite of degree -2 and $\mathbf{E}^\wedge \in C^\infty(R^3 \setminus 0, R^3 \otimes R^3)$.

The inverse Fourier transform applied to the formula (2.4) leads to

PROPOSITION 1. Fundamental solution of the equilibrium equations (2.1) is positive definite of degree -1 and $\mathbf{E} \in C^\infty(R^3 \setminus 0, R^3 \otimes R^3)$.

REMARK 1. It should be noted that, while for some specific groups of elastic symmetry the Fourier inversion of the formula (2.4) can be performed analytically, in the general case of elastic anisotropy it can be done only numerically [6].

3. Representation of solution

The displacement field produced by a crack is represented by the double-layer potential

$$\mathbf{u}(\mathbf{x}) = \int_\Omega \mathbf{b}(\mathbf{y}') \cdot \mathbf{T}(\partial_y, \nu_y) \mathbf{E}(\mathbf{x} - \mathbf{y}') dy', \tag{3.1}$$

where **b** is the crack opening, **T** is the operator of surface tractions, dy' is the Lebesgue measure on the Π_ν-plane, and $\Omega \subset \Pi_\nu$ is the bounded plane region occupied by the crack at the Π_ν-plane.

Surface tractions acting at the Π_ν-plane are determined by the limits (evaluated in non-tangential direction)

$$\mathbf{t}(\mathbf{x}') = \lim_{\mathbf{x} \to \mathbf{x}'} \mathbf{T}(\partial_x, -\nu_{x'}) \int_\Omega \mathbf{b}(\mathbf{y}') \cdot \mathbf{T}(\partial_y, \nu_y) \mathbf{E}(\mathbf{x} - \mathbf{y}') dy', \qquad \mathbf{x}' \in \Pi_\nu \,. \tag{3.2}$$

These limits are correctly determined according to the Lyapunov–Tauber theorem for elastic potentials [7].

Application of the Fourier transform to (3.2) gives the amplitude [9] of the corresponding pseudo-differential operator

$$\mathbf{G}^\wedge(\xi) = (2\pi)^2 \nu_y \cdot \mathbf{C} \cdots \xi \otimes \mathbf{E}^\wedge(\xi) \otimes \xi \cdots \mathbf{C} \cdot \nu_x \,. \tag{3.3}$$

Properties of the amplitude (3.3) and the associated principal symbol were investigated in [7, 8] where it was proved that condition of strong ellipticity for the elasticity tensor **C** ensures strong ellipticity for the amplitude (3.3) and principal symbol.

Reduction of the amplitude (3.3) to the Π_ν-plane gives the principal symbol we are looking for, which depends on $\xi' \in \Pi_\nu$ variables alone:

$$\mathbf{G}^\sim(\xi') = (2\pi)^2\, \text{F.P.} \int_{-\infty}^{\infty} \mathbf{G}^\wedge(\xi)\, d\xi'', \tag{3.4}$$

where $\xi \in R^3$, $\xi' = \mathrm{Pr}_{\Pi_\nu}\xi$, $\xi'' = \mathrm{Pr}_\nu(\xi)$ so $\xi = \xi' + \xi''\nu$. In (3.4) F.P. stands for the Finite Part of the diconvergent improper integral.

4. Regularization technique

To evaluate integral in (3.4) we observe that the integrand in (3.4) has an obvious asymptotic property due to (2.4)

$$\|\mathbf{G}^\wedge(\xi)\| = O(|\xi|^0), \qquad |\xi| \to \infty. \tag{4.1}$$

Moreover, the limit of the integrand when $|\xi''| \to \infty$ can be easily obtained from (3.4) using Eqs. (2.3) and (2.4), that is

$$\underset{\xi' \neq 0}{\forall}\ \xi' \ \lim_{|\xi''| \to \infty} \mathbf{G}^\wedge(\xi) = \nu \cdot \mathbf{C} \cdot \nu. \tag{4.2}$$

Now, from Eqs. (4.1) and (4.2) it follows that

$$\|\mathbf{G}^\wedge(\xi) - \nu \cdot \mathbf{C} \cdot \nu\| = O(|\xi''|^{-1}), \qquad |\xi''| \to \infty. \tag{4.3}$$

Analysis of the expressions (4.2) and (4.3) shows that the asymptotic terms of the highest order $O(|\xi''|^{-1})$, $|\xi''| \to \infty$ are odd functions of ξ''. So, the improper integral in (3.3) exists in the Principal Value sense at any $\xi' \neq 0$:

$$\mathbf{G}^\sim(\xi') = \text{P.V.} \int_{-\infty}^{\infty} (\mathbf{G}^\wedge(\xi) - \nu \cdot \mathbf{C} \cdot \nu)\, d\xi''. \tag{4.4}$$

Taking into account oddness (with respect to ξ''-variable) of the highest asymptotic expression in (4.3), the integral in (4.4) can be finally rewritten in the following form, which can be more convenient in computations

$$\mathbf{G}^\sim(\xi') = \int_{-\infty}^{\infty} [\mathbf{G}^\wedge(\xi) - \mathbf{G}^\wedge(-\xi) - 2\nu \cdot \mathbf{C} \cdot \nu]\, d\xi'', \qquad \xi' \neq 0. \tag{4.5}$$

5. Properties and structure of the operator $\mathbf{G}^{\sim}$

Proof of the following proposition can be found in [7, 8]:

PROPOSITION 2. a) Operator $\mathbf{G}^{\sim}$ is symmetric; b) $\mathbf{G}^{\sim}$ is positive definite of degree 1 with respect to $|\xi'|$; c) symbol $\mathbf{G}^{\sim}$ is strongly elliptic; d) work produced by the surface loadings $\mathbf{t}_0 \in H_{-1/2}(\Omega, R^3)$ acting on the crack faces

$$\int_{\Omega} \mathbf{t}_0 \cdot \mathbf{b} \, dx' > 0 \tag{5.1}$$

is positive, where $H_{-1/2}$ is the Hörmander functional space; e) quadratic functional

$$F(\mathbf{b}) \equiv \int_{\Pi_\nu} \mathbf{b}^{\sim}(\xi') \cdot \mathbf{G}^{\sim}(\xi') \cdot \overline{\mathbf{b}^{\sim}(\xi')} \, d\xi' \tag{5.2}$$

representing the elastic energy is coercive in Hörmander's space $H_{1/2}$.

COROLLARY 1. Normal loading on the crack surface $\mathbf{t}_0 = p\nu$, $p > 0$ produces crack opening and increases the crack volume, independently of the elastic anisotropy.

COROLLARY 2. Variational problem

$$\inf_{V \subset H_{1/2}} \left[\frac{1}{2} F(b) - l(b) \right], \qquad l(b) \equiv \int_{\Omega} t_0 \cdot b \, dx' = \int_{\Pi_\nu} t_0^{\sim} \cdot \overline{b}^{\sim} d\xi' \tag{5.3}$$

has a unique solution provided V is a closed subspace in $H_{1/2}(\Omega, R^3)$.

REMARK 1. It should be noted that for an anisotropic medium, normal loading of the crack surface can also produce components of displacement lying in the crack plane (together with necessarily present normal components, due to Corollary 1).

PROPOSITION 3. If anisotropic material possesses a plane of elastic symmetry and the crack lies in it, then $\mathbf{G}^{\sim}$ may be represented in the form

$$\mathbf{G}^{\sim}(\xi') = \mathbf{g}_1(\xi') + g_2(\xi') \nu \otimes \nu, \tag{5.4}$$

where $\mathbf{g}_1$ is a tensor with components lying in the Π_ν-plane: $\nu \cdot \mathbf{g}_1 = 0$, $\mathbf{g}_1 \cdot \nu = 0$, and $g_2(\xi')$ is a scalar-valued function.

P r o o f. At first we remark that if ν is the unit normal to the plane of elastic symmetry, then the fourth-order elasticity tensor $\mathbf{C}$ can have only an even number of indices corresponding to the ν-direction. Otherwise it would not satisfy the symmetry condition. Now it becomes obvious that term $\nu \cdot \mathbf{C} \cdot \nu$ in (4.3), (4.4) does not contain mixed indices, that is referring to ν and in the Π_ν-plane.

Similar considerations based on the decomposition (5.4), show that both operators $\mathbf{A}^\wedge$ and $\mathbf{E}^\wedge$ consist of odd or even components with respect to ξ''-variable, provided these components have, respectively, odd or even number of indices corresponding to the ν-direction. Analysis of the formula (3.3) with the preceding remark yields the conclusion that the mixed components of $\mathbf{G}^\wedge$ are odd functions of the ξ''-variable. This, together with (4.3), completes the proof.

REMARK 2. When the crack lies in the plane of elastic symmetry, then in contrast to the general case noted in the Remark 1, normal loading produces only normal components of the crack opening. The preceding proposition shows that the inverse statement is also true.

6. Construction of the p.d.o.

Fourier inversion in the Π_ν-plane of the operator $\mathbf{G}^\sim$ which gives the p.d.o. of the crack theory, can be done by the method similar to the multipolar expansion method [6].

Let the operator $\mathbf{G}^\sim$ be expanded into harmonic series on the unit circle $S \subset \Pi_\nu$,

$$
\begin{gathered}
\mathbf{G}^\sim(\xi') = |\xi'|^2 \frac{\sum\limits_n \mathbf{G}_n \exp(in\varphi)}{|\xi'|}, \\
\xi' = (\xi_1, \xi_2), \qquad \xi_1 = |\xi'|\cos\varphi, \qquad \xi_2 = |\xi'|\sin\varphi.
\end{gathered} \tag{6.1}
$$

Matrix coefficients $\mathbf{G}_n$ in Eq. (6.1) are determined by integration along the circle S (at $|\xi'| = 1$):

$$
\mathbf{G}_n = \pi^{-1} \int\limits_0^{2\pi} \mathbf{G}^\sim(\varphi) \exp(in\varphi)\, d\varphi\,. \tag{6.2}
$$

REMARK 3. In the expansion (6.1) are presented harmonic functions of even order only. That is due to positive definiteness of the operator $\mathbf{G}^\sim$.

Now the inverse Fourier transform of (6.1) can be obtained by Bochner's inversion formula which leads to an operator with weak singularity. This gives the p.d.o. we are looking for

$$
\begin{gathered}
\mathbf{G}(\mathbf{x}') = (2\pi)^{-2} \frac{\sum\limits_n i^n \mathbf{G}_n \exp(in\varphi)}{|\mathbf{x}'|} \cdot \mathbf{I}\Delta_{x'}, \\
\mathbf{x}' = (x_1, x_2), \qquad x_1 = |\mathbf{x}'|\cos\varphi, \qquad x_2 = |\mathbf{x}'|\sin\varphi.
\end{gathered} \tag{6.3}
$$

Remark 3 shows that formula (6.3) defines a weakly singular operator with the zero imaginary part.

7. Numerical method

The variational equation relating the crack closure to the surface tractions may be written in the form

$$
\begin{gathered}
W(\mathbf{b}) - l(\mathbf{b}) = 0, \\
W(\mathbf{b}) \equiv \int_{\Omega} \mathbf{b}\cdot\mathbf{G}\cdot\mathbf{b}\, dx', \qquad l(\mathbf{b}) \equiv \int_{\Omega} \mathbf{t}_0\cdot\mathbf{b}\, dx',
\end{gathered}
\tag{7.1}
$$

where W is the quadratic functional defining the energy necessary for the crack opening. The condition of vanishing of the gradient of expression (7.1) leads to the Euler equation

$$\nabla_{\mathbf{b}}[W(\mathbf{b}) - l(\mathbf{b})] = 0,$$

which coincides with (3.2). Equation (7.1) may be represented by means of the integral Fourier transform and Parseval's identity in the form

$$\int_{\Pi} \overline{b^{\sim}(\xi')}\cdot G^{\sim}(\xi')\cdot b^{\sim}(\xi')\, d\xi' = \int_{\Pi} b^{\sim}\overline{(\xi')}\cdot t_0{}^{\sim}(\xi')\, d\xi', \tag{7.2}$$

where Π is the plane of the crack Ω. We will find the Fourier-transform of the crack opening in the series form [10]

$$b^{\sim}(\xi') = \sum_m b_m \varphi^{\sim}{}_m(\xi'), \tag{7.3}$$

where $\varphi_m \in H_{1/2}$ are the coordinates, and the unknown vectorial coefficients b_m are defined from the condition of minimization of the quadratic functional (7.1). It gives the linear system enabling the determination of b_m [10]:

$$\sum_m b_m \int_{\Pi} \varphi^{\sim}{}_m(\xi')\overline{\varphi^{\sim}{}_k(\xi')G^{\sim}(\xi')}\, d\xi' = \int_{\Pi} \varphi^{\sim}{}_k(\xi')\overline{t^{\sim}{}_0(\xi')}\, d\xi', \tag{7.4}$$

or in a coordinate form

$$\sum_m b^{\beta}{}_m \int_{\Pi} \varphi^{\sim}{}_m(\xi')\overline{\varphi^{\sim}{}_k(\xi')G^{\sim\alpha\beta}(\xi')}\, d\xi' = \int_{\Pi} \varphi^{\sim}{}_k(\xi')\overline{t^{\alpha\sim}{}_0(\xi')}\, d\xi', \tag{7.5}$$

where indices α, β run from 1 to 3.

8. Example of numerical calculation

A crystal of $MgAl_2O_4$ (spinel) was taken for model calculations with the following anisotropy coefficients:

$$
\begin{gathered}
C_{1111} = 1, \qquad C_{1122} = 0.548, \qquad C_{1133} = 0.548, \\
C_{2222} = 1, \qquad C_{2233} = 0.548, \qquad C_{3333} = 1, \\
C_{1212} = C_{3232} = C_{1313} = 0.548,
\end{gathered}
$$

which correspond to the cubic crystal. The crack is placed in one of the main symmetry planes, and the crack is subjected to the normal loading.

FIG. 1. Opening of a crack of elliptical form. The semi-axes ratio 1 : 1.

FIG. 2. Opening of a crack of elliptical form. The semi-axes ratio 2 : 1.

Two examples of a circular crack and an elliptical crack with semiaxes ratio 1 : 2 were calculated. The computer results showed that in case of this loading and crack position in the cubic crystal, only normal crack opening occurs (the tangential displacement jumps are equal to zero), what is also in a good agreement with the theoretical results. The cracks openings are represented graphically in Fig. 1 and 2.

References

1. J.R. WILLIS, *Boussinesq problems for an anisotropic halfspace*, J. Mech. Phys. Solids, **15**, 331–339, 1967.
2. J.R. WILLIS, *The stress field around an elliptic crack in an anisotropic elastic medium*, Int. J. Engng. Sci., **6**, 253–263, 1968.
3. A.E. ANDREIKIV, *Spatial problems of the crack theory* [in Russian], Naukova Dumka, Kiev 1982.

4. R.V. GOLDSTEIN and V.M. ENTOV, *Variational estimates for the SIF at the front of the plane crack* [in Russian], Izv. AN SSSR, Mech. Tverdogo Tela, 3, 59–64, 1975.
5. R.V. GOLDSTEIN, V.M. ENTOV and A.F. ZAZOVSKI, *Solution of the mixed boundary value problems by the direct variational method* [in Russian], Chisl. Meth. Mech. Splosh. Sredy, 7, 5–13, 1976.
6. S.V. KUZNETSOV, *Fundamental solutions for Lamé's equations in anisotropic elasticity* [in Russian], Izv. AN SSSR, Mech. Tverdogo Tela, 4, 50–54, 1989.
7. S.V. KUZNETSOV, *Spatial problems for cracks in anisotropic media* [in Russian], Izv. AN SSSR, Mech. Tverdogo Tela, 4, 50–54, 1989.
8. S.K. KANAUN, *On the 3D problem of crack in anisotropic elastic media*, Appl. Math. Mech., **45**, 2, 1981.
9. F. TREV, *Introduction to pseudodifferential and Fourier integral operators*. Vol. 1. *Pseudodifferential operators*, Plenum Press, N. Y. and London 1982.
10. A.V. BALUEVA, *Three-dimensional problem on crack kinetics growth under gas diffusion*, Mech. of Solids, 6, 123–131, 1993.

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