Arch. Mech., 49, 1, pp. 77–101, Warszawa 1997

An idea of thin-plate thermal mirror
I. Mirror created by a heat pulse

Z. PŁOCHOCKI (WARSZAWA)
and A. MIODUCHOWSKI (EDMONTON)

AN IDEA AND THE THEORY of thermal mirrors created on the surfaces of a simply supported thin plane circular plate of an isotropic thermoelastic solid material by a uniform heat pulse, which is applied to one of the plate surfaces, is presented. Such a thermal mirror is – within the approximations applied for obtaining the solutions of the heat conduction and thermoelasticity equations – an ideal (aberration-free) optical mirror. The optical properties of the thermal mirror and their time evolution are derived and discussed in two asymptotical time regimes: the short-time and the long-time ones. Observability conditions for optical characteristics of the thermal mirror are estimated. Theoretical possibilities of an application of the thermal mirror to experimental determination of the temperature conductivity of a material are discussed. The theory presented can be also used for estimations of distortions of optical properties of pulse high power optical systems, originated by absorption of light by optical mirrors in such systems.

1. Introduction

IN THE PREVIOUS PAPER by the Authors [5] the idea of thermal mirror was presented following an example of the thermal mirror created by a focused heat pulse on the surface of an isotropic thermoelastic solid material half-space. In the present paper an opposite (in some sense) case is examined, namely – the thermal mirror created on the surfaces of a simply supported thin plane circular plate of a material of the same kind by a heat pulse, which is applied to one of the plate surfaces and is homogeneous across the surface. The aim is to calculate the fundamental optical properties of the mirror (i.e. – its aberration characteristic, optical power, and focal length), and their time evolution.

All the fundamental assumptions adopted here are the same as in the previous paper [5]; these are: thermal stresses theory approximation (rigid heat conductor approximation), quasi-static treatment of all the mechanical phenomena, and linearization of: the thermoelasticity and the heat conduction equations, and suitable boundary conditions (which are formulated at the undeformed surfaces of the plate); the plate is also assumed to be adiabatically insulated on its surfaces. Criteria of applicability of the thermal stresses theory approximation and the quasi-static displacement field one will be discussed in a separate paper by the Authors; here we note only that the former approximation depends on neglecting the influence of deformation rate on heat conduction processes, and the latter one denotes, that all the phenomena are observed in the time scale specific for heat conduction processes (the time scale specific for dynamic mechanical processes is much shorter). Some comments on the quasi-static displacement field approximation and on the adiabatic insulation are given in Secs. 7 and 8, respectively.

Main symbols

- c_p specific heat (the value of c_p for the numerical estimations is assumed together with ϱ_0),
- $D = 1/f$ optical power,
- E Young's modulus,
- f focal length,
- h half-thickness of the (unperturbed) plate (the numerical estimations are performed for $2h \doteq 10^{-3}$ m, and 10^{-2} m),
- $\mathrm{ierfc}(x)$ integral complementary error function: $\mathrm{ierfc}(x) = \int_x^\infty \mathrm{erfc}(t)\,dt$, $\mathrm{erfc}(t) = 1 - \mathrm{erf}(t) = \frac{2}{\sqrt{\pi}}\int_t^\infty \exp[-y^2]\,dy$,
- l (subscript) refers to the lower surface of the plate,
- M_T see suitable equation at the beginning of Sec. 4 and Eq. $(4.1)_2$,
- N_T see suitable equation at the beginning of Sec. 4 and Eq. $(4.1)_1$,
- O^* assumed small number ($\ll 1$), determining the accuracy of a given approximation (the value of the order of 0.01 is assumed for the numerical estimations),
- Q_{tot} total energy of the heat pulse,
- r, φ, z cylindrical coordinates,
- r_0 radius of the (unperturbed) plate (for the numerical estimations r_0 is assumed to be of the order of $10\cdot(2h)$),
- t time,
- T temperature, measured from an initial (constant) value,
- T_∞ final temperature, defined by Eq. (3.1),
- u (superscript) refers to the upper surface of the plate,
- u_α α-th coordinate of the displacement vector,
- U vertical displacement (shift) of the surface with respect to its initial (unperturbed) level (Fig. 1),
- z see r, φ, z,
- α linear heat expansion coefficient (the value of the order of 10^{-5} 1/K is assumed for the numerical estimations),
- $\delta(x - x_0)$ Dirac's delta distribution,
- δ^u, δ_l small terms (Eqs. (4.4)),
- $\Delta U(r)$ $:= U(0) - U(r)$,
- ε deflection angle (Fig. 2),
- ζ $:= z/(2h)$ – dimensionless z-coordinate,
- Θ $= T/T_\infty$ – dimensionless temperature,
- κ $:= \lambda/(\varrho_0 c_p)$ – temperature conductivity (heat diffusivity), (λ – heat conductivity), (the values of the order of $(10^{-7} - 10^{-4})\,\mathrm{m^2/s}$ are assumed for the numerical estimations, where the first value refers to the worst temperature conductors, and the second to the best ones),
- ν Poisson's coefficient,
- ϱ_0 mass density of the (unperturbed) material (the value of $\varrho_0 c_p$, as being of the order of $5\cdot10^6$ J/(m^{3}K), is assumed for the numerical estimations),
- τ $:= t\,\kappa/(2h)^2$ – dimensionless time,
- φ see r, φ, z,
- $\doteq$ reads: is of the order of.

2. Statement of the problem

Let us consider a plane circular plate of an isotropic thermoelastic solid material of thickness $2h$ and of radius r_0 (Fig. 1). The plate is described using the cylindrical coordinate system with the origin located in the center of the plate and with z-axis perpendicular to the main surfaces of the plate (before deformation). The plate is perturbed thermally by a heat pulse (in Fig. 1 the pulse is applied to the upper surface), which is homogeneous across the surface.

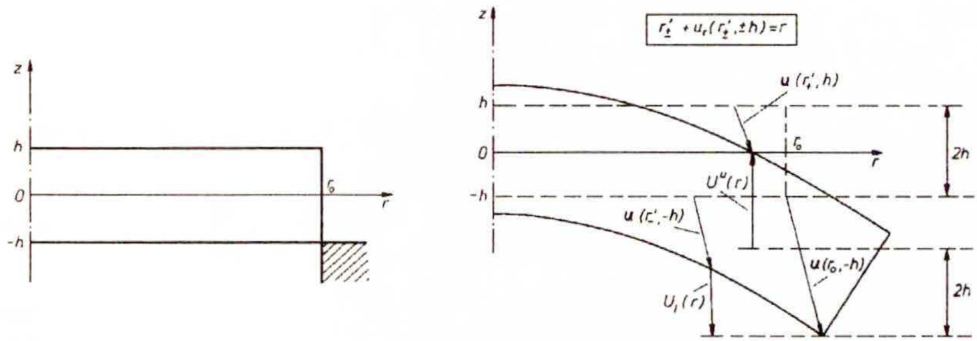

FIG. 1. Geometry of displacements.

The aim is to calculate the fundamental optical properties of the thermal mirror, i.e. – its aberration characteristic and optical power (focal length).

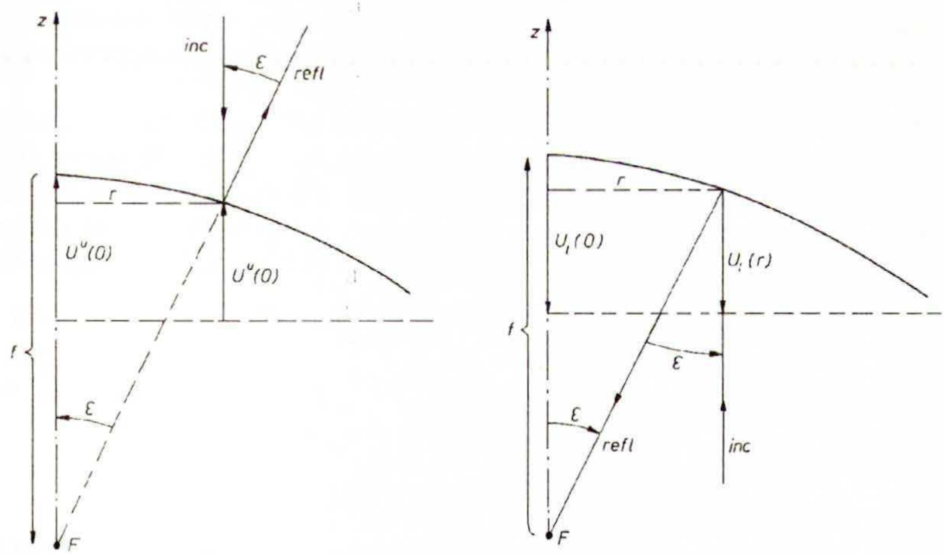

FIG. 2. Geometry of light rays.

The aberration characteristic is understood as a dependence: $\varepsilon = \varepsilon(r)$, where ε is an angle between incident testing light ray parallel to the symmetry axis and this ray after reflection from the mirror (Fig.2). The deflection angle ε is understood to be negative in the case of defocusing mirror (the upper surface in our case), and positive in the case of focusing mirror (the lower surface in our case).

The relationship between the deflection angle ε and the function U, which describes the vertical displacement of the surface with respect to its initial (unperturbed) level (Figs. 1 and 2), is (for both the upper and lower surfaces):

$$\tan\frac{\varepsilon}{2} = \frac{\partial U}{\partial r} \qquad \text{or} \qquad \tan\varepsilon = \frac{2\dfrac{\partial U}{\partial r}}{1-\left(\dfrac{\partial U}{\partial r}\right)^2},$$

therefore the aberration characteristic of the mirror is given by the formula:

$$\varepsilon = 2\arctan\frac{\partial U}{\partial r} = \arctan\frac{2\dfrac{\partial U}{\partial r}}{1-\left(\dfrac{\partial U}{\partial r}\right)^2} \cong 2\frac{\partial U}{\partial r}, \tag{2.1}$$

where the approximation is valid if:

$$\left(\frac{\partial U}{\partial r}\right)^2 \leq \frac{3O^*}{1+O^*} \cong 3\,O^*, \tag{2.2}$$

where, in turn, O^* stands for an assumed small number, which determines an approximation accuracy in the sense, that a relative error of an approximation does not exceed O^*.

The classical definition of the focal length is used [2], namely: the focal length f of the mirror is defined as a distance of the focal point F from the mirror along the mirror symmetry axis (Fig. 2); the focal length is understood to be negative in the case of defocusing mirror (the uppper surface in our case), and positive in the case of focusing mirror (the lower surface in our case). According to this definition we have (Fig. 2):

$$\tan\varepsilon = \frac{r}{f+\Delta U(r)},$$

where

$$\Delta U(r) := U(0) - U(r); \tag{2.3}$$

therefore the optical power D and the focal length f of the mirrors are given by the formula:

$$D = \frac{1}{f} = \frac{\tan\varepsilon}{r}\,\frac{1}{1-\Delta U\dfrac{\tan\varepsilon}{r}} = \frac{2}{r}\frac{\partial U}{\partial r}\,\frac{1}{1-\left(\dfrac{\partial U}{\partial r}\right)^2-\dfrac{2}{r}\dfrac{\partial U}{\partial r}\Delta U} \cong \frac{2}{r}\frac{\partial U}{\partial r}, \tag{2.4}$$

where the approximation holds, if

$$\left|\left(\frac{\partial U}{\partial r}\right)^2 + \frac{2}{r}\frac{\partial U}{\partial r}\Delta U\right| \le O^*. \tag{2.5}$$

In an ideal case both D and f do not depend on r, i.e. – each of these two functions has the same value for each testing ray, or – the focal point F is the same for all the testing rays, independently of r. Such a situation takes place when U is simply proportional to r^2 (parabolic mirror) ([1]).

Thus, in order to find the fundamental optical characteristics of the mirror and their time evolution, it is sufficient to find the function $U(r,t)$.

The function $U(r,t)$ is determined by both coordinates u_z and u_r of the displacement field in the material at a given surface (at $z = \pm h$, Fig. 1):

$$\begin{aligned} U^u(r) &= u_z(r'_+, h) - u_z(r_0, -h), \\ U_l(r) &= u_z(r_0, -h) - u_z(r'_-, -h), \end{aligned} \tag{2.6}$$

where $r'_+(r)$ and $r'_-(r)$ are solutions of the equations: $r'_\pm + u_r(r'_\pm, \pm h) = r$ with respect to $r'_\pm$, respectively (criteria of linearization of these formulae, which depend on the approximation: $r'_\pm \cong r$, are given in Sec. 6).

Thus, in order to find the fundamental optical characteristics of the mirror, it is sufficient to find the displacement field (the vertical displacement u_z only, if linearized Eqs. (2.6) are applied) at a given surface. This information will be deduced from the solution of the Lamé thermoelasticity equation, for which we need the solution of the heat conduction equation first. Thus, we will examine, first, the thermal part of the problem, and next – the thermoelastic part. Having suitable information we will come back to the analysis of the optical properties of the mirror.

3. Thermal problem

Following the specification of the thermal perturbation, the temperature field in the material is assumed to be dependent on z and t only: $T = T(z,t)$. Therefore, according to the general assumptions adopted, the heat conduction equation is:

$$\frac{\partial \Theta}{\partial \tau} = \frac{\partial^2 \Theta}{\partial \zeta^2} + \delta(\tau - 0)\,\delta\left(\zeta - \frac{1}{2}\right),$$

([1]) Both criteria expressed by Ineqs. (2.2) and (2.5) determine the so-called paraxial optics approximation:

$$D = \frac{1}{f} = \frac{\varepsilon}{r} = \frac{2}{r}\frac{\partial U}{\partial r}.$$

An ideal case in this approximation is characterized by simple proportionality of ε to r.

It will be proved later that this approximation is not necessary for the mirrors examined, because for such mirrors the left-hand side of Ineq. (2.5) is identically equal to zero (and only the approximation $\arctan x \cong x$ may be applied).

where

$$\tau := \frac{\kappa}{(2h)^2}\,t\,, \qquad \zeta := \frac{z}{2h}$$

stand for dimensionless time and z-coordinate, respectively, $\kappa = \lambda/(\varrho_0 c_p)$ is the temperature conductivity (heat diffusivity) of a given material, λ, ϱ_0 and c_p stand for heat conductivity, density and specific heat of a given material, respectively, $\delta(x - x_0)$ stands for the Dirac's delta distribution, and

$$\Theta(\zeta,\tau) = \frac{T[z = z(\zeta), t = t(\tau)]}{T_\infty}$$

stands for dimensionless temperature (as a function of dimensionless variables), where, in turn,

$$T_\infty := T(t = \infty) = \frac{Q_{\mathrm{tot}}}{\pi r_0^2}\,\frac{1}{2h\varrho_0 c_p}\,, \tag{3.1}$$

and Q_{tot} stands for the total energy of the heat pulse. The boundary and initial conditions are:

$$\frac{\partial\Theta}{\partial\zeta}\left(\zeta = \pm\frac{1}{2}\right) = 0 = \Theta(\tau = 0).$$

The Green function for the thermal problem in the whole space is known [3]. Applying therefore the method of sources and sinks one may write the solution of our problem in the form:

$$\begin{aligned}\Theta &= \frac{1}{\sqrt{\pi\tau}}\sum_{m=0}^{\infty}\left\{\exp\left[-\frac{\left(2m + \frac{1}{2} - \zeta\right)^2}{4\tau}\right] + \exp\left[-\frac{\left(2m + \frac{3}{2} + \zeta\right)^2}{4\tau}\right]\right\} \\ &= 1 + 2\sum_{k=1}^{\infty}(-1)^k \exp[-k^2\pi^2\tau]\cos k\pi\left(\zeta + \frac{1}{2}\right),\end{aligned} \tag{3.2}$$

where the first line represents the original solution obtained using the method mentioned(2), and the second one – that solution after expansion into Fourier cosine series(3) (the function $\Theta(\zeta,\tau)$ is symmetric with respect to $\zeta + 1/2$, and it satisfies the Dirichlet conditions).

(2) The same result is obtainable by applying the Laplace transformation method to solve the following equivalent problem:

$$\frac{\partial\Theta}{\partial\tau} = \frac{\partial^2\Theta}{\partial\zeta^2}\,, \qquad \frac{\partial\Theta}{\partial\zeta}\left(\zeta = -\frac{1}{2}\right) = 0, \qquad \frac{\partial\Theta}{\partial\zeta}\left(\zeta = \frac{1}{2}\right) = \delta(\tau - 0), \qquad \Theta(\tau = 0) = 0.$$

(3) The same result is obtainable by applying the Fourier method of separation of independent variables to

4. The thermoelastic problem

The solution of the Lamé thermoelasticity equation for a simply supported plane finite thin plate ([4]) with $T = T(z)$ and with no external forces is known [1] (in the approximation, which depends od replacing the local boundary conditions for the stress tensor coordinates at the side surface of the plate by suitable integral ones); in the case of circular plate we have:

$$u_r(r,z) = \frac{r}{E}\left[\frac{N_T}{2h} + \frac{3z}{2h^3}M_T\right],$$
$$u_z(r,z) = -\frac{3M_T}{4h^3E}r^2 + \frac{1}{1-\nu}\left[(1+\nu)\alpha\int\limits_0^z T\,dz - \frac{\nu z}{hE}N_T - \frac{3\nu z^2}{2h^3E}M_T\right],$$

where α stands for the (linear) heat expansion coefficient, E – for the Young's modulus, ν – for the Poisson's coefficient, and

$$N_T := \alpha E\int\limits_{-h}^{h} T\,dz\,, \qquad M_T := \alpha E\int\limits_{-h}^{h} Tz\,dz\,.$$

Using the formulae representing the solution of the thermal problem (Eq. (3.2)) we have:

$$\begin{aligned} N_T &= 2hE\alpha T_\infty\,, \\ M_T &= 2h^2E\alpha T_\infty\left[1 - \frac{4}{\sqrt{\pi}}\sqrt{\tau} - 8\sqrt{\tau}\sum_{m=1}^{\infty}(-1)^m \operatorname{ierfc}\frac{m}{2\sqrt{\tau}}\right] \\ &= 2h^2E\alpha T_\infty\sum_{k=1}^{\infty}\frac{8}{\pi^2}\frac{1}{(2k-1)^2}\exp[-(2k-1)^2\pi^2\tau], \end{aligned} \tag{4.1}$$

where $\operatorname{ierfc}(x)$ stands for the integral complementary error function:

$$\operatorname{ierfc}(x) = \int\limits_x^\infty \operatorname{erfc}(t)\,dt\,, \qquad \operatorname{erfc}(t) = 1 - \operatorname{erf}(t) = \frac{2}{\sqrt{\pi}}\int\limits_t^\infty \exp[-y^2]\,dy\,.$$

solve the following equivalent problem:

$$\frac{\partial\Theta}{\partial\tau} = \frac{\partial^2\Theta}{\partial\zeta^2}, \qquad \frac{\partial\Theta}{\partial\zeta}\left(\zeta = \pm\frac{1}{2}\right) = 0, \qquad \Theta(\tau=0) = \delta\left(\zeta - \frac{1}{2}\right).$$

It may be useful to note that, if the initial condition is not specified, then the solution of the heat conduction equation has the same form with coefficients $2(-1)^k$ replaced by unknown coefficients a_k (which are determinable from the initial condition after it will be specified), i.e. the structure of time-dependence of the solution (in the Fourier cosine representation) does not depend on the initial condition.

([4]) The plate is understood to be thin in the sense that the following approximate conditions for the stress tensor coordinates are valid: $\sigma_{rz} = \sigma_{\varphi z} = \sigma_{zz} = 0$.

Using these formulae one may rewrite Eqs. (2.6) in the form:

$$
\begin{aligned}
U^u &= \frac{N_T}{E} + U_{\max}\left[1 - \left(\frac{r}{r_0}\right)^2 \frac{1}{(1+\delta^u)^2}\right] \cong \frac{N_T}{E} + U_{\max}\left[1 - \left(\frac{r}{r_0}\right)^2\right],\\
U_l &= -U_{\max}\left[1 - \left(\frac{r}{r_0}\right)^2 \frac{1}{(1-\delta_l)^2}\right] \qquad \cong -U_{\max}\left[1 - \left(\frac{r}{r_0}\right)^2\right],
\end{aligned} \tag{4.2}
$$

where

$$
U_{\max} = \frac{3M_T}{4h^3E}\, r_0^2, \tag{4.3}
$$

$$
\delta_l^u = \pm \frac{1}{r}\, u_r(r, \pm h) = \frac{1}{2hE}\left[\pm N_T + \frac{3}{h} M_T\right], \tag{4.4}
$$

where, in turn: the superscript u and the upper sign refer to the upper surface of the plate; the subscript l and the lower sign refer to the lower surface; N_T is given by Eq. $(4.1)_1$ and M_T – by Eq. $(4.1)_2$ or Eq. $(4.1)_3$; and the approximations in Eqs. (4.2) (which correspond to the linearization of Eqs. (2.6)) are valid if the functions δ can be neglected (see Sec. 6).

5. The optical problem

After substitution of Eqs. (4.2) into Eq. (2.1), the aberration characteristic of the thermal mirrors examined is obtained:

$$
\begin{aligned}
\varepsilon_l^u &= \mp 2 \arctan\left[\frac{2U_{\max}}{r_0} \frac{r}{r_0} \frac{1}{(1 \pm \delta_l^u)^2}\right]\\
&\cong \mp 2 \arctan\left[\frac{2U_{\max}}{r_0} \frac{r}{r_0}\right] \cong \mp \frac{4U_{\max}}{r_0} \frac{r}{r_0},
\end{aligned} \tag{5.1}
$$

where (as previously): the superscript u and the upper sign refer to the upper surface of the plate; the subscript l and the lower sign refer to the lower surface; $U_{\max}$ is given by Eq. (4.3) with Eqs. $(4.1)_{2,3}$; δ are given by Eq. (4.4) with Eqs. (4.1); the first approximation (which corresponds to the linearization of Eqs. (2.6)) is valid, if the functions δ can be neglected (see Sec. 6), and the second one (the paraxial optics approximation) – if (in addition)

$$
\left(\frac{2U_{\max}}{r_0}\right)^2 \frac{r^2}{r_0^2} \le \frac{3\,O^*}{1 + O^*} \cong 3\,O^*, \tag{5.2}
$$

where O^* is an assumed small number.

It may be useful to comment at this place on the condition of applicability of the paraxial optics approximation, as given by Ineq. (5.2). The functions δ are assumed to be negligible. As it follows from Eqs. $(4.1)_{2,3}$, the function M_T is a monotonically decreasing one from $2h^2E\alpha T_\infty$ to 0, as time varies from 0 to ∞, respectively. Therefore, according to Eq. (4.3) we have:

$$0 \le \frac{U_{\max}}{r_0} \le 3\frac{r_0}{2h}\,\alpha\,T_\infty\ ,$$

where the right-hand side of this inequality represents the value of $U_{\max}$ at $\tau = 0$, and the left-hand one – at $\tau = \infty$. The criterion of applicability of the paraxial optics approximation can be therefore written in the form:

$$\left(\frac{r}{r_0}\right)^2 \le \frac{1}{12}\,\frac{O^*}{1+O^*}\left(\frac{2h}{r_0}\right)^2\frac{1}{(\alpha\,T_\infty)^2}\ .$$

Assuming

- $O^* \doteq 10^{-2}$,
- $r_0 \doteq 10\cdot(2h)$,
- $\alpha \doteq 10^{-5}$ 1/K,
- $T_\infty \doteq (1-10)$ K,

(the sign $\doteq$ reads: is of the order of) and taking into account that the maximum value of r is very close to r_0, one can see, that the right-hand side of the inequality given above is of the order of $10^5 - 10^3$, so the criterion examined is well satisfied (it can be violated only in a case of very strong requirements; if for instance: $O^* \doteq 10^{-4}$, $r_0 \doteq 10^2\cdot(2h)$, and $\alpha T_\infty \doteq 10^{-4}$, then the right-hand side of the inequality given above may be even of the order of 10^{-1} in the worst case).

The aberration characteristic obtained represents an ideal case, therefore both the upper and lower surfaces of the plate considered represent an ideal (parabolic) mirror (the upper – defocusing mirror, and the lower – focusing one). In fact, substituting Eqs. (4.2) into Eq. (2.4) we obtain the optical power D and the focal length f of the mirror as independent of distance r:

$$D_l^u = \frac{1}{f_l^u} = \mp\frac{4}{r_0^2}\,U_{\max}\,\frac{1}{(1\pm\delta_l^u)^2} \cong \mp\frac{4}{r_0^2}\,U_{\max}\,, \tag{5.3}$$

where $U_{\max}$ is given by Eq. (4.3) with Eq. $(4.1)_{2,3}$, and δ are given by Eqs. (4.4) with Eqs. (4.1); and the approximation holds, if the functions δ can be neglected (see Sec. 6).

The results expressed by Eqs. (5.3) denote, that the mirrors considered are aberration-free, and no paraxial optics approximation is needed to idealize them (although this approximation may be applied for simplifying the formulae for the functions ε, if it is allowable (see comment given above)). It should be noted, that our results are not valid for an arbitrary plate, because they were obtained under defined assumptions.

As it is seen from the formulae given above, the time evolution of the displacement function U and the optical properties of the thermal mirror is governed by the dependence of the function $U_{\max}$ (Eqs. (4.3) and (4.1)) and, in addition, by that of the functions δ – on time. This dependence is complicated and difficult for a simple interpretation. It can be simplified in two steps: first, by neglecting the functions δ in the suitable expressions (see Sec. 6); then, second, significant simplification can be obtained for sufficiently short or long time (see Secs. 7 or 8, respectively).

6. Criterion for neglecting the functions δ

Because the quantity N_T (Eqs. $(4.1)_1$) is independent of time, and the quantity M_T (Eqs. $(4.1)_{2,3}$) is a monotonically decreasing function of time, which varies from $2h^2E\alpha T_\infty$ at $\tau = 0$ to 0 at $\tau = \infty$, therefore the functions δ vary within the limits:

$$\alpha T_\infty \le \delta^u \le 4\alpha T_\infty \,,$$
$$-\alpha T_\infty \le \delta_l \le 2\alpha T_\infty \,,$$

where the right-hand side limits correspond to $\tau = 0$, and the left-hand side ones – to $\tau = \infty$. Adopting the previously assumed values for α and T_∞ one has:

$$\delta^u \le 4\cdot(10^{-5} - 10^{-4}),$$
$$|\delta_l| \le 2\cdot(10^{-5} - 10^{-4}),$$

where the first value in the brackets corresponds to $T_\infty = 1\,\mathrm{K}$, and the second one – to $T_\infty = 10\,\mathrm{K}$.

Thus, in practical cases the functions δ are in fact small quantities in comparison with unity. Criteria for neglecting these fuctions in each of the formulae for U^u, U_l, ε^u, ε_l and $D = 1/f$ are examined in details in the Appendix. This discussion suggests the following assumption as the common criterion for neglecting the functions δ in all the formulae mentioned (in the sense, that a relative error of an approximation in any case does not exceed O^*, if this criterion is satisfied)([5]):

$$|\delta_l^u| \le 4\,\alpha T_\infty \le \frac{1}{2}O^* \tag{6.1}$$

(which is approximated in some cases, with a reasonable accuracy however, as it is pointed out in the Appendix). This assumption implies no limitation for the distance r in the case of the functions ε and $D = 1/f$, whereas in the case of

([5]) If this criterion would be formulated for the upper and the lower surfaces separately, then for the upper surface it would have the form as given by Ineq. (6.1), and for the lower one – by the same inequality with only number 4 replaced by number 2.

the functions U it is (approximately) equivalent to the following condition for r (see Appendix):

$$r \le \frac{1}{\sqrt{2}} r_0 \cong 0.7071\, r_0 .$$

It may be useful to note here, that using Eq. (3.1) one can rewrite Ineq. (6.1) as a criterion for the maximum pulse energy Q_{tot}, for which Ineq. (6.1) is satisfied. Assuming (in addition to the assumptions of this kind adopted previously):

- $\varrho_0 c_p \doteq 5 \cdot 10^6$ J/(m^{3}K)

we obtain in this way

$$Q_{\text{tot}} \le \begin{cases} 2 \cdot 10^2 \text{ J}, & \text{for } \; 2h \doteq 10^{-3}\,\text{m}, \\ 2 \cdot 10^5 \text{ J}, & \text{for } \; 2h \doteq 10^{-2}\,\text{m}; \end{cases}$$

(6.2)

$$\frac{Q_{\text{tot}}}{\pi r_0^2} \le \begin{cases} 6 \cdot 10^5 \text{ J/m}^2, & \text{for } \; 2h \doteq 10^{-3}\,\text{m}, \\ 6 \cdot 10^6 \text{ J/m}^2, & \text{for } \; 2h \doteq 10^{-2}\,\text{m}. \end{cases}$$

7. Short-time regime

For sufficiently short time the sum in the brackets in Eq. $(4.1)_2$ can be truncated after the second term. Let us note, that because $\operatorname{ierfc}(x)$ is a monotonically decreasing function, therefore $\operatorname{ierfc}(m/2\sqrt{\tau}) > \operatorname{ierfc}[(m+1)/2\sqrt{\tau}]$. In addition, if $\tau < \pi/16 \cong 0.196$, then $\operatorname{ierfc}(1/2\sqrt{\tau}) < 1/2\sqrt{\pi}$. The whole sum in the brackets in Eq. $(4.1)_2$ can be therefore treated as a Leibniz-type series([6]). Then, the sum considered can be approximated by the first two terms only with an accuracy to O^*, if

$$8\sqrt{\tau}\,\operatorname{ierfc}\frac{1}{2\sqrt{\tau}} \le O^* \left(1 - \frac{4\sqrt{\tau}}{\sqrt{\pi}}\right).$$

This inequality is satisfied, if

$$\tau \le \tau_2 := \frac{1}{4x_0^2},$$

([6]) The Leibniz-type series (LS) is understood to be a convergent series of the type:

$$LS := \sum_{m=0}^{\infty} (-1)^m a_m, \qquad a_m > a_{m+1} > 0.$$

Such a series can be precisely estimated as follows (Leibniz's theorem):

$$\sum_{m=0}^{2k} (-1)^m a_m > LS > \sum_{m=0}^{2k-1} (-1)^m a_m .$$

In particular case one may obtain

$$a_0 - a_1 + a_2 > LS > a_0 - a_1 ,$$

therefore $LS \cong a_0 - a_1$ with an accuracy to O^*, if $a_2 \le O^*(a_0 - a_1)$.

where x_0 stands for a solution of the equation: $\operatorname{ierfc} x = O^*\left(\frac{x}{4} - \frac{1}{2\sqrt{\pi}}\right)$ with respect to x.

Assuming (as previously) $O^* = 0.01$, one may find $x_0 \cong 1.87$, and

$$\tau \leq \tau_{\text{short}} = \tau_2 \cong 7 \cdot 10^{-2}. \tag{7.1}$$

Assuming, in addition

- $\kappa \doteq (10^{-7} - 10^{-4})\ \mathrm{m^2/s}$,

where the first value in the brackets refers to the worst temperature conductors and the second one – to the best temperature conductors, one may rewrite the criterion expressed by Ineq. (7.1) in dimensional form([7])

$$t \leq t_{\text{short}} = t_2 \doteq \begin{cases} 7 \cdot (10^{-1} - 10^{-4})\,\mathrm{s}, & \text{for} \quad 2h \doteq 10^{-3}\,\mathrm{m}, \\ 7 \cdot (10 - 10^{-2})\,\mathrm{s}, & \text{for} \quad 2h \doteq 10^{-2}\,\mathrm{m}. \end{cases} \tag{7.2}$$

By the way let us note here that all the mechanical phenomena are treateted in the quasi-static approximation, i.e. observation time τ should be sufficiently large. The following criterion is assumed:

$$\tau \geq \tau_{\min} = \frac{1}{O^*}\frac{r_0}{c}\frac{\kappa}{4h^2}, \qquad t \geq t_{\min} = \frac{1}{O^*}\frac{r_0}{c}, \tag{7.3}$$

where the first condition is written in the dimensionless form (in the time scale applied in the paper), the second condition is written in the usual dimensional form, and c stands for velocity of sound in a given material. Assuming (in addition to the assumptions of this kind adopted previously):

- $c \doteq 2 \cdot 10^3\ \mathrm{m/s}$,

we have (in dimensionless and in dimensional forms):

$$\begin{aligned} \tau \geq \tau_{\min} &\doteq \begin{cases} 5 \cdot (10^{-5} - 10^{-2}), & \text{for} \quad 2h \doteq 10^{-3}\,\mathrm{m}, \\ 5 \cdot (10^{-6} - 10^{-3}), & \text{for} \quad 2h \doteq 10^{-2}\,\mathrm{m}, \end{cases} \\ t \geq t_{\min} &\doteq \begin{cases} 5 \cdot 10^{-4}\,\mathrm{s}, & \text{for} \quad 2h \doteq 10^{-3}\,\mathrm{m}, \\ 5 \cdot 10^{-3}\,\mathrm{s}, & \text{for} \quad 2h \doteq 10^{-2}\,\mathrm{m}. \end{cases} \end{aligned} \tag{7.4}$$

Comparing Ineqs. (7.4) and (7.1) [(7.2)] one can see, that within the quasi-static displacement fields approximation, there exists a relatively large field for the short-time regime approximation([8]).

([7]) For $O^* = 10^{-3}$ or 10^{-4} one may find $x_0 \cong 2.25$ or 2.61, respectively, and the number 7 in Ineqs. (7.1) and (7.2) is replaced by the number 5 or 4, respectively.

([8]) Let us note in addition, that the perturbing heat pulse is assumed to be instantaneous, therefore the observation time has to be much longer than the time of duration of the real physical pulse.

If the criterion expressed by Ineq. (7.1) (or (7.2)) is satisfied, then the sum in the brackets in Eq. $(4.1)_2$ can be approximated by its first two terms only, which is decreasing from 1 to about 0.4 as τ is increasing from 0 to $\tau_{\text{short}} = 7 \cdot 10^{-2}$.

Thus, if the criteria expressed by Ineqs. (7.1) (or (7.2)) and (6.1) are satisfied, then the sum in the brackets in Eq. $(4.1)_2$ can be truncated after the second term, and the functions δ can be ignored (the total relative error of this double approximation does not exceed $(1 + O^*)^2 - 1 \cong 2O^*$). In this approximation, the function $U_{\max}$, and therefore also U, D, and f are linear functions of $\sqrt{\tau}$:

$$
(7.5) \qquad
\begin{aligned}
U_{\max} &= U_{\max}(0)\left(1 - \frac{4}{\sqrt{\pi}}\sqrt{\tau}\right),\\
U^u &= U_{\max}(0)\left\{\frac{1}{3}\left(\frac{2h}{r_0}\right)^2 + \left[1 - \left(\frac{r}{r_0}\right)^2\right]\left(1 - \frac{4}{\sqrt{\pi}}\sqrt{\tau}\right)\right\},\\
U_l &= -U_{\max}(0)\left[1 - \left(\frac{r}{r_0}\right)^2\right]\left(1 - \frac{4}{\sqrt{\pi}}\sqrt{\tau}\right),\\
D_l^u &= \frac{1}{f_l^u} = \mp\frac{4}{r_0^2}U_{\max}(0)\left(1 - \frac{4}{\sqrt{\pi}}\sqrt{\tau}\right),
\end{aligned}
$$

where the superscript u and the upper sign refer to the upper surface of the plate, and the subscript l and the lower sign – to the lower surface,

$$U_{\max}(0) := 3\alpha\,\frac{r_0^2}{(2h)^2}\,\frac{Q_{\text{tot}}}{\pi\, r_0^2\, \varrho_0 c_p}.$$

The deflection angle

$$
(7.6) \qquad
\begin{aligned}
\varepsilon_l^u &= \mp 2\arctan\left[2\,\frac{U_{\max}(0)}{r_0}\,\frac{r}{r_0}\left(1 - \frac{4}{\sqrt{\pi}}\sqrt{\tau}\right)\right]\\
&\cong \mp 4\,\frac{U_{\max}(0)}{r_0}\,\frac{r}{r_0}\left(1 - \frac{4}{\sqrt{\pi}}\sqrt{\tau}\right)
\end{aligned}
$$

is a linear function of $\sqrt{\tau}$ only in the paraxial optics approximation (the approximated part of Eq. (7.6)), which holds (with an accuracy to O^*), if (cf. Ineq. (2.2))

$$\frac{4}{r_0^2}U_{\max}^2(0)\left(\frac{r}{r_0}\right)^2\left(1 - \frac{4}{\sqrt{\pi}}\sqrt{\tau}\right)^2 \le \frac{3O^*}{1 + O^*} \cong 3O^*$$

(the total relative error of this triple approximation does not exceed $(1 + O^*)^3 - 1 \cong 3O^*$).

Thus, the short-time approximation seems to be realistic (except for very thin plates with the best temperature conductors) and offering simple interpretation of the time evolution of the optical properties of the mirror considered.

8. Long-time regime

Although the short-time regime, discussed in the previous section, seems to be sufficient for use and interpretation of the results obtained earlier, we will discuss shortly the opposite regime – the long-time one for the completeness of the picture. For this purpose it is more convenient to use the second version of the solution of the thermal problem (Eq. $(3.2)_2$), and therefore – also the second version of the function M_T (Eq. $(4.1)_3$).

The idea of the long-time approximation is similar to that used previously in the case of the short-time approximation. We have to find criteria, which allow us to simplify the expression for the function M_T as far as possible (the assumption, that the functions δ can be ignored, will also be used).

For sufficiently long time, the series in Eq. $(4.1)_3$ can be approximated by its first term only with an accuracy to an assumed small number O^*. For this purpose it is sufficient to require:

• the second term of the series to be much smaller than the first one in the following sense:

$$\frac{1}{9}\exp[-8\pi^2\tau] \le 0.9\, O^*,$$

• and the $(k+1)$-th term, $k \ge 2$, to be not larger than 0.1 of the k-th term:

$$\exp[-8k\pi^2\tau] \le 0.1\left(\frac{2k+1}{2k-1}\right)^2.$$

These inequalities are satisfied if, respectively:

$$\tau \ge \frac{1}{8\pi^2}[-\ln 8.1\, O^*],$$

$$\tau \ge \frac{1}{8\pi^2 k}\ln\left[10\left(\frac{2k-1}{2k+1}\right)^2\right].$$

The latter inequality is the strongest one for $k = 2$, therefore we have:

$$\tau \ge \tau_{2/1} := \frac{1}{8\pi^2}[-\ln 8.1\, O^*],$$

$$\tau \ge \tau_{3/2} := \frac{1}{16\pi^2}\ln 3.6 \cong 8.1\cdot 10^{-3}.$$

Because $\tau_{2/1} \cong \tau_{3/2}$ for $O^* \cong 6.5\cdot 10^{-2}$, therefore for $O^* < 6.5\cdot 10^{-2}$ the first of these two conditions is stronger than the second one, and inversely for $O^* > 6.5\cdot 10^{-2}$.

Assuming (as previously) $O^* = 0.01$ we have([9]):

$$\tau \ge \tau_{\text{long}} = \tau_{2/1} \cong 3.2\cdot 10^{-2}; \tag{8.1}$$

([9]) For $O^* = 10^{-3}$ or 10^{-4} the number 3.2 in Ineqs. (8.1) and (8.2) is replaced by the number 6.1 or 9.0, respectively.

assuming also (as previously) $\kappa \doteq (10^{-7} - 10^{-4})$ m^2/s, we rewrite criterion expressed by Ineq. (8.1) in the dimensional form:

$$t \geq t_{\text{long}} = t_{2/1} \doteq \begin{cases} 3.2\cdot(10^{-1} - 10^{-4}) \text{ s}, & \text{for} \quad 2h \doteq 10^{-3}\,\text{m}, \\ 3.2\cdot(10 - 10^{-2}) \text{ s}, & \text{for} \quad 2h \doteq 10^{-2}\,\text{m}. \end{cases} \tag{8.2}$$

Let us note by the way, comparing Ineqs. (8.1) [(8.2)] and (7.4), that the latter one is always fulfilled in the long-time regime.

If the criterion expressed by Ineq. (8.1) (or (8.2)) is satisfied, then the series in Eq. $(4.1)_3$ can be approximated by its first term only, which for $\tau = \tau_{2/1} \cong 3.2\cdot 10^{-2}$ is equal to about 0.59, whereas the whole series for $\tau = 0$ is equal to unity (see [4]).

Thus, if the criteria expressed by Ineqs. (8.1) (or (8.2)) and (6.1) are satisfied, then the series in Eq. $(4.1)_3$ reduces to the first term, and the functions δ in Eqs. (5.1) and (5.3) are neglected (the total relative error of this double approximation does not exceed $(1 + O^*)^2 - 1 \cong 2O^*$). Then the quantity $U_{\max}$, and therefore also the functions $U^u - U^u(r = 0)$, U_l, D and f – depend on time exponentially:

$$\begin{aligned} U_{\max} &= U_{\max}(0)\,\frac{8}{\pi^2}\exp[-\pi^2\tau], \\ U^u &= U_{\max}(0)\left\{\frac{1}{3}\left(\frac{2h}{r_0}\right)^2 + \left[1 - \left(\frac{r}{r_0}\right)^2\right]\frac{8}{\pi^2}\exp[-\pi^2\tau]\right\}, \\ U_l &= -U_{\max}(0)\left[1 - \left(\frac{r}{r_0}\right)^2\right]\frac{8}{\pi^2}\exp[-\pi^2\tau], \\ D_l^u &= \frac{1}{f_l^u} = \mp\frac{4}{r_0^2}\,U_{\max}(0)\,\frac{8}{\pi^2}\exp[-\pi^2\tau], \end{aligned} \tag{8.3}$$

where the superscript u and the upper sign refer to the upper surface of the plate, and the subscript l and the lower sign – to the lower surface, and $U_{\max}(0)$ is defined by the equation following Eqs. (7.5). The deflection angle

$$\begin{aligned} \varepsilon_l^u = \mp 2 \arctan\left[\frac{16}{\pi^2}\,\frac{U_{\max}(0)}{r_0}\,\frac{r}{r_0}\exp[-\pi^2\tau]\right] \\ \cong \mp\frac{32}{\pi^2}\,\frac{U_{\max}(0)}{r_0}\,\frac{r}{r_0}\exp[-\pi^2\tau] \end{aligned} \tag{8.4}$$

depends exponentially on time only in the paraxial optics approximation (the approximated part of Eq. (8.4)), which holds (with an accuracy to O^*), if (cf. Ineq. (2.2))

$$\frac{256}{\pi^4}\,\frac{U_{\max}^2(0)}{r_0^2}\left(\frac{r}{r_0}\right)^2\exp[-2\pi^2\tau] \leq \frac{3O^*}{1 + O^*} \cong 3O^*$$

(the total relative error of this triple approximation does not exceed $(1 + O^*)^3 - 1 \cong 3O^*$).

Let us note by the way, that τ can not be too large. The plate is assumed to be adiabatically insulated on all its surfaces. This assumption can be violated, after sufficiently long time, at least by the radiation heat exchange between the plate and its surroundings. The (dimensionless) relaxation time for the latter process $\tau_{\rm rad}$ (in the time scale applied in the paper) may be estimated as follows. We start from the heat conduction equation with no heat sources, assuming the boundary conditions in the form (see Footnote 3):

$$\frac{\partial \Theta}{\partial \zeta}\left(\zeta = \frac{1}{2}\right) = -\beta_1 \Theta\left(\zeta = \frac{1}{2}\right), \qquad \frac{\partial \Theta}{\partial \zeta}\left(\zeta = -\frac{1}{2}\right) = \beta_2 \Theta\left(\zeta = -\frac{1}{2}\right),$$

where β_1, β_2 stand for (dimensionless) coefficients of surface losses (assumed to be constants). The solution of the heat conduction equation with these boundary conditions (as obtained using the Fourier method of separating the independent variables) has the form:

$$\Theta = \sum_{k=1}^{\infty} \exp[-\mu_k^2 \tau] \left[A_k \cos \mu_k \left(\zeta + \frac{1}{2}\right) + B_k \sin \mu_k \left(\zeta + \frac{1}{2}\right)\right],$$

where $\beta_2 A_k = \mu_k B_k$, coefficients A_k (or B_k) are (in principle) determinable from an initial condition, and μ_k stands for positive solutions of the following characteristic equation:

$$\tan \mu = \frac{\mu(\beta_1 + \beta_2)}{\mu^2 - \beta_1 \beta_2}.$$

For small surface losses (β_1, $\beta_2 \ll 1$) one may obtain (in the linear approximation):

$$\mu_k \cong k\pi + \frac{1}{k\pi}(\beta_1 + \beta_2),$$

therefore:

$$\exp[-\mu_k^2 \tau] \cong \exp[-k^2\pi^2 \tau] \exp[-2(\beta_1 + \beta_2)\tau].$$

Thus, the (dimensionless) relaxation time connected with the surface losses is

$$\tau_{\rm rel} \cong \frac{1}{2(\beta_1 + \beta_2)}.$$

If the plate loses its energy through its surfaces by thermal radiation only, then using the linearized Stefan–Boltzmann law one may write:

$$\beta_1 = \beta_2 = \frac{2h}{\kappa} \frac{4b\,\sigma_{SB} T_0^3}{\varrho_0 c_p},$$

where σ_{SB} stands for the Stefan - Boltzmann constant, b – for a correction factor for a real body as compared with the perfectly black one, and T_0 – for the initial temperature (before the perturbation); thus,

$$\tau_{\mathrm{rad}} = \frac{\kappa}{2h}\,\frac{\varrho_0 c_p}{8b\,\sigma_{SB}\,T_0^3}\,.$$

The thermal radiation losses can be therefore neglected, if the observation time τ is much shorter than τ_{rad}:

$$\tau \le \tau_{\max} := O^*\,\tau_{\mathrm{rad}} = O^*\,\frac{\kappa}{2h}\,\frac{\varrho_0 c_p}{8b\,\sigma_{SB}\,T_0^3}\,,$$

where O^* stands for an assumed small number.

Assuming (in addition to the assumptions of this kind adopted previously):

- $b \doteq 0.1$,
- $\sigma_{SB} \cong 5.67\cdot 10^{-8}\,\mathrm{J/(m^2 s\,K^4)}$,
- $T_0 \doteq 3\cdot 10^2\,\mathrm{K}$,

we have (in dimensionless and in dimensional forms):

$$\tau \le \tau_{\max} \doteq \begin{cases} 4\cdot(1-10^3), & \text{for}\quad 2h \doteq 10^{-3}\,\mathrm{m},\\ 4\cdot(10^{-1}-10^2), & \text{for}\quad 2h \doteq 10^{-2}\,\mathrm{m},\end{cases}$$

$$t \le t_{\max} \doteq \begin{cases} 40\,\mathrm{s}, & \text{for}\quad 2h \doteq 10^{-3}\,\mathrm{m},\\ 4\cdot 10^2\,\mathrm{s}, & \text{for}\quad 2h \doteq 10^{-2}\,\mathrm{m}.\end{cases} \tag{8.5}$$

This criterion restricts the applicability of the theory presented, however there still remains a relatively large field for application of the long-time regime (as it is seen by comparison of Ineqs. (8.5) with (8.1) [(8.2)]). Thus, the long-time regime seems to be a realistic and useful supplement to the short-time regime([10]). It starts relatively quickly. The values of $U_{\max}$ at the beginning of this regime are only a dozen percent lower than the initial value of $U_{\max}$. By comparing Ineqs. (8.1) and (7.1) one may see, that for $O^* = 0.01$ both regimes – the short- and the long-time ones – cover the full time range from $\tau_{\min}$ to $\tau_{\max}$ (for smaller O^* the situation is not so comfortable – see Footnotes 7 and 9).

9. Estimations for possible experiments

9.1. Introductory remarks

In principle, the thermal mirror considered may be experimentally studied by investigating the functions: U, ε and f. Each of these quantities can be experimentally investigated and interpreted using the theoretical scheme presented, if some conditions are fulfilled.

([10]) Supplement only, because of the restriction mentioned in Subsec. 9.2 (see also estimations given in Subsecs. 9.3, 9.4, 9.5, and cf. Ineq. (9.1) and Ineqs. (9.3), (9.4), (9.5)).

9.2. General conditions

Some general conditions, which should be taken into account in any experiment, were discussed earlier. Here the last such a condition will be mentioned. It follows from the requirement that the heat perturbation can not significantly change the properties of the material. Assuming the perturbation region to be a layer of thickness Δh, and the temperature not to exceed some critical value T^*, we can write this requirement in the form:

$$Q_{\text{tot}} \leq Q_{\max} := \varrho_0 c_p T^* \Delta h\, \pi r_0^2 .$$

Assuming (in addition to the assumptions of this kind adopted previously):

- $T^* \doteq 2.10^2\,\mathrm{K}$,
- $\Delta h \doteq 0.05 \cdot (2h)$,

we have:

$$\text{(9.1)} \qquad \begin{aligned} Q_{\text{tot}} \;\leq\; Q_{\max} &\doteq \begin{cases} 15\,\mathrm{J}, & \text{for} \quad 2h \doteq 10^{-3}\,\mathrm{m}, \\ 1.5 \cdot 10^4\,\mathrm{J}, & \text{for} \quad 2h \doteq 10^{-2}\,\mathrm{m}, \end{cases} \\ \frac{Q_{\text{tot}}}{\pi r_0^2} \leq \frac{Q_{\max}}{\pi r_0^2} &\doteq \begin{cases} 5 \cdot 10^4\,\mathrm{J/m^2}, & \text{for} \quad 2h \doteq 10^{-3}\,\mathrm{m}, \\ 5 \cdot 10^5\,\mathrm{J/m^2}, & \text{for} \quad 2h \doteq 10^{-2}\,\mathrm{m}. \end{cases} \end{aligned}$$

Comparing the conditions expressed by Ineqs. (9.1) and (6.2) one may see, that the latter is weaker than the former one, i.e. if Ineq. (9.1) is satisfied, then the functions δ can be neglected in all the previous formulae.

9.3. Observability conditions for U

According to Eqs. (7.5) and (8.3) (for the short- and the long-time regimes, respectively), the condition for the minimum pulse energy Q_{tot} allowing U to be observable on the level at least of U^* can be written in the form:

$$Q_{\text{tot}} \geq Q^U_{\min} := U^* \frac{\varrho_0 c_p}{\alpha} (2h)^2 \frac{\pi}{3} \frac{1}{1 - \left(\dfrac{r}{r_0}\right)^2} \psi(\tau),$$

where

$$\text{(9.2)} \qquad \psi(\tau) \cong \begin{cases} \left[1 - \dfrac{4}{\sqrt{\pi}} \sqrt{\tau}\right]^{-1} & \text{in the short-time regime}, \\ \dfrac{\pi^2}{8} \exp[\pi^2 \tau] & \text{in the long-time regime}, \end{cases}$$

and the contribution of N_T to U^u was neglected.

Assuming (in addition to the assumptions of this kind adopted previously):

- $U^* \doteq 10^{-6}\,\mathrm{m}$,
- $r \ll r_0$,
- $\tau \doteq 7 \cdot 10^{-2}$ (see Ineqs. (7.1) and (8.1))

we have:

$$
\begin{aligned}
&Q_{\mathrm{tot}} \geq Q^U_{\min} \doteq \begin{cases} 1\ \mathrm{J}, & \text{for} \quad 2h \doteq 10^{-3}\,\mathrm{m}, \\ 10^2\ \mathrm{J}, & \text{for} \quad 2h \doteq 10^{-2}\,\mathrm{m}, \end{cases} \\
&\frac{Q_{\mathrm{tot}}}{\pi r_0^2} \geq \frac{Q^U_{\min}}{\pi r_0^2} \doteq 4 \cdot 10^3\,\mathrm{J/m^2}
\end{aligned}
\tag{9.3}
$$

(cf. Ineqs. (9.1), (9.4) and (9.5)).

9.4. Observability conditions for ε

According to Eqs. (7.6) and (8.4) (for the short- and the long-time regimes, respectively) the condition for the minimum pulse energy Q_{tot} allowing ε to be observable on the level at least of ε^* can be written in the form:

$$
Q_{\mathrm{tot}} \geq Q^{\varepsilon}_{\min} := \varepsilon^* \, \frac{\varrho_0 c_p}{\alpha} \, (2h)^2 \, \frac{1}{12} \, \frac{\pi r_0^2}{r} \, \psi(\tau),
$$

where $\psi(\tau)$ is given by Eq. (9.2).

Assuming (in addition to the assumptions of this kind adopted previously):

- $\varepsilon^* \doteq 10^{-4}\,\mathrm{rad}$,
- $r \cong r_0$

we have:

$$
\begin{aligned}
&Q_{\mathrm{tot}} \geq Q^{\varepsilon}_{\min} \doteq \begin{cases} 3 \cdot 10^{-1}\,\mathrm{J}, & \text{for} \quad 2h \doteq 10^{-3}\,\mathrm{m}, \\ 3 \cdot 10^{2}\,\mathrm{J}, & \text{for} \quad 2h \doteq 10^{-2}\,\mathrm{m}, \end{cases} \\
&\frac{Q_{\mathrm{tot}}}{\pi r_0^2} \geq \frac{Q^{\varepsilon}_{\min}}{\pi r_0^2} \doteq \begin{cases} 10^3\,\mathrm{J/m^2}, & \text{for} \quad 2h \doteq 10^{-3}\,\mathrm{m}, \\ 10^4\,\mathrm{J/m^2}, & \text{for} \quad 2h \doteq 10^{-2}\,\mathrm{m} \end{cases}
\end{aligned}
\tag{9.4}
$$

(cf. Ineqs. (9.1), (9.3) and (9.5)).

9.5. Observability conditions for f

According to Eqs. (7.5) and (8.3) (for the short- and the long-time regimes, respectively), the minimum pulse energy Q_{tot} allowing f to be observable on the level not higher than f^* can be written in the form:

$$
Q_{\mathrm{tot}} \geq Q^{f}_{\min} := \frac{1}{f^*} \, \frac{\varrho_0 c_p}{\alpha} \, (2h)^2 \, \frac{1}{12} \, \pi r_0^2 \, \psi(\tau),
$$

where the function $\psi(\tau)$ is given by Eq. (9.2).

Assuming (in addition to the assumptions of this kind adopted previously)

- $f^* \doteq 40\,\mathrm{m}$

we have:

$$
(9.5)\qquad
\begin{aligned}
&Q_{\mathrm{tot}} \geq Q^f_{\min} \doteq \begin{cases} 0.8\,\mathrm{J}, & \text{for } 2h \doteq 10^{-3}\,\mathrm{m},\\ 8\cdot 10^3\,\mathrm{J}, & \text{for } 2h \doteq 10^{-2}\,\mathrm{m},\end{cases}\\
&\frac{Q_{\mathrm{tot}}}{\pi r_0^2} \geq \frac{Q^f_{\min}}{\pi r_0^2} \doteq \begin{cases} 3\cdot 10^3\,\mathrm{J/m^2}, & \text{for } 2h \doteq 10^{-3}\,\mathrm{m},\\ 3\cdot 10^5\,\mathrm{J/m^2}, & \text{for } 2h \doteq 10^{-2}\,\mathrm{m}\end{cases}
\end{aligned}
$$

(cf. Ineqs. (9.1), (9.3) and (9.4)).

10. Possible applications for determining the temperature conductivity (and the surface losses coefficients)

As it is seen from the suitable formulae given above (after coming back to dimensional time $t = \tau\,(2h)^2/\kappa$), the time evolution of the thermal mirror depends, among others, on temperature conductivity κ of the material. Measuring suitable properties of the mirror it is therefore possible to determine κ. However, as it is seen from the formulae mentioned, such a procedure performed in arbitrary conditions may require some additional information (which should be known or measured), and may prove to be complicated for interpretation.

The problem simplifies in the short-time and the long-time regimes. In fact, as it follows from Eqs. (7.5) and (7.6), in the short-time regime the quantities: U, $\tan(\varepsilon/2)$, and f are linear functions of $\sqrt{t}$ with coefficient (at $\sqrt{t}$) equal to $4\sqrt{\kappa}/(2h\sqrt{\pi})$. Measuring the evolution of these quantities one may therefore determine this coefficient and, knowing it and the plate thickness $2h$ of the plate – find κ of a given material.

Analogously, as it follows from Eqs. (8.3) and (8.4), logarithms of the following quantities: $U^u(r=0) - U^u(r)$, $|U_l|$, $|\tan(\varepsilon/2)|$ and $|f|$ in the long-time regime are linear functions of time t with the coefficient (at t) equal to $\pi^2\kappa/(2h)^2$. Measuring the evolution of these quantities one may therefore determine this coefficient, and knowing it and the plate thickness $2h$ – determine κ of a given material.

By the way let us note shortly, that one may think also on applying the thermal mirrors considered for experimental determining the surface losses coefficients β_1, or β_2 (see the end of Sec.8), if the temperature conductivity κ of a given material is known. Using equations given at the end of Sec. 8 for Θ and suitable equations for the optical characteristics of the mirror, and applying the same argumentation as it was used for specification the long-time regime, one may conclude that for sufficiently long time the suitable optical characteristics F of the mirror are simply proportional to $\exp[-\mu_1^2\,\tau]$. From measurements of the time evolution of $\ln|F|$ one may therefore determine the quantity μ_1. Then from the characteristic equation for μ one may determine: $\beta_2 = \mu_1 \tan\mu_1$, if $\beta_1 = 0$ (an ideal thermal insulation on the perturbed surface); $\beta_1 = \mu_1 \tan\mu_1$, if $\beta_2 = 0$ (an ideal thermal insulation on the opposite surface); $\beta_1 = -\mu_1/\tan\mu_1$, if $\beta_2 = \infty$ (ideal losses on the opposite surface, realized for instance by a thermostate).

11. Remark on distortion of properties of optical mirrors

Absorption of light by mirrors in high power optical systems causes thermal deformation of the mirrors, and therefore changes their optical properties. The theory presented may be useful for estimations of such effects in light-pulse optical systems. In particular, the criteria given in Subsecs. 9.3, 9.4 and 9.5 may be useful (in reversed form) for estimation of the maximum allowable energy of light pulse, which do not distort optical properties of the mirrors over an assumed level.

12. Conclusions

The thermal mirrors created on the surfaces of a thin plate of an isotropic thermoelastic solid material by a heat pulse, which is applied to one of the plate surface and is homogeneous across this surface, is – within the approximations applied in the paper – an ideal (aberration-free) optical mirror. These mirror effects are relatively very small, however they may be studied experimentally using high precision optics. The variations of the optical properties of the mirror considered are comparable with those of the half-space thermal mirror [5], however, because the thin-plate thermal mirror is free of aberrations, therefore it seems to be easier for experimental research.

In general, the time dependence of the thin-plate thermal mirror is complicated. However, there exist two regimes: the short-time and the long-time ones, in which this dependence becomes much simpler and easy for interpretation. In these conditions the thermal mirror considered may be, in principle, used for experimental determination of the temperature conductivity of a material.

Appendix. Detailed criteria for neglecting the functions δ

A.1. Criterion for neglecting δ^u in the formula for U^u

The relative error of neglecting the function δ^u in Eq. (4.2)$_1$ does not exceed O^*, if the following criterion is satisfied:

$$\left[\left(\frac{r}{r_0}\right)^2 - O^*\left(1+\frac{N_T}{E\,U_{\max}}\right)\right]\delta^u(2+\delta^u) \le O^*\left[\left(1+\frac{N_T}{E\,U_{\max}}\right) - \left(\frac{r}{r_0}\right)^2\right]$$

or

$$\left(\frac{r}{r_0}\right)^2\left[O^* + \delta^u(2+\delta^u)\right] \le O^*\left(1+\delta^u\right)^2\left(1+\frac{N_T}{E\,U_{\max}}\right).$$

Three cases should be considered to analyze this criterion. If

$$\left(\frac{r}{r_0}\right)^2 \le O^* \left(1 + \frac{N_T}{E\,U_{\max}}\right),$$

then the criterion considered is always satisfied for an arbitrary δ^u, i.e. – for sufficiently small r the function δ^u can be always neglected in Eq. $(4.2)_1$.

If

$$O^* \left(1 + \frac{N_T}{E\,U_{\max}}\right) < \left(\frac{r}{r_0}\right)^2 < \left(1 + \frac{N_T}{E\,U_{\max}}\right),$$

then the criterion considered is satisfied for

$$\delta^u < \left(\frac{r}{r_0}\right) \frac{\sqrt{1 - O^*}}{\sqrt{\left(\frac{r}{r_0}\right)^2 - O^* \left(1 + \frac{N_T}{E\,U_{\max}}\right)}} - 1$$

or

$$\left(\frac{r}{r_0}\right)^2 < \frac{O^* (1 + \delta^u)^2 \left(1 + \frac{N_T}{E\,U_{\max}}\right)}{O^* + \delta^u (2 + \delta^u)} .$$

If, in particular,

$$\delta^u < \frac{1}{2} O^* ,$$

then the latter inequality is satisfied for

$$\left(\frac{r}{r_0}\right)^2 \le \frac{1}{2} .$$

If

$$\left(\frac{r}{r_0}\right)^2 \ge \left(1 + \frac{N_T}{E\,U_{\max}}\right),$$

then there exists no function δ^u satisfying the criterion considered, i.e. – for sufficiently large r the function δ^u can not be neglected in Eq. $(4.2)_1$ (however, this case may have only symbolical meaning, because of the approximation applied for solving the thermoelasticity equation, as it was mentioned at the beginning of Sec. 4).

A.2. Criterion for neglecting δ_l in the formula for U_l

Because the function δ_l decreases from $2\alpha T_\infty$ to $-\alpha T_\infty$ as τ varies from 0 to ∞ (see Sec. 6), therefore the criterion for neglecting the function δ_l in the formula for U_l should be examined separately for $\delta_l \ge 0$ and $\delta_l \le 0$.

A.2.1. The case of $\delta_l \geq 0$. The relative error of neglecting the function $\delta_l \geq 0$ in Eq. $(4.2)_2$ does not exceed O^*, if the following criterion is satisfied.

$$\left[\left(\frac{r}{r_0}\right)^2 + O^*\right]\delta_l(2-\delta_l) \leq O^*\left[1-\left(\frac{r}{r_0}\right)^2\right]$$

or

$$\left(\frac{r}{r_0}\right)^2\left[O^* + \delta_l(2-\delta_l)\right] \leq O^*\,(1-\delta_l)^2 .$$

If $r = 0$, then this criterion is satisfied for an arbitrary δ_l.

If $r \neq 0$, then the criterion considered is satisfied for([11])

$$\delta_l \leq 1 - \frac{r}{r_0}\frac{\sqrt{1+O^*}}{\sqrt{\left(\dfrac{r}{r_0}\right)^2 + O^*}}$$

or

$$\left(\frac{r}{r_0}\right)^2 \leq \frac{O^*\,(1-\delta_l)^2}{O^* + \delta_l(2-\delta_l)} .$$

If, in particular,

$$\delta_l \leq \frac{1}{2}O^* ,$$

then the latter inequality is satisfied for

$$\left(\frac{r}{r_0}\right)^2 \leq \frac{1}{2}\frac{\left(1-\dfrac{1}{2}O^*\right)^2}{1-\dfrac{1}{8}O^*} \cong \frac{1}{2} ,$$

(exactly: for $O^* = 0.01,\ 0.001,\ 0.0001$ the double right-hand side of this inequality is equal to 0.99126, 0.999125, 0.9999125, respectively).

A.2.2. The case of $\delta_l \leq 0$. The discussion and the conclusion in this case are exactly the same as in the case examined in Subsec. A.1 with $N_T = 0$ and δ^u replaced by $|\delta_l|$.

A.3. Criterion for neglecting δ^u in the formula for ε^u

The relative error of neglecting the function δ^u in Eq. (5.1) for ε^u does not exceed O^*, if the following criterion is satisfied:

$$\arctan\left[\frac{2U_{\max}}{r_0}\frac{r}{r_0}\right] \leq (1+O^*)\arctan\left[\frac{2U_{\max}}{r_0}\frac{r}{r_0}\frac{1}{(1+\delta^u)^2}\right] .$$

([11]) This is a very fair condition in case of small r. If, for instance, $r = 0.1r_0$ and $O^* = 0.01$, then this inequality reads: $\delta \leq 0.2893$ (see Sec. 6 and cf. Sec.A.1).

Because x arctan $y \leq$ arctan xy for $x \leq 1$, $y \ll 1$, therefore this criterion may be replaced by the following stronger one:

$$(\delta^u)^2 + 2\delta^u - O^* \leq 0\,,$$

which is satisfied for

$$\delta^u \leq \sqrt{1+O^*} - 1 \cong \frac{1}{2}\,O^*$$

(exactly: for $O^* = 0.01$, 0.001, 0.00001 the double right-hand side of this inequality divided by O^* is 0.9975, 0.99975, 0.99997, respectively).

A.4. Criterion for neglecting δ_l in the formula for ε_l

The relative error of neglecting the function $\delta_l \geq 0$ in Eq. (5.1) for ε_l does not exceed O^*, if the following criterion is satisfied:

$$\arctan\left[\frac{2U_{\max}}{r_0}\,\frac{r}{r_0}\right] \geq (1 - O^*)\,\arctan\left[\frac{2U_{\max}}{r_0}\,\frac{r}{r_0}\,\frac{1}{(1-\delta_l)^2}\right].$$

Because x arctan $y \geq$ arctan xy for $x \geq 1$, $y \ll 1$, therefore this criterion may be replaced by the following stronger one:

$$(\delta^u)^2 - 2\delta^u + O^* \geq 0,$$

which is satisfied for

$$\delta^u \leq \frac{1}{2}\,O^* \leq 1 - \sqrt{1-O^*}\,.$$

The discussion and the conclusion in the case of $\delta_l < 0$ are exactly the same as in the case examined in Subsec. A.3 with only δ^u replaced by $|\delta_l|$.

A.5. Criteria for neglecting δ in the formulae for $D = 1/f$

The relative error of neglecting the functions δ in Eqs. (5.3) does not exceed O^*, if the following criteria are satisfied:

$$\begin{aligned}
&\delta^2 + 2\delta - O^* \leq 0, \qquad && \delta = \delta^u, \qquad -\delta_l > 0,\\
&\delta_l^2 - 2\delta_l + O^* \geq 0, \qquad && \delta_l > 0,
\end{aligned}$$

for the upper surface and the lower one, respectively. These inequalities are satisfied for

$$\begin{aligned}
&\delta \leq \sqrt{1+O^*} - 1 \cong \frac{1}{2}O^*, \qquad && \delta = \delta^u, \qquad -\delta_l > 0,\\
&\delta_l \leq \frac{1}{2}\,O^* \leq 1 - \sqrt{1-O^*}, \qquad && \delta_l > 0
\end{aligned}$$

(see and cf. Subsecs. A.3, A.4).

A.6. Conclusion

The criteria for neglecting the functions δ in the suitable formulae are different in various cases. In order to discuss this problem in a uniform way for all the cases, one needs a common criterion, which will be satisfied in all the cases. Such a criterion is proposed in Sec. 6 (see Ineq. (6.1)).

Acknowledgment

The Authors would like to express thanks to Prof. Dr. M. SOKOŁOWSKI and his coworkers, Institute of Fundamental Technological Research, Warszawa, for fruitful and helpful discussions.

References

1. B.A. BOLEY and J.H. WEINER, *Theory of thermal stresses*, J. Wiley, New York-London 1960.
2. M. BORN and E. WOLF, *Principles of optics*, Pergamon Press, Oxford-London 1968.
3. H.S. CARSLAW and J.C. JAEGER, *Conduction of heat in solids*, 2nd ed., Clarendon Press, Oxford, 1989; see also: A.V. LUIKOV, *Analytical heat diffusion theory* [transl. from Russian], Academic Press, New York-London 1968.
4. I.S. GRADSHTEYN and I.M. RYZHIK, *Tables of integrals, series and products* [transl. from Russian], Academic Press, New York-San Francisco-London 1965.
5. Z. PŁOCHOCKI, A. MIODUCHOWSKI and J.F. MĄCZYŃSKI, *An idea of half-space thermal mirror*, ZAMP, **46**, 339–354, 1995.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH
ul. Świętokrzyska 21, 00-049 WARSZAWA, POLSKA
and
UNIVERSITY OF ALBERTA
DEPARTMENT OF MECHANICAL ENGINEERING, EDMONTON, CANADA
e-mail: mioducho@sneezy.mece.ualberta.ca

Received July 5, 1996.