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Uniqueness in nonlinear theory of porous elastic materials

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THIS NOTE is concerned with static deformations in a nonlinear theory of elastic materials with voids. First we extend some conservation laws to the nonlinear theory. A uniqueness result is presented under a condition related to the quasi-convexity assumptions.

1. Introduction

IN [1], KNOPS AND STUART proved the uniqueness of the solutions to certain displacement boundary-value problems in the context of the nonlinear theory of homogeneous hyperelasticity for a body occupying a star-shaped bounded region. Recently, this result has been extended to the theory of interacting continua [22]. In this paper we extend some of these results to the theory of nonlinear elastic materials with voids.

The theory of elastic materials with voids is a recent extension of the classical theory of elasticity. The nonlinear theory has been established by NUNZIATO and COWIN [2]. In this theory the bulk density is the product of two scalar fields, the matrix material density and the volume fraction field. An intensive work on this kind of materials is developing currently [3–9]. An extensive review on elastic materials with voids has been presented in [10].

Existence and uniqueness results in the statical linear theory of an elastic material with voids have been presented [10, 11]; meanwhile many other theorems have been presented for the dynamic case [12–14], and in [15] for the dynamical nonlinear problem. We remark that in [10] CIARLETTA and IEŞAN have obtained a uniqueness and existence theorem for the static equations of porous elastic materials, but the authors noted that their results apply the one-dimensional case only.

We consider the homogeneous deformation $(\mathbf{x}, \nu) : \mathbf{X} \to (\mathbf{M}\,\mathbf{X} + \mathbf{b}, \nu_0)$, where $\mathbf{M}$ is a fixed regular square matrix such that $\det(\mathbf{M}) > 0$, $\mathbf{b}$ is a fixed vector, $0 < \nu_0 \leq 1$ is a constant number and $\mathbf{X}$ represents the material point. We suppose that this deformation is a solution to the equilibrium problem with boundary conditions $(\mathbf{x}, \nu) : \mathbf{X} \to (\mathbf{M}\,\mathbf{X} + \mathbf{b}, \nu_0)$. For star-shaped elastic materials we will prove, under suitable assumptions concerning the energy function Σ, that there is no other solution satisfying these boundary conditions.

The method follows the ideas of [1]. We first extend a conservation law established by GREEN [16] in the case of hyperelasticity.

Following the method used in [1], we impose start with a basic assumption on the energy to obtain our result. We suppose that the energy satisfies a condition related with the quasi-convexity, an assumption introduced by MORREY [17] and employed in the classical works of BALL [18–20]. Nevertheless, in this paper we are not concerned with the problem of existence of solutions.

In Sec. 2 we state the basic equations and the assumptions. We also extend some conservation laws to the nonlinear theory of elastic materials with voids. The uniqueness result is presented in Sec. 3.

2. Preliminaries

We consider a body which occupies a bounded regular region B of the Euclidean n-dimensional space with the boundary surface ∂B. We assume that B is star-shaped and that ∂B is sufficiently regular to ensure the validity of the usual laws of transformation of surface integrals.

Throughout this paper we employ the usual summation and differentiation conventions: subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate; ∇ is the gradient operator with respect to the position $\mathbf{X}$. We let N_A be the components of the outward unit normal to ∂B and denote the scalar product of two tensors by an interposed dot. By $\mathbf{a} \otimes \mathbf{b}$ we denote the tensor product of the vectors $\mathbf{a}$ and $\mathbf{b}$.

We assume that B is occupied by an elastic material with voids. A deformation in B is described by the spatial position field $\mathbf{x}$ and the volume fraction field ν. The deformations determine the deformation gradient $\mathbf{F} = \nabla\mathbf{x}$, and the gradient of the volume fraction $\mathbf{G} = \nabla\nu$. By $\mathcal{M}^+$ we denote the set of all real square matrices $\mathbf{F}$ of order n such that $\det(\mathbf{F}) > 0$. As usual, we suppose that $\mathbf{F} \in \mathcal{M}^+$ and $0 < \nu \leq 1$ for all deformations.

We also assume that the material possesses internal energy Σ per unit initial volume. We denote by $\mathbf{T}$ the first Piola–Kirchhoff stress tensor, $\mathbf{S}$ the equilibrated stress and by g the equilibrated body force per unit volume. In what follows, occasionally it will be convenient to write various expressions in component form and to represent the vector and tensor fields by their components referred to the considered system of Cartesian axes. Thus, the components of the deformation $(\mathbf{x}, \nu)$ will be denoted by (x_i, ν), while the components of the deformation gradient fields $\mathbf{F}$ and $\mathbf{G}$ will be denoted by F_{iA} and G_A, respectively.

A deformation $(\mathbf{x}, \nu)$ in B, defined for all $\mathbf{X}$ in B, is a smooth equilibrium solution provided $x_i, \nu \in C^2(B, R^n) \cap C^1(\overline{B}, R^n)$ and the equilibrium equations (see, e.g. [10])

$$
\begin{aligned}
T_{Ai,A} &= 0, \\
S_{A,A} + g &= 0,
\end{aligned}
\tag{1}
$$

are satisfied.

The material at the point $\mathbf{X}$ is characterized by the constitutive relations

$$(2)\qquad \begin{aligned} \Sigma &= \Sigma^*(\mathbf{F},\mathbf{G},\nu), &\qquad \mathbf{T} &= \mathbf{T}^*(\mathbf{F},\mathbf{G},\nu),\\ \mathbf{S} &= \mathbf{S}^*(\mathbf{F},\mathbf{G},\nu), & g &= g^*(\mathbf{F},\mathbf{G},\nu), \end{aligned}$$

where Σ^*, $\mathbf{T}^*$, $\mathbf{S}^*$, g^* are smooth functions.

We suppose that the Piola–Kirchhoff stress tensor, the equilibrated stress and equilibrated body force are related to the energy in the following manner:

$$(3)\qquad \mathbf{T} = \left(\frac{\partial\Sigma}{\partial\mathbf{F}}\right)^{\top}, \qquad \mathbf{S} = \left(\frac{\partial\Sigma}{\partial\mathbf{G}}\right)^{\top}, \qquad g = -\frac{\partial\Sigma}{\partial\nu}.$$

We recall that equalities (3) are used in the analysis of elastic materials with voids in the absence of dissipation (see [10]).

Let us assume that $\mathbf{M}$ is a fixed regular square matrix in $\mathcal{M}^+$, $\mathbf{b}$ is a fixed vector and $0 < \nu_0 \le 1$ is a constant number. In this paper we suppose that the motion

$$(4)\qquad \mathbf{x} = \mathbf{M}\mathbf{X} + \mathbf{b}, \qquad \nu = \nu_0 \quad \text{in } B,$$

is a solution of the problem determined by the equilibrium equations (1) and the boundary conditions

$$(5)\qquad \mathbf{x} = \mathbf{M}\mathbf{X} + \mathbf{b}, \qquad \nu = \nu_0 \quad \text{in } \partial B.$$

It is clear that the equality

$$(6)\qquad \frac{\partial\Sigma}{\partial\nu}(\mathbf{M},0,\nu_0) = 0,$$

is the necessary and sufficient condition for the energy function Σ to ensure that the deformation (4) is a solution to the problem determined by the equilibrium equations (1) and the boundary conditions (5).

Let us also note for later use that the divergence theorem applied to the equilibrium equations gives

$$(7)\qquad \int\limits_{\partial B}\mathbf{T}\cdot\mathbf{N}\,ds = 0, \qquad \text{and} \qquad \int\limits_{\partial B}\mathbf{S}\cdot\mathbf{N}\,ds + \int\limits_{B} g\,dv = 0.$$

In order to obtain the uniqueness result, we will introduce some assumptions on the energy function. We suppose that

(i) Σ is rank-one convex at $(\mathbf{M},0,\nu_0)$, i.e. the following inequality holds

$$(8)\qquad \Sigma(\mathbf{M}+\mathbf{a}\otimes\mathbf{d},\widetilde{\mathbf{a}},\nu_0) \ge \Sigma(\mathbf{M},0,\nu_0) + \frac{\partial\Sigma}{\partial\mathbf{F}}(\mathbf{M},0,\nu_0)\mathbf{a}\otimes\mathbf{d} + \frac{\partial\Sigma}{\partial\mathbf{G}}(\mathbf{M},0,\nu_0)\widetilde{\mathbf{a}},$$

for all $\mathbf{a}$, $\mathbf{d}$, $\widetilde{\mathbf{a}}$ in an n-dimensional Euclidean space, and

(ii) Σ satisfies the inequality

$$\int_D \Big[\Sigma(\mathbf{M} + \nabla\phi(\mathbf{X}), \nabla\psi(\mathbf{X}), \nu_0 + \eta(\mathbf{X})) - \frac{1}{n}\frac{\partial\Sigma}{\partial\nu}(\mathbf{M} + \nabla\phi(\mathbf{X}), \nabla\psi(\mathbf{X}), \nu_0 + \eta(\mathbf{X})) \cdot \eta(\mathbf{X})\Big]\, dv \geq \Sigma(\mathbf{M}, 0, \nu_0) \quad \text{volume } (D). \tag{9}$$

for all non-empty bounded subsets D and for all Lipschitz-continuous vectorial fields η, ϕ and ψ which vanish on the boundary of D, such that $\mathbf{M} + \nabla\phi(\mathbf{X}) \in \mathcal{M}^+$ for all $\mathbf{X} \in B$ and $\nabla\eta = \nabla\psi$. Furthermore we suppose that equality holds only when $\eta = \psi = 0$ and $\phi = 0$.

We remark that the last condition is related to a quasi-convexity assumption. The rank-one convexity and quasi-convexity assumptions are usual in the studies of nonlinear elasticity [1, 18–21]. One expects that the energetic condition:

(ii′) Σ satisfies the inequality

$$\int_D [\Sigma(\mathbf{M} + \nabla\phi(\mathbf{X}), \nabla\psi(\mathbf{X}), \nu_0 + \eta(\mathbf{X}))]\, dv \geq \Sigma(\mathbf{M}, 0, \nu_0) \quad \text{volume } (D)$$

and (i) could be sufficient to allow our uniqueness result, but our analysis does not guarantee it.

We can obtain a family of functions satisfying (i) and (ii). Let $W(\mathbf{F}, \mathbf{G}, \nu)$ be a function satifying (i) and (ii′) and $\partial W/\partial\nu(\mathbf{M}, 0, \nu_0) = 0$, and let $\Sigma(\mathbf{F}, \mathbf{G}, \nu)$ be the solution of the equation

$$n\Sigma + (\nu_0 - \nu)\partial\Sigma/\partial\nu = W.$$

Then Σ satisfies conditions (i) and (ii). An easy quadrature shows that

$$\Sigma(\mathbf{F}, \mathbf{G}, \nu) = (\nu - \nu_0)^n \int_\nu^{\nu_0} W(\mathbf{F}, \mathbf{G}, \xi)(\xi - \nu_0)^{-(n+1)}\, d\xi.$$

We finish this section by stating a Lemma on equalities of the conservation type.

LEMMA 1. Let $(\mathbf{x}, \nu)$ be a solution to the equations of equilibrium (1). Then the following equalities are satisfied:

(i) $(T_{Ai}x_i + S_A\nu)_{,A} = T_{Ai}x_{i,A} + S_A\nu_{,A} - g\nu$,

(ii) $\Sigma_{,K} = (T_{Ai}x_{i,K} + S_A\nu_{,K})_{,A}$,

(iii) $n\Sigma - g\nu + (X_K(T_{Ai}x_{i,K} + S_A\nu_{,K}))_{,A} = (X_K\Sigma)_{,K} + (T_{Ai}x_i + S_A\nu)_{,A}$.

P r o o f. The first equality follows from multiplying the first equation (1) by x_i and the second by ν. After addition we have

$$0 = T_{Ai,A}x_i + (S_{A,A} + g)\nu = (T_{Ai}x_i + S_A\nu)_{,A} - \{T_{Ai}x_{i,A} + S_A\nu_{,A} - g\nu\}.$$

Thus, the first equality is proved.

To obtain the second equality we proceed in a similar way, but multiply by $x_{i,K}$ and $\mu_{,K}$, respectively, to obtain

$$\begin{aligned}0 = T_{Ai,A}x_{i,K} &+ (S_{A,A} + g)\nu_{,K}\\ &= (T_{Ai}x_{i,K} + S_A\nu_{,K})_{,A} - (T_{Ai}x_{i,AK} + S_A\nu_{,KA} - g\nu_{,K}),\end{aligned}$$

which on using (3), becomes

$$0 = (T_{Ai}x_{i,K} + S_{Ai}y_{i,K})_{,A} - \Sigma_{,K},$$

and the second equality is proved.

The third equality is obtained from the second one by multiplying by X_K. We have

$$\begin{aligned}0 &= X_K\{(T_{Ai}x_{i,K} + S_A\nu_{,K})_{,A} - \Sigma_{,K}\}\\ &= (X_K(T_{Ai}x_{i,K} + S_A\nu_{,K}))_{,A} + n\Sigma - (T_{Ai}x_{i,A} + S_A\nu_{,A}) - (X_K\Sigma)_{,K}.\end{aligned}$$

From the equality (i), we finally obtain

$$0 = (n\Sigma - g\nu) + (X_K(T_{Ai}x_{i,K} + S_A\nu_{i,K}))_{,A} - (X_K\Sigma)_{,K} - (T_{Ai}x_i + S_A\nu)_{,A},$$

which implies (iii).

3. The uniqueness result

In this section we obtain a uniqueness theorem to the problem determined by the equilibrium equations (1) and the boundary conditions (5). To this end, it will be useful to introduce the function

$$\mathcal{J}(\mathbf{x}, \nu) = \int\limits_B \Sigma\, dv - \frac{1}{n}\int\limits_B g\nu\, dv. \tag{10}$$

Throughout this section, we suppose that B is an open bounded domain of the three-dimensional Euclidean space and that B is star-shaped with respect the origin which is located in B. It is clear that

$$\mathbf{X}\cdot\mathbf{N} \geq 0, \qquad \text{for all} \quad \mathbf{X} \in \partial B. \tag{11}$$

We have the following result:

LEMMA 2. Let B be defined as above. Let $(\mathbf{x}, \nu)$ be a smooth equilibrium solution to the system (1). Then

$$(12)\qquad n\mathcal{J}(\mathbf{x},\nu) = \int_{\partial B} \left\{ (\mathbf{N}\cdot\mathbf{X})\Sigma + \mathbf{T}^{\top}\cdot\left[\mathbf{N}\otimes\left(\mathbf{x} - r\frac{\partial\mathbf{x}}{\partial r}\right)\right] + \mathbf{S}^{\top}\cdot\left[\mathbf{N}\otimes\left(\nu - r\frac{\partial\nu}{\partial r}\right)\right]\right\} ds,$$

where $r = (\mathbf{X}\cdot\mathbf{X})^{1/2}$.

P r o o f. The proof follows by application of the divergence theorem to equality (iii) and use of the identities $r(\partial\mathbf{x}/\partial r) = \mathbf{X}\cdot\nabla\mathbf{x}$ and $r(\partial\nu/\partial r) = \mathbf{X}\cdot\nabla\nu$.

Our uniqueness theorem follows by considering the difference between two solutions and using the function $\mathcal{J}$.

Let $(\mathbf{x}, \nu)$ and $(\overline{\mathbf{x}}, \overline{\nu})$ be two solutions to the equilibrium equations (1) satisfying the same boundary conditions (5). Then we have

$$\begin{aligned} n(\mathcal{J}(\mathbf{x},\nu) - \mathcal{J}(\overline{\mathbf{x}},\overline{\nu})) = &\int_{\partial B} (\mathbf{N}\cdot\mathbf{X})\left\{\Sigma(\nabla\mathbf{x},\nabla\nu,\nu) - \Sigma(\nabla\overline{\mathbf{x}},\nabla\overline{\nu},\overline{\nu})\right\} ds \\ &+ \int_{\partial B}\left(\mathbf{T}^{\top}(\nabla\mathbf{x},\nabla\nu,\nu)\cdot\left[\mathbf{N}\otimes\left(\mathbf{x} - r\frac{\partial\mathbf{x}}{\partial r}\right)\right]\right. \\ &\left. - \mathbf{T}^{\top}(\nabla\overline{\mathbf{x}},\nabla\overline{\nu},\overline{\nu})\cdot\left[\mathbf{N}\otimes\left(\overline{\mathbf{x}} - r\frac{\partial\overline{\mathbf{x}}}{\partial r}\right)\right]\right) ds \\ &+ \int_{\partial B}\left(\mathbf{S}^{\top}(\nabla\mathbf{x},\nabla\nu,\nu)\cdot\left[\mathbf{N}\otimes\left(\nu - r\frac{\partial\nu}{\partial r}\right)\right]\right. \\ &\left. - \mathbf{S}^{\top}(\nabla\overline{\mathbf{x}},\nabla\overline{\nu},\overline{\nu})\cdot\left[\mathbf{N}\otimes\left(\overline{\nu} - r\frac{\partial\overline{\nu}}{\partial r}\right)\right]\right) ds. \end{aligned}$$

Now, on ∂B the two solutions $(\mathbf{x}, \nu)$ and $(\overline{\mathbf{x}}, \overline{\nu})$ coincide, so that

$$\mathbf{x} = \overline{\mathbf{x}} = \mathbf{M}\,\mathbf{X} + \mathbf{b} \qquad \text{and} \qquad \nu = \overline{\nu} = \nu_0, \quad \text{on } \partial B$$

and we deduce

$$\begin{aligned} \mathbf{T}^{\top}(\nabla\mathbf{x},\nabla\nu,\nu)\cdot\left[\mathbf{N}\otimes\left(\mathbf{x} - r\frac{\partial\mathbf{x}}{\partial r}\right)\right] - \mathbf{T}^{\top}(\nabla\overline{\mathbf{x}},\nabla\overline{\nu},\nu)\cdot\left[\mathbf{N}\otimes\left(\overline{\mathbf{x}} - r\frac{\partial\overline{\mathbf{x}}}{\partial r}\right)\right] \\ = \mathbf{T}^{\top}(\nabla\mathbf{x},\nabla\nu,\nu)\cdot\left[\mathbf{N}\otimes\left(r\frac{\partial\overline{\mathbf{x}} - \partial\mathbf{x}}{\partial r}\right)\right] \\ + \left[\mathbf{T}^{\top}(\nabla\mathbf{x},\nabla\nu,\nu) - \mathbf{T}^{\top}(\nabla\overline{\mathbf{x}},\nabla\overline{\nu},\nu)\right]\cdot\left[\mathbf{N}\otimes\left(\overline{\mathbf{x}} - r\frac{\partial\overline{\mathbf{x}}}{\partial r}\right)\right] \end{aligned}$$

and

$$\mathbf{S}^{\top}(\nabla\mathbf{x},\nabla\nu,\nu)\cdot\left[\mathbf{N}\otimes\left(\nu-r\frac{\partial\nu}{\partial r}\right)\right]-\mathbf{S}^{\top}(\nabla\overline{\mathbf{x}},\nabla\overline{\nu},\nu)\cdot\left[\mathbf{N}\otimes\left(\overline{\nu}-r\frac{\partial\overline{\nu}}{\partial r}\right)\right]$$
$$=\mathbf{S}^{\top}(\nabla\mathbf{x},\nabla\nu,\nu)\cdot\left[\mathbf{N}\otimes\left(r\frac{\partial\overline{\nu}-\partial\nu}{\partial r}\right)\right]$$
$$+\left[\mathbf{S}^{\top}(\nabla\mathbf{x},\nabla\nu,\nu)-\mathbf{S}^{\top}(\nabla\overline{\mathbf{x}},\nabla\overline{\nu},\nu)\right]\cdot\left[\mathbf{N}\otimes\left(\overline{\nu}-r\frac{\partial\overline{\nu}}{\partial r}\right)\right].$$

We also recall the following identities on ∂B (see [1])

$$r\frac{\partial(\overline{\mathbf{x}}-\mathbf{x})}{\partial r}=(\mathbf{N}\cdot\mathbf{X})\frac{\partial(\overline{\mathbf{x}}-\mathbf{x})}{\partial\mathbf{N}},\qquad r\frac{\partial(\overline{\nu}-\nu)}{\partial r}=(\mathbf{N}\cdot\mathbf{X})\frac{\partial(\overline{\nu}-\nu)}{\partial\mathbf{N}}$$

and

$$\nabla\overline{\mathbf{x}}=\nabla\mathbf{x}+\nabla\overline{\mathbf{x}}-\nabla\mathbf{x}=\nabla\mathbf{x}+\frac{\partial(\overline{\mathbf{x}}-\mathbf{x})}{\partial\mathbf{N}}\otimes\mathbf{N},$$
$$\nabla\overline{\nu}=\nabla\nu+\nabla\overline{\nu}-\nabla\nu=\nabla\nu+\frac{\partial(\overline{\nu}-\nu)}{\partial\mathbf{N}}\otimes\mathbf{N}.$$

From the previous equalities we deduce

$$n(\mathcal{J}(\mathbf{x},\nu)-\mathcal{J}(\overline{\mathbf{x}},\overline{\nu}))=\int_{\partial B}(\mathbf{N}\cdot\mathbf{X})\Big\{\Sigma(\nabla\mathbf{x},\nabla\nu,\nu)$$
$$-\Sigma\left(\nabla\mathbf{x}+\frac{\partial\overline{\mathbf{x}}-\partial\mathbf{x}}{\partial\mathbf{N}}\otimes\mathbf{N},\ \nabla\nu+\frac{\partial\overline{\nu}-\partial\nu}{\partial\mathbf{N}}\otimes\mathbf{N},\ \nu\right)$$
$$+\mathbf{T}(\nabla\overline{\mathbf{x}},\nabla\overline{\nu},\nu)\cdot\left[\frac{\partial\overline{\mathbf{x}}-\partial\mathbf{x}}{\partial\mathbf{N}}\otimes\mathbf{N}\right]+\mathbf{S}(\nabla\overline{\mathbf{x}},\nabla\overline{\nu},\nu)\cdot\left[\frac{\partial\overline{\nu}-\partial\nu}{\partial\mathbf{N}}\otimes\mathbf{N}\right]\Big\}ds$$
$$+\int_{\partial B}\Big\{[\mathbf{T}(\nabla\mathbf{x},\nabla\nu,\nu)-\mathbf{T}(\nabla\overline{\mathbf{x}},\nabla\overline{\nu},\nu)]\cdot\left[\left(\overline{\mathbf{x}}-r\frac{\partial\overline{\mathbf{x}}}{\partial r}\right)\otimes\mathbf{N}\right]$$
$$+\ [\mathbf{S}(\nabla\mathbf{x},\nabla\nu,\nu)\mathbf{X}-\mathbf{S}(\nabla\overline{\mathbf{x}},\nabla\overline{\nu},\nu)]\cdot\left[\left(\overline{\nu}-r\frac{\partial\overline{\nu}}{\partial r}\right)\otimes\mathbf{N}\right]\Big\}\,ds. \tag{13}$$

Now, we may state:

Lemma 3. Let B be defined as above and let $(\mathbf{x},\nu)$ be a smooth solution to the equilibrium system (1) such that boundary conditions (5) are satisfied. Let us suppose that Σ satisfies the condition (8). Then

$$\mathcal{J}(\mathbf{x},\nu)+\frac{\overline{\nu}}{n}\int_{B}g\,dv\le\mathcal{J}(\overline{\mathbf{x}},\overline{\nu}), \tag{14}$$

where $(\overline{\mathbf{x}},\overline{\nu})$ is a solution defined by (4).

P r o o f. We apply the inequalities (8) and (11) to the first integrand on the right-hand side of equality (13) to conclude that

$$
\begin{aligned}
(15) \qquad & n(\mathcal{J}(\mathbf{x},\nu) - \mathcal{J}(\bar{\mathbf{x}},\bar{\nu})) \\
&\leq \int_{\partial B} \Big\{ [\mathbf{T}(\nabla\mathbf{x},\nabla\nu,\nu) - \mathbf{T}(\nabla\bar{\mathbf{x}},\nabla\bar{\nu},\nu)] \cdot \Big[\Big(\bar{\mathbf{x}} - r\frac{\partial\bar{\mathbf{x}}}{\partial r}\Big) \otimes \mathbf{N}\Big] \\
&\quad + [\mathbf{S}(\nabla\mathbf{x},\nabla\nu,\nu) - \mathbf{S}(\nabla\bar{\mathbf{x}},\nabla\bar{\nu},\nu)] \cdot \Big[\Big(\bar{\nu} - r\frac{\partial\bar{\nu}}{\partial r}\Big) \otimes \mathbf{N}\Big]\Big\}\, ds.
\end{aligned}
$$

On the other hand, from (4), we have $\nabla\bar{\mathbf{x}} = \mathbf{M}$ for all $\mathbf{X} \in B$. Then it follows that

$$
\bar{\mathbf{x}} - r\frac{\partial\bar{\mathbf{x}}}{\partial r} = \mathbf{b} \qquad \text{and} \qquad \bar{\nu} - r\frac{\partial\bar{\nu}}{\partial r} = \nu_0,
$$

and inequality (15) therefore yields

$$
\begin{aligned}
n(\mathcal{J}(\mathbf{x},\nu) - \mathcal{J}(\bar{\mathbf{x}},\bar{\nu})) \leq \int_{\partial B} & [\mathbf{T}(\nabla\mathbf{x},\nabla\nu,\nu) - \mathbf{T}(\nabla\bar{\mathbf{x}},\nabla\bar{\nu},\bar{\nu})] \cdot [\mathbf{b} \otimes \mathbf{N}]\, ds \\
& + [\mathbf{S}(\nabla\mathbf{x},\nabla\nu,\nu_0) - \mathbf{S}(\nabla\bar{\mathbf{x}},\nabla\bar{\nu},\nu_0)] \cdot [\nu_0 \otimes \mathbf{N}]\, ds.
\end{aligned}
$$

Inequality (14) follows from (7) on recalling that $\mathbf{b}$ and ν_0 are constants.

Now, we may state the uniqueness result:

THEOREM 1. *Let $B, (\mathbf{x},\nu), (\bar{\mathbf{x}},\bar{\nu})$, $\mathbf{M}$ and ν_0 be as in the previous Lemma, and let the energy Σ satisfy the condition* (9). *Then $(\mathbf{x},\nu)$ is a solution defined by* (4).

P r o o f. Let us suppose that $(\mathbf{x},\nu) \neq (\bar{\mathbf{x}},\bar{\nu}) = (\mathbf{M}\,\mathbf{X} + \mathbf{b}, \nu_0)$. Then assumption (9) implies

$$
\mathcal{J}(\bar{\mathbf{x}},\bar{\nu}) < \mathcal{J}(\mathbf{x},\nu) + \frac{\bar{\nu}}{n}\int_B g\, dv,
$$

which contradicts Lemma 3. Hence $(\mathbf{x},\nu) = (\bar{\mathbf{x}},\bar{\nu})$ for all $\mathbf{X} \in B$.

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